(M_t) is a càdlàg L^p-martingale $(p>1)$ if it is a càdlàg martingale and $M_t \in L^p$, $\forall t \geq 0$

Prop (Doob's maximal inequality) If M is a càdlàg L^p mtgle for some $p>1$ then

$$E\left(\sup_{s\in[0,t]} |M_s|^p\right) \leq \left(\frac{p}{p-1}\right)^p E(|M_t|^p), \text{ for every } t \geq 0$$

Sketch of proof: if F is an arbitrary finite set in $[0,t]$ then $(M_s)_{s\in F}$ is a discrete-time martingale in its own filtration

Therefore discrete theory tells us that

$$E\left(\max_{s\in F} |M_s|^p\right) \leq \left(\frac{p}{p-1}\right)^p \max_{s\in F} E\left(|M_s|^p\right) \leq \left(\frac{p}{p-1}\right)^p E(|M_t|^p)$$

(why the last step?)

Now replace F by $F_n = \left\{\frac{jt}{2^n} : 0 \leq j \leq 2^n\right\}$ and take limits $n \to \infty$

By the dominated convergence theorem $\lim_{n\to\infty} E\left(\max_{s\in F_n} |M_s|^p\right) =$

$= E\left(\sup_{n\geq 1} \max_{s\in F_n} |M_s|^p\right)$ and the sup is equal to $\sup_{s\in[0,t]} |M_s|^p$ because M and $s \mapsto |M_s|^p$ is càdlàg. $\square$

Similarly we can derive the following from the discrete-parameter theory of martingales

Prop Let M be a càdlàg martingale s.t. either

a) $M_t \geq 0$ a.s. $\forall t \geq 0$ or b) $\sup_{t\geq 0} E(|M_t|) < \infty$

(or more general uniformly integrable)

Then $\lim_{t\to\infty} M_t$ exists a.s and it is finite a.s.

Propo Let M be a martingale (hence $t \mapsto E(M_t)$ is constant)
Then there exist a unique modification of M which is càdlàg
(see Protter p 8)

As an application recall the following fact: let Y be an integrable r.v. Then $M_t = \mathbb{E}(Y \mid \mathcal{F}_t)$ defines a martingale. Recall that T is a stopping time if $\{T \leq t\} \in \mathcal{F}_t$, $\forall t \geq 0$ and the σ-algebra $\mathcal{F}_T$ is defined as in discrete parameter theory $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$
If T is a simple stopping time with possible values in a finite (non-random) set F then for all $A \in \mathcal{F}_T$

$$\mathbb{E}(\mathbb{E}(Y / \mathcal{F}_T) \mathbb{1}_A) = \mathbb{E}(Y \mathbb{1}_A) = \sum_{t \in F} \mathbb{E}(Y \mathbb{1}_{A \cap \{T = t\}})$$

Since $A \cap \{T = t\} \in \mathcal{F}_t$ it follows that

$$\mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_T) \mathbb{1}_A) = \sum_{t \in F} \mathbb{E}(M_t \mathbb{1}_{A \cap \{T = t\}}) = \mathbb{E}(M_T \mathbb{1}_A) \quad \forall A \in \mathcal{F}_T$$

Therefore $M_T = \mathbb{E}(Y \mid \mathcal{F}_T)$ a.s. for all simple stopping times T. If T is a bounded stopping time then we can find simple stopping times $T_n \downarrow T$. Therefore the càdlàg version of M satisfies $M_T = \mathbb{E}(Y \mid \mathcal{F}_T)$ a.s. for all bounded (hence a.s-finite) stopping times T.

Let us end this section with two useful results

Thm Every Lévy process has a càdlàg modification that is itself a Lévy process

Proof Let X be a Lévy process that is adapted to $(\mathcal{F}_t^X)$.
Let D be a countable dense set in $[0,\infty)$. At each $t>0$ the left and right limits $M_u(t-)$ and $M_u(t+)$ of M_u exist along D a.s. (see Revuz-Yor p. 63-65)
For each $u \in \mathbb{R}^d$ let $\Omega'_u \subset \Omega$ the subset for which these limits fail to exist. Then $\Omega' = \bigcup_{u\in\mathbb{R}^d} \Omega'_u$ is also a set of $\mathbb{P}$-meas zero.
Fix $\omega \in \Omega\setminus\Omega'$ and for each $t\geq 0$ let $(s_n) \subset D$, $s_n \uparrow t$.
It can be proved that $\{X_{s_n}(\omega)\}$ has a unique accumulation point$^{(*)}$, hence X has a unique left limit along D at every $t>0$ on $\Omega\setminus\Omega'$. Similarly X has a unique right limit along D at every $t\geq 0$ on $\Omega\setminus\Omega'$.
Set
$$Y_t(\omega) = \begin{cases} \lim_{s\in D, s\downarrow t} X_s(\omega) & \text{if } \omega \in \Omega\setminus\Omega' \\ 0 & \text{if } \omega \in \Omega' \end{cases}$$
Then (Y_t) is càdlàg and Y is a modification of X since for every $t\geq 0$
$$\mathbb{E}\left(e^{i\langle u, Y_t - X_t\rangle}\right) = \lim_{\substack{s\in D \\ s\downarrow t}} \mathbb{E}\left(e^{i\langle u, X_s - X_t\rangle}\right) = 1$$
by b) and c) in definition of a Lévy process. Hence $\mathbb{P}(Y_t = X_t) = 1$. The result follows from the following Lemma $\square$

<u>Lemma</u> If Y is a modification of a Levy process X then Y is a Levy process with same caracteristics as X.

<u>Proof</u> b) is clear. To prove c) fix $0 \le s < t$ and denote $N(s,t) = \{\omega \in \Omega : X_s(\omega) = Y_s(\omega) \text{ and } X_t(\omega) = Y_t(\omega)\}$

Then $\mathbb{P}(N(s,t)^c) \le \mathbb{P}(X_s \ne Y_s) + \mathbb{P}(X_t \ne Y_t) = 0$

so $\mathbb{P}(N(s,t)) = 1$. Let $A \in \mathcal{B}(\mathbb{R}^d)$

$$\mathbb{P}(Y_t - Y_s \in A) = \mathbb{P}(Y_t - Y_s \in A, N(s,t)) + \mathbb{P}(Y_t - Y_s \in A, N(s,t)^c)$$
$$= \mathbb{P}(X_t - X_s \in A, N(s,t)) \le \mathbb{P}(X_t - X_s \in A)$$
$$= \mathbb{P}(X_{t-s} \in A) = \mathbb{P}(Y_{t-s} \in A)$$

The reverse inequality is similar. The same type of reasoning gives d) and a). It is easy to see that X and Y have the same characteristics. $\square$

<u>End of proof of Thm (*)</u>: suppose that $x_t^1(\omega), x_t^2(\omega)$ are distinct accumulation points of $\{X_{s_n}(\omega)\}$ limits with (s_{n_j}) resp (s_{n_k}). But since $\exists\, M_u(t-)$, the limit $\lim_{s_n \uparrow t} e^{i\langle u, X_{s_n}(\omega)\rangle}$ exists hence $x_t^1(\omega), x_t^2(\omega)$ are finite

Choose $u \in \mathbb{Q}^d$ s.t. $\langle u, x_t^1(\omega) - x_t^2(\omega)\rangle \ne 2n\pi$ for any $n \in \mathbb{Z}$

By continuity $\lim_{s_{n_j} \uparrow t} e^{i\langle u, X_{s_{n_j}}(\omega)\rangle} = e^{i\langle u, x_t^1(\omega)\rangle}$ and

$\lim_{s_{n_k} \uparrow t} e^{i\langle u, X_{s_{n_k}}(\omega)\rangle} = e^{i\langle u, x_t^2(\omega)\rangle}$. This gives a contradiction.

a) for each $T>0$, $\exists$ a subsequence (deterministic) $(\varepsilon_n^T)_{n\geq 1}$ tending towards 0 s.t.

$$\mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq T} \left|M_s^{\varepsilon_n^T} - M_s\right|^2 = 0\right) = 1$$

b) (M_t) is $(\mathcal{F}_t)_{t\geq 0}$-adapted

c) (M_t) has càdlàg paths a.s.

d) it has stationary and independent increments

e) it has, at most, a countable number of discontinuities on $[0,T]$ a.s

Rk With other words M is a Lévy process which is also a martingale (and it has a countable number of discontinuities on each finite interval)

Proof Let us prove a). Choose $0<\delta<\varepsilon<1$, and fix $T>0$

A similar calculation as in Lemma 2, second part gives

(think $d=1$ for simplicity then norm on $\mathbb{R}^d$)

$$E\left[\left|M_T^{\varepsilon} - M_T^{\delta}\right|^2\right] = E\left[\left|\int_{[0,T]}\int_{\delta\leq|z|<\varepsilon} z\, N(ds,dz)\right|^2\right]$$

$$= T\int_{\delta\leq|z|<\varepsilon} |z|^2\, N(ds,dz)$$

Note that the left hand side above is also equal to

$\|M^\varepsilon - M^\delta\|^2_{\mathcal{M}^2_T}$. Thanks to the assumption $\int_{B_1} |z|^2 \, \nu(dz) < \infty$

we can see that $\lim_{\varepsilon,\delta \downarrow 0} \|M^\varepsilon - M^\delta\|_{\mathcal{M}^2_T} = 0$, hence $\{M^\varepsilon, 0<\varepsilon<1\}$

is a Cauchy family in $\mathcal{M}^2_T$. By Lemma 3 there exists a martingale

in $\mathcal{M}^2_T$, $(M_t)_{t\in[0,T]}$ s.t. $\lim_{\varepsilon \downarrow 0} \|M^\varepsilon - M\|_{\mathcal{M}^2_T} = 0$

We apply again the Doob maximal inequality:

$$\mathbb{E}\left(\sup_{0\le s\le T} |M^\varepsilon_s - M_s|^2\right) \le 4\, \mathbb{E}\left(|M^\varepsilon_T - M_T|^2\right)$$

hence

$$(*) \quad \lim_{\varepsilon\downarrow 0} \mathbb{E}\left(\sup_{0\le s\le T} |M^\varepsilon_s - M_s|^2\right) \le 4 \lim_{\varepsilon\downarrow 0} \|M^\varepsilon - M\|^2_{\mathcal{M}^2_T} = 0$$

Let us prove that the limit does not depend on T. Suppose it did: $(M_{t,T})_{t\in[0,T]}$ the limit. Then, by $(*)$, for any $0 < T' < T$

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left(\sup_{0\le s\le T'} |M^\varepsilon_s - M_{s,T'}|^2\right) = 0$$

By a simpler upper bound

$$\mathbb{E}\left(\sup_{0\le s\le T'} |M^\varepsilon_s - M_{s,T}|^2\right) \le \mathbb{E}\left(\sup_{0\le s\le T} |M^\varepsilon_s - M_{s,T}|^2\right)$$

and both have limit 0 as $\varepsilon \downarrow 0$

Since $\sup |u_n|^2 = (\sup |u_n|)^2$, $\sup |u_n + v_n| \leq \sup |u_n| + \sup |v_n|$ for (u_n), (v_n) sequences of real numbers, by Minkowski's inequality we get

$$E\left(\sup_{0 \leq s \leq T'} |M_{s,T'} - M_{s,T}|^2\right)^{1/2}$$

$$\leq \lim_{\varepsilon \downarrow 0} E\left(\sup_{0 \leq s \leq T'} |M_s^\varepsilon - M_{s,T'}|^2\right)^{1/2}$$

$$+ \lim_{\varepsilon \downarrow 0} E\left(\sup_{0 \leq s \leq T'} |M_s^\varepsilon - M_{s,T}|^2\right)^{1/2} = 0$$

Therefore $M_{\cdot,T'}$ and $M_{\cdot,T}$ are a.s. uniformly equal on $[0,T]$ and since T' and T may be arbitrarily chosen, we can conclude that the martingale limit (M_t) is well defined on $[0,\infty)$

By using again $(*)$ we may deduce that there exists a deterministic subsequence $(\varepsilon_n^T)_{n \geq 0}$ s.t.

$$\lim_{\varepsilon_n^T \downarrow 0} \sup_{0 \leq s \leq T} |M_s^{\varepsilon_n^T} - M_s|^2 = 0 \quad \text{a.s.}$$

(L^2-convergence implies convergence in probability and so a.s. convergence on a subsequence)

Let us prove now b): fix $0 < t < T$. Clearly

$M_t^{\varepsilon_n^T} \in \mathcal{F}_t$ and by a) this sequence has an a.s. 62

limit as $n \to \infty$. By standard measure theory the limit is also $\mathcal{F}_t$-measurable

Let us prove c). Paths of M^{ε} are càdlàg. The a.s. convergence along a subsequence on finite time intervals implies that the limiting process, M, has paths which are càdlàg. Indeed this a consequence of the following deterministic lemma

Lemma A | Let $D([0,1];\mathbb{R})$ the set of càdlàg real functions. Then $D([0,1];\mathbb{R})$ is a closed subspace of $C([0,1];\mathbb{R})$ under the metric $d(x,y) = \sup_{0\le t\le 1} |x(t)-y(t)|$, $x,y \in D([0,1];\mathbb{R})$

[$(x_n) \subset D([0,1];\mathbb{R})$, and $\overbrace{d(x_n,x)}^{x:[0,1]\to\mathbb{R} \text{ is s.t}} \to 0$, then $x \in D([0,1];\mathbb{R})$]

(exe)

Let us prove d) Uniform a.s. convergence along a subsequence guarantees convergence in distribution for a collection of fixed times along the same subsequence. For any $0\le u\le v\le s\le t\le T$ and any $\lambda_1,\lambda_2 \in \mathbb{R}^d$ one can write

$$\mathbb{E}\left(e^{i\langle\lambda_1, M_v - M_u\rangle} e^{i\langle\lambda_2, M_t - M_s\rangle}\right)$$

$$= \lim_{\varepsilon_n^T \downarrow 0} \mathbb{E}\left(e^{i\langle\lambda_1, M_v^{\varepsilon_n^T} - M_u^{\varepsilon_n^T}\rangle} e^{i\langle\lambda_2, M_t^{\varepsilon_n^T} - M_s^{\varepsilon_n^T}\rangle}\right)$$

$$= \lim_{\varepsilon_n^T \downarrow 0} \mathbb{E}\left(e^{i\langle\lambda_1, M_{v-u}^{\varepsilon_n^T}\rangle}\right)\mathbb{E}\left(e^{i\langle\lambda_2, M_{t-s}^{\varepsilon_n^T}\rangle}\right)$$

$= \mathbb{E}\left(e^{i\langle\lambda_1, M_{v-u}\rangle}\right)\mathbb{E}\left(e^{i\langle\lambda_2, M_{t-s}\rangle}\right)$ by using dominated convergence.

Finally to prove e) it suffices to note that càdlàg functions have discontinuities of the first kind and there are necessarily a countable number of such discontinuities:

Lemma B: | If $x \in D([0,1];\mathbb{R})$ then the following sets are countable

$$\Delta_\theta = \{t \in [0,1] : |x(t) - x(t-)| > \theta\}, \forall \theta > 0$$

$$\Delta = \{t \in [0,1] : |x(t) - x(t-)| \neq 0\}$$

(exe)

$\boxed{\equiv}$