of the distribution of the unique solution (X_t) of the (L,D)-mp with initial law δ_x is measurable w.r.t x.

Rk A Feller process (X_t) solves the (A,D)-m.p where A is its i.g and $D \subset \mathcal{D}(A)$. If the (L,D)-mp is well posed there can exist no more than one Feller process with a generator being an extension of L

Propo (equivalent formulations) Suppose that D is closed under addition of constants ($L1=0$). Then the following are equivalent

a) X is a solution of the (L,D)-m.p starting from μ

b) $X_0 \sim \mu$ and $\exists (\mathcal{F}_t)$ s.t $\forall \lambda > 0, \forall f \in D$

$$e^{-\lambda t} f(X_t) - f(X_0) - \int_0^t e^{-\lambda s} (\lambda f(X_s) - Lf(X_s)) ds$$

is a $(\mathcal{F}_t)$-martingale

c) $X_0 \sim \mu$ and $\exists (\mathcal{F}_t)$ s.t $\forall f \in D$, $\inf_{y \in \mathbb{R}^d} f(y) > 0$

$$R_t^f = \frac{f(X_t)}{f(X_0)} \exp\left\{ - \int_0^t \frac{Lf(X_s)}{f(X_s)} ds \right\}$$

is a $(\mathcal{F}_t)$-martingale

Proof Let us verify a) $\Leftrightarrow$ c)

$$f(X_t) \exp\left\{ - \int_0^t \frac{Lf(X_s)}{f(X_s)} ds \right\} =$$

f i.p

$\downarrow$

$$= f(X_0) + \int_0^t \exp\left\{-\int_0^r \frac{Lf(X_s)}{f(X_s)}\,ds\right\} dM_r^f$$

where (M_t^f) is given in (1)

So if (M_t^f) is a martingale then (R_t^f) is a martingale

Conversely, if (R_t^f) is a martingale

$$M_t^f = f(X_0) + \int_0^t \exp\left\{\int_0^r \frac{Lf(X_s)}{f(X_s)}\right\} dR_r^f$$

is a martingale

(here f is strictly positive that can be done by adding a constant to f)

$\blacksquare$

Being a solution of a (L,D)-m.p. is an assertion about the finite dimensional distributions of X: indeed (X_t) solves (L,D)-m.p. iff

(2) $\mathbb{E}\, g(X) = 0$

for any partition $0 \le t_1 < t_2 < \dots < t_{m+1}$ of $\mathbb{R}_+$, $f \in D$ and $h_1, \dots, h_m \in C(\mathbb{R}^d)$, where

(3) $$g(X) = g(X, f, \{h_i\}, \{t_i\}) = \left(f(X_{t_{m+1}}) - f(X_{t_m}) - \int_{t_m}^{t_{m+1}} Lf(X_s)\,ds\right) \prod_{k=1}^{m} h_k(X_{t_k})$$

(exe)

(see Ethier-Kurtz p. 174)

Propo (uniqueness of 1-dim distb implies uniqueness fdd.)

Let X, Y any two solutions of the (L,D)-mp s.t. for any $\mu \in \mathcal{P}(\mathbb{R}^d)$

$$X_0 \sim \mu, \quad Y_0 \sim \mu$$

and

$$(4) \quad \mathbb{P}(X_t \in B) = \mathbb{P}(Y_t \in B), \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \ \forall t \geq 0$$

Then X and Y have the same fdd (uniqueness)

Proof Assume X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and Y on $(\Omega', \mathcal{F}', \mathbb{P}')$ and we show that

$$(*)_m \quad \mathbb{E}^{\mathbb{P}}\left(\prod_{i=1}^m h_i(X_{t_i})\right) = \mathbb{E}^{\mathbb{P}'}\left(\prod_{i=1}^m h_i(Y_{t_i})\right)$$

for all choices of $0 \leq t_0 < t_1 < \dots < t_m < \infty$ and

$h_1, \dots, h_m \in C(\mathbb{R}^d)$ positives

For $m=1$ $(*)_m$ is (4). Proceed by induction on m. Assume $(*)_m$ true for all $m \leq n$ and fix $t_1, \dots, t_m$, $h_1, \dots, h_m$, $h_j > 0$

Set $$\tilde{\mathbb{P}}(B) = \frac{\mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_B \prod_{j=1}^n h_j(X_{t_j})\right)}{\mathbb{E}^{\mathbb{P}}\left(\prod_{j=1}^n h_j(X_{t_j})\right)}, \quad B \in \mathcal{F}$$

$$\tilde{\mathbb{P}}'(B) = \frac{\mathbb{E}^{\mathbb{P}'}\left(\mathbb{1}_B \prod_{j=1}^n h_j(Y_{t_j})\right)}{\mathbb{E}^{\mathbb{P}'}\left(\prod_{j=1}^n h_j(Y_{t_j})\right)}, \quad B \in \mathcal{F}'.$$

Set $\tilde{X}_t = X_{t_n+t}$, $\tilde{Y}_t = Y_{t_n+t}$. We can prove that:

(#) $\tilde{X}$ and $\tilde{Y}$ are solutions of (L,D)-mp on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ resp $(\Omega', \mathcal{F}', \tilde{\mathbb{P}}')$

(see next result)

Furthermore by $(*_m)$ with $m = n$

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[f(\tilde{X}_0)\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[f(X_{t_n}) \prod_{j=1}^{n} h_j(X_{t_j})\right]}{\mathbb{E}^{\mathbb{P}}\left(\prod_{j=1}^{n} h_j(X_{t_j})\right)}$$

$$\overset{\binom{\#}{\#}}{=} \frac{\mathbb{E}^{\mathbb{P}'}\left[f(Y_{t_n}) \prod_{j=1}^{n} h_j(Y_{t_j})\right]}{\mathbb{E}^{\mathbb{P}'}\left(\prod_{j=1}^{n} h_j(X_{t_j})\right)} = \mathbb{E}^{\tilde{\mathbb{P}}'}\left(f(\tilde{Y}_0)\right)$$

$\forall f$ Borel fct.

Hence $\tilde{X}_0, \tilde{Y}_0$ have the same initial distribution

By (#) we get $\mathbb{E}^{\tilde{\mathbb{P}}}\left(f(\tilde{X}_t)\right) = \mathbb{E}^{\tilde{\mathbb{P}}'}\left(f(\tilde{Y}_t)\right)$ $\forall f, \forall t$

As in $\binom{\#}{\#}$ the last equality implies

$$\mathbb{E}^{\mathbb{P}}\left[f(X_{t_n+t}) \prod_{j=1}^{n} h_j(X_{t_j})\right]$$

$$= \mathbb{E}^{\mathbb{P}'}\left[f(Y_{t_n+t}) \prod_{j=1}^{n} h_j(Y_{t_j})\right]$$

Setting $t_{n+1} = t_n + t$ we have $(*)_m$ for $m = n+1$ ▩

Propo (Markov property)

Let X, Y any two solutions of the (L, D)-m.p starting from μ and suppose $(\mathcal{F}_t)$ (this means well posed m.p)

Then any solution of the (L,D)-m.p w.r.t $(\mathcal{F}_t)$ has the Markov prop w.r.t to $\mathcal{F}_t$

$$(5) \quad \mathbb{E}\left(f(X_{r+t}) \mid \mathcal{F}_r\right) = \mathbb{E}\left(f(X_{r+t}) \mid X_r\right)$$

$$\forall f \text{ Borel function}, r, t \geq 0$$

and even the strong Markov property (τ stopping time finite)

Proof (5) is equivalent with $\forall F \in \mathcal{F}_r$

$$(6) \quad \mathbb{E}\left(\mathbb{1}_F \, \mathbb{E}(f(X_{r+t}) \mid \mathcal{F}_r)\right) = \mathbb{E}\left(\mathbb{1}_F \, \mathbb{E}(f(X_{r+t}) \mid X_r)\right)$$

It is sufficient to show for F s.t $\mathbb{P}(F) > 0$.

Set $\mathbb{P}_1(B) = \dfrac{\mathbb{E}(\mathbb{1}_F \, \mathbb{E}(\mathbb{1}_B \mid \mathcal{F}_r))}{\mathbb{P}(F)}$ $\quad B \in \mathcal{F}$

$$\mathbb{P}_2(B) = \frac{\mathbb{E}(\mathbb{1}_F \, \mathbb{E}(\mathbb{1}_B \mid X_r))}{\mathbb{P}(F)}$$

and we show $\mathbb{P}_1 = \mathbb{P}_2$. If $B \in \sigma(X_r) \subset \mathcal{F}_r$

$$\mathbb{P}_1(B) = \mathbb{P}_2(B) = \frac{\mathbb{P}(B \cap F)}{\mathbb{P}(F)} = \mathbb{P}(B \mid F)$$

Set $Y_t = X_{t+r}$

$$\mathbb{P}_1(Y_0 \in \Gamma) = \frac{\mathbb{E}(\mathbb{1}_F \mathbb{E}(\mathbb{1}_{Y_0 \in \Gamma} | \mathcal{F}_r))}{\mathbb{P}(F)}$$

$$= \frac{\mathbb{E}(\mathbb{1}_F \mathbb{E}(\mathbb{1}_{X_r \in \Gamma} | \mathcal{F}_r))}{\mathbb{P}(F)}$$

$$= \frac{\mathbb{E}(\mathbb{1}_F \mathbb{1}_{X_r \in \Gamma})}{\mathbb{P}(F)} = \frac{\mathbb{E}(\mathbb{1}_F \mathbb{E}(\mathbb{1}_{X_r \in \Gamma} | X_r))}{\mathbb{P}(F)}$$

$$= \mathbb{P}_2(Y_0 \in \Gamma)$$

Since X solves m.p. $\mathbb{E}(g(X_{r+\cdot}) | \mathcal{F}_r) = 0$ where g is given by (3). But then $\mathbb{E}(g(X_{r+\cdot}) | X_r) = 0$ implying

$$\mathbb{E}_1(g(Y)) = \mathbb{E}_2(g(Y)) = 0$$

Hence Y is sol of (L, D) - m.p. on $(\Omega, \mathcal{F}, \mathbb{P}_1)$ and on $(\Omega, \mathcal{F}, \mathbb{P}_2)$. This means by (4)

$\mathbb{E}_1 f(Y_t) = \mathbb{E}_2 f(Y_t) \quad \forall t \geq 0, \forall f$ Borel funct

This exactly (5)

By stopping thm $\mathbb{E}[g(X_{T+\cdot}) | \mathcal{F}_T] = 0$

so the same proof work to get strong Markov thm $\blacksquare$

Corollary (convergence and continuous dependence on initial conditions)

Let X^n solving the m.p. for operators L^n with common domain $D \subset C(\mathbb{R}^d)$ that forms an algebra and is dense in $C(\mathbb{R}^d)$ for the topology of uniform convergence on compacts (typically $D = C_c^2(\mathbb{R}^d)$)

Suppose that X^n satisfies the <u>compact containment condition</u>: $\forall T>0, \forall \varepsilon>0, \exists R = R_{T,\varepsilon}$ s.t

(CCC) $\inf_n \mathbb{P}\left(\sup_{s\in[0,T]} |X_s^n| \leq R \right) \geq 1-\varepsilon$)

Then the family of laws of X^n is tight

If additionally $\|(L^n - L)f\| \xrightarrow[n\to\infty]{} 0$, $\forall f \in D$

then any convergent subsequence of laws of X^n converges to a solution of the (L,D)-mp

To prove this corollary we need to

- study convergence of cadlag processes
- study convergence of martingales

in the next section. The proof of the corollary is postponed at the end of Section 20 (see page

Proof of Corollary page 149 As X^n solves the m.p

$$f(X_t^n) = f(X_0^n) + M_t^{f,n} + \int_0^t L^n f(X_s^n)\,ds$$

so we are in the setting of the Corollary of Rebolledo's criterion (p 158)

At this level we need the following lemma

Lemma Let (X_t) be a solution of the (L,D)-m.p such that D forms an algebra under pointwise multiplication. Then, $\forall f,g \in D$

(*) $$\langle f(X), g(X)\rangle_t = \langle M^f, M^g\rangle_t = \int_0^t \left(L(fg) - fLg - gLf\right)(X_s)\,ds$$

where $M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$, $M_t^g = g(X_t) - g(X_0) - \int_0^t Lg(X_s)\,ds$

Admit the lemma and let us finish the proof of Corollary:

$$\langle f(X^n)\rangle_t = \langle f(X^n), f(X^n)\rangle_t = \int_0^t \left(L^n(f^2) - 2fL^n f\right)(X_s^n)\,ds$$

We have that

(#) $\int_0^\cdot L^n f(X_s^n)\,ds$ and $\int_0^\cdot \left(L^n f^2(X_s^n) - 2f(X_s^n) L^n f(X_s^n)\right)ds$ satisfy the Aldous condition (A).

Indeed the integrands are uniformly bounded, by taking $f \in D = C_c^2(\mathbb{R}^d)$. For instance

$$\left|\int_{\tau_n}^{\tau_n+\delta_n} L^n f(X_s^n)\,ds\right| \leq \int_{\tau_n}^{\tau_n+\delta_n} |L^n f(X_s^n)|\,ds \leq C\delta_n \longrightarrow 0$$

(the other term is similar)

By (#) we deduce that $f(X^n)$ satisfies also (A). We use now (CCC) and we deduce $f(X^n)$ is tight for any $f \in D$ (see Fact 2 p 151)

For any $\varepsilon > 0$, there exists a compact set K s.t the probability that any X^n has values outside K does not exceed ε and there exists $f \in D$ s.t. $\sup_{x\in K} |f(x) - x| \leq \varepsilon$. This clearly implies (A) for X^n and we conclude as previously. By dominated convergence it is clear that any limiting point of X^n solves the (L,D)-m.p $\blacksquare$

<u>Proof of Lemma</u> The processes M^f, M^g and

$$M_t^{fg} = (fg)(X_t) - (fg)(X_0) - \int_0^t L(fg)(X_s)\,ds$$ are well-defined

martingales. By properties of quadratic variation $[f(X), g(X)]_t = [M^f, M^g]_t$ hence the first equality in $(*)$ is verified. By integration by parts formula p 86, if $X_t^1 = M_t^1 + \int_0^t b_s^1\,ds$, $X_t^2 = M_t^2 + \int_0^t b_s^2\,ds$ with M^1, M^2 square integrable martingales and b^1, b^2 bounded càdlàg processes, then

$$(X^1X^2)_t - [X^1, X^2]_t - \int_0^t (X_s^1 b_s^2 + X_s^2 b_s^1)\,ds$$

is a martingale

We deduce that

$$[f(X), g(X)]_t = f(X_t)g(X_t) - \int_0^t \big(f(X_s)\,Lg(X_s) + g(X_s)\,Lf(X_s)\big)ds + M_t$$

where M_t is a martingale. Hence

$$[f(X), g(X)]_t = \int_0^t \big(L(fg)(X_s) - f(X_s)\,Lg(X_s) - g(X_s)\,Lf(X_s)\big)ds$$

with $\tilde{M}_t$ a martingale and we get the second equality in $(*)$. $\blacksquare$