18) Feller processes / Levy-type processes

Recall that the semigroup of a Levy process is a Feller process but the translation invariance gives the convolution structure of the semigroup. What should be the changes when processes whose semigroups are Feller semigroups but not translation invariant (still stay Markov processes)

- First remark: we can construct a modification of a Feller process s.t its paths are càdlàg. As for Levy processe instead look to $t \mapsto X_t$ we look to $t \mapsto u(X_t)$ with $u : \mathbb{R} \to \mathbb{R}$ (it suffices to consider the coordinates of X_t). The idea is to use a martingale, so choose u st $u(X_t)$ is a supermartingale. We use the resolvent:
  If $f \geq 0$, $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ then $M_t = e^{-\lambda t} R_\lambda f(X_t)$ is a supermartingale

- Second remark: every Feller process is a strong Markov process

$$\mathbb{E}_x\left(f(X_{t+T}) \mid \mathcal{F}_T\right) = \mathbb{E}_{X_T} f(X_t),$$

$$\mathbb{P}_x\text{-a.s on } \{T < \infty\}, \ T \text{ stopping time}.$$

Proof: Approximate T from above by discrete stopping times $T_n = \dfrac{\lfloor 2^n T \rfloor + 1}{2^n}$. If $A \in \mathcal{F}_T \cap \{T < \infty\}$ we can write dominated conv

$$\mathbb{E}_x\left(\mathbb{1}_A f(X_{t+T})\right) \overset{\downarrow}{=} \lim_{n\to\infty} \mathbb{E}_x\left(\mathbb{1}_A f(X_{t+T_n})\right)$$ (8)

$$= \lim_{n\to\infty} \mathbb{E}_x\left(\mathbb{1}_A \mathbb{E}_{X_{T_n}} f(X_t)\right)$$

Strong Markov for discrete

$$= \mathbb{E}_x\left(\mathbb{1}_A \mathbb{E}_{X_T} f(X_t)\right)$$

Feller continuity f)

since $\{T_n < \infty\} = \{T < \infty\}$ we get

$$\mathbb{E}_x\left(\mathbb{1}_A f(X_{t+T_n})\right) = \sum_{k=1}^{\infty} \mathbb{E}_x\left(\underbrace{\mathbb{1}_{A\cap\{T_n = \frac{k}{2^n}\}}}_{\in \mathcal{F}_{k/2^n}} f\left(X_{t+\frac{k}{2^n}}\right)\right)$$

$$= \sum_{k=1}^{\infty} \mathbb{E}_x\left(\mathbb{1}_{A\cap\{T_n=\frac{k}{2^n}\}} \mathbb{E}_{X_{\frac{k}{2^n}}} f(X_t)\right)$$

Markov prop

$$= \mathbb{E}_x\left(\mathbb{1}_A \mathbb{E}_{X_{T_n}} f(X_t)\right).$$ ■

<u>Third remark</u> If (X_t) is a Feller process with generator $(A, \mathcal{D}(A))$ then $\forall f \in \mathcal{D}(A)$ the process

(Dynkin martingale)

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds, \quad t \geq 0$$

is a martingale w.r.t. $\mathcal{F}_t^X$ and $\mathbb{P}_x$, $x \in \mathbb{R}^d$ (HW4)

<u>Fourth remark</u> If X is a Feller process then A is i.g. satisfies PMP

Indeed, let $f \in \mathcal{D}(A)$ and suppose that there exists x_0 s.t.

$$f(x_0) = \sup_{y \in \mathbb{R}^d} f(y) \geq 0 .$$

Since $T_t f(x_0) = \int f(y) p_t(x_0, dy)$ we get

$$Af(x_0) = \lim_{t \downarrow 0} \frac{1}{t} \int (f(y) - f(x_0)) p_t(x_0, dy) \leq 0$$

since

$$f(y) - f(x_0) \leq f(y) - \sup_{y \in \mathbb{R}^d} f(y) \leq 0 \quad , \forall y \in \mathbb{R}^d$$

$\blacksquare$

Fifth remark : Courrège theorem : if A is a linear operator on $C_0(\mathbb{R}^d)$ and if $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$ then A satisfies PMP iff $\exists\, c, b, \Gamma$ and μ s.t.

$$Af(x) = c(x) f(x) + \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \langle \nabla, \Gamma(x) \nabla f(x) \rangle$$

Lévy type

$$+ \int_{\mathbb{R}^d \setminus \{x\}} \left[f(y) - f(x) - h(x,y) \langle y - x, \nabla f(x) \rangle \right] \mu(x, dy) \quad , \forall f \in C_c^\infty(\mathbb{R}^d), \forall x \in \mathbb{R}^d$$

where $h(x,y) = 1$, $\forall (x,y)$ in a neighbourhood of $\{(x,x) : x \in \mathbb{R}^d\}$

and $\mu(x, \cdot)$ is a Levy kernel, ie

$$x \longmapsto \int_{\mathbb{R}^d \setminus \{x\}} |y - x|^2 f(y) \mu(x, dy) \quad \text{meas, bounded}$$

and $\mu(x, \mathbb{R}^d \setminus V_x) < \infty$, $\forall V_x$ neigh. of x

Sixth remark : if A lin opé , $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$ and if A satisfies PMP, then

$A|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator

$$Af(x) = \eta(x,D)f(x) = \int \eta(x,u)\mathcal{F}f(u)\, e_u(x)\,du$$

whose symbol is

$$\eta(x,u) = \eta(x,0) + i\langle b(x),u\rangle - \frac{1}{2}\langle u,\Gamma(x)u\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left[e^{i\langle u,z\rangle} - 1 - i\langle u,z\rangle \mathbb{1}_{B_1}(z)\right]\nu(x,dz)$$

and $(b(x),\Gamma(x),\nu(x,dy))$ is a Levy triplet $\forall x\in\mathbb{R}^d$

(here $h(x,y) = (y-x)\mathbb{1}_{B_1}(y-x)$)

As in the previous section, $\exists\ K:\mathbb{R}^d\to[0,\infty)$ locally bounded s.t. $|\eta(x,u)| \le K(x)(1+|u|^2)$, $\forall x,u\in\mathbb{R}^d$

Seventh remark: It is not simple to see that a Feller process does not explode in finite time. We need conditions on the symbol. Technical assumptions:

- bounded coefficients $\sup \eta(x,0) + \sup|b(x)| + \sup\|\Gamma(x)\| + \sup \int_{y\neq 0} \frac{|y|^2}{1+|y|^2}\nu(x,dy) < \infty$
- continuity: $x\mapsto \eta(x,u)$ continuous $\forall u$
  $\iff x\mapsto\eta(x,0)$ continuous
  $\iff$ tightness $\lim_{R\to\infty}\sup_{x\in K}\nu(x,B_R^C) = 0$ $\forall K$ comp
  $\iff \sup_{x\in K}|\eta(x,u)-\eta(x,0)| \xrightarrow[|u|\to 0]{} 0$

Rq 1) If the Feller process has bounded jumps then it can be proved that $\eta(\cdot, u)$ is continuous

2) The boundedness of coefficients is important if coefficients grow too fast we may observe explosion in finite time even if $\eta(x,0)=0$:

Stroock-Varadhan p. 260 $\quad d=3$

$$Af(x) = \frac{1}{2} a(x) \Delta f(x), \quad a \text{ continuous}$$

$$a(x) = \alpha(|x|), \quad \int_1^{\infty} \frac{1}{\alpha(\sqrt{r})} dr < \infty$$

$$\eta(x,u) = \frac{1}{2} a(x) |u|^2$$

$\Rightarrow$ the process explodes in finite time (a time changed BM)

Examples of Lévy-type operators / processes

a) diffusion operators

$$Af(x) = \langle b(x), \nabla f(x) \rangle - \frac{1}{2} \langle \nabla, \Gamma(x) \nabla f(x) \rangle$$

Conditions on b, Γ s.t. the process is Feller

b) pseudo-Poisson process

$$Af(x) = \int_{\mathbb{R}^d} [f(y) - f(x)] \lambda q(x,dy)$$

where

$$q(x,B) = P(S(n) \in B \mid S(0) = x), \quad \lambda > 0$$

$(S(n))_{n \geq 0}$ a $\underbrace{\ }_{\text{homogenous}}$ Markov chain in $\mathbb{R}^d$

If Q is the transition operator $Qf(x) = \int f(y) q(x,dy)$

then $Af(x) = \lambda ((Q-I)f)(x)$ and the semigroup is

$T_t f(x) = e^{t[\lambda(Q-I)]} f(x)$ of the process

$X_t = S(N_t)$, (N_t) a Poisson-proc $\perp\!\!\!\perp$ $(S(n))$ of intensity λ.

(it is a Feller process)

c) stable-like processes: symbol

$$\eta(x,u) = -|u|^{\alpha(x)}, \quad \alpha : \mathbb{R}^d \to (0,2) \text{ continuous}$$

$$\Rightarrow Af(x) = -(2\pi)^{d/2} \int_{\mathbb{R}^d} |u|^{\alpha(x)} \mathcal{F}f(u)\, e_u(x)\, du$$

$$\Rightarrow Af(x) = \int_{\mathbb{R}^d \setminus \{0\}} [f(y+x) - f(x) - \langle y, \nabla f(x)\rangle \mathbb{1}_{B_1}(|y|)] \underbrace{\frac{K(x)\,dy}{|y|^{d+\alpha(x)}}}_{\mu(x,dy)}$$

$$Af(x) = \left(-(-\Delta)^{\alpha(x)/2} f\right)(x)$$

How to prove that is a Feller process? $\to$ Bass (martingale pb) / Tsuchiya (SDE's)

Recall that the symbol of a Levy process $\eta(u) = \lim_{t\to 0} \frac{e^{t\eta(u)}-1}{t} =$
$= \lim_{t\to 0} \frac{E_x(e^{i\langle u, X_t - x\rangle}) - 1}{t}$. For Feller proc

Thm | If $x \mapsto \eta(x,u)$ is cont and η has bounded coeff then
$\eta(x,u) = \lim_{t\to 0} \frac{E_x(e^{i\langle u, X_t - x\rangle}) - 1}{t}$, where (X_t) Feller proc $(A, \mathcal{D}(A))$