12) Stochastic integration

a) Let (A_t) a càdlàg process of finite variation (it is known that if a function is of finite variation then it can be written as the difference of two non-decreasing functions

$$g = \frac{V(g)+g}{2} - \frac{V(g)-g}{2} \Big)$$

The integral of $F_t = F(t,\omega)$ bounded, jointly measurable w.r.t. to A is

$$\int_0^t F_s(\omega)\, dA_s(\omega) \overset{\text{not}}{=} I_t(\omega) \overset{\text{not}}{=} (F.A)_t(\omega)$$

is defined ω-by-ω. I is càdlàg jointly meas.

$$|A|_t = \sup_{n\geq 1} \sum_{k=1}^{2^n} \left| A_{\frac{kt}{2^n}} - A_{\frac{(k-1)t}{2^n}} \right| < \infty \text{ if } A \text{ is f.v.}$$

$t \mapsto |A|_t$ increasing process

It can be shown that there exists a jointly meas adapted proc H, $-1 \leq H \leq 1$ s.t.

$$|A| = H.A \text{ and } A = H.|A|$$

If $s \mapsto F_s(\omega)$ is a.s. cont then $F.A$ is limit of Riemann sums. Finally if A is a f.v. process with continuous paths and if $f \in C^1$ then $(f(A_t))$ is a f.v. process and

$$f(A_t) - f(A_0) = \int_0^t f'(A_s)\, dA_s$$

If A is f.v. process but only càdlàg then, if $f \in C^1$

$$f(A_t) - f(A_0) = \int_0^t f'(A_{s-})\, dA_s + \sum_{0<s\leq t} \left(f(A_s) - f(A_{s-}) - f'(A_{s-})\Delta A_s \right)$$

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b) X an adapted process with càdlàg paths is a semimartingale if $X_t = X_0 + M_t + A_t$, $M_0 = A_0 = 0$, M_t a local martingale (often square integrable), A càdlàg adapted, f.v. on compacts.

X will be a "good" integrator.

We will also need a new type of convergence

Def $(H^n)_{n\geq 1}$ sequence of processes converges to H ucp uniformly on compacts in probability if for each $t>0$

$$\sup_{0\leq s\leq t} |H^n_s - H_s| \xrightarrow[n\to\infty]{\text{prob}} 0.$$

Def H is simple predictable if $H_t = H_0 + \sum_{j=1}^n H_j \mathbb{1}_{(T_j, T_{j+1}]}$ (*)

$0 < T_1 \leq \dots \leq T_{n+1} < \infty$ stopping times

$H_j \in \mathcal{F}_{T_j}$, $|H_j| < \infty$ a.s. : this family $\mathcal{S}$

Def $J_X : \mathcal{S} \longrightarrow \mathbb{D}$ (adapted càdlàg processes)

$$J_X(H) = H_0 X_0 + \sum_{j=1}^n H_j \left(X^{T_{j+1}} - X^{T_j} \right)$$ Stoch integral of H w.r.t X

$$\overset{\text{not}}{=} \int H_s\, dX_s = H \cdot X$$

Propo 1 Let X be a semimartingale. Then $J_X : \mathcal{S}_{ucp} \longrightarrow \mathcal{D}_{ucp}$ is continuous

Proof We set $H_t^* = \sup_{0 \le s \le t} |H_s|$. Assume first $H^k \in \mathcal{S}$, $H^k \xrightarrow[k\to\infty]{u} 0$ and uniformly bounded. We show that $J_X(H^k) \xrightarrow{ucp} 0$. Let $\delta > 0$ and set $T^k = \inf\{t : |(H^k \cdot X)_t| \geq \delta\}$

Then $H^k \mathbb{1}_{[0,T^k]} \in \mathcal{S}$ and $\xrightarrow{u} 0$. Hence

$$\mathbb{P}\left((H^k \cdot X)_t^* > \delta\right) \le \mathbb{P}\left(|H^k \cdot X_{T^k \wedge t}| \geq \delta\right)$$

$$= \mathbb{P}\left(|(H^k \mathbb{1}_{[0,T^k]} \cdot X)_t| \geq \delta\right)$$

$$= \mathbb{P}\left(|I_X(H^k \mathbb{1}_{[0,T^k \wedge t]})| \geq \delta\right) \longrightarrow 0$$

car $X_T : \mathcal{S}_u \to L^0$ continue

Suppose now $H^k \xrightarrow{ucp} 0$, let $\delta, \varepsilon, t > 0$: $\exists \eta$ s.t. $\|H\|_u \le \eta$ implies $\mathbb{P}(J_X(H)_t^* > \delta) < \frac{\varepsilon}{2}$

Set $R_k = \inf\{s : |H_s^k| > \eta\}$, $\tilde{H}^k = H^k \mathbb{1}_{[0,R_k]} \mathbb{1}_{R_k > 0}$

$\tilde{H}^k \in \mathcal{S}$, $\|\tilde{H}^k\| < \eta$. Since $R^k \geq t$ implies $(\tilde{H}^k \cdot X)_t^* = (H^k \cdot X)_t^*$ we have

$$\mathbb{P}\left((H^k \cdot X)_t^* > \delta\right) \le \mathbb{P}\left((\tilde{H}^k \cdot X)_t^* > \delta\right) + \mathbb{P}(R^k < t)$$

$$\le \frac{\varepsilon}{2} + \underbrace{\mathbb{P}\left((H^k)_t^* > \eta\right)}_{\to 0 \ (k\to\infty)} < \varepsilon$$

Rq $\int_0^t A_s\,dA_s = \frac{1}{2}A_t^2$, $\int_0^t B_s\,dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$

Prop 2 | If T is a stopping time $(H\cdot X)^T = H\,\mathbb{1}_{[0,T]}\cdot X = H\cdot(X^T)$

Rq X semimartingale with finite var path $H\cdot X$ is the Lebesgue – Stieltjes integral

Prop 3 | Let X be a locally square integrable local martingale and let $H \in \mathbb{L}$ (càglàd). Then the stoch integral is locally square integrable martingale

Rq Let $M_t = N_t - \lambda t$ a compensated Poisson proc (martingale, $M_t \in L^p, \forall t \geq 0, \forall p \geq 1$) Let (T_i) be the jump times of N and $H_t = \mathbb{1}_{[0,T_1)}(t) \in \mathbb{D}$
The Lebesgue – Stieltjes integral is

$$\int_0^t H_s\,dM_s = \int_0^t H_s\,dN_s - \lambda\int_0^t H_s\,ds$$

$$= \sum_{i=1}^{\infty} H_{T_i}\mathbb{1}_{t \geq T_i} - \lambda\int_0^t H_s\,ds = -\lambda(t \wedge T_1)$$

which is not a martingale

Notation $(Y_-)_s = \lim_{u \uparrow s} Y_u$ with $(Y_-)_0 = 0$ by convention

Def The quadratic variation process of X semimart.

$$[X,X] = X^2 - 2\int X_- dX$$

It is a càdlàg increasing adapted

Prop 4 | $[X,X]_0 = X_0^2$, $\Delta[X,X] = (\Delta X)^2$
$[X^T, X] = [X, X^T] = [X^T, X^T] = [X,X]^T$

For instance

$$(\Delta X_s)^2 = (X_s - X_{s-})^2 = X_s^2 - 2X_s X_{s-} + X_{s-}^2$$

$$= X_s^2 - X_{s-}^2 + 2X_{s-}(X_{s-} - X_s)$$

$$= (\Delta X^2)_s + 2X_{s-}(\Delta X_s) \quad \text{etc}$$

Rq $[B,B]_t = t$; if X semimart with continuous f.v paths then $[X,X] \equiv X_0^2$

Propo 5 | $XY = \int X_- dY + \int Y_- dX + [X,Y]$
where $[X,Y] = \frac{1}{2}\left([X+Y, X+Y] - [X,X] - [Y,Y]\right)$.

Propo 6 (K-W inequality) X, Y semimartingales,

H, K two meas proc

$$\int_0^\infty |H_s||K_s||d[X,Y]_s| \leq \left(\int_0^\infty H_s^2\, d[X,X]_s\right)^{1/2} \cdot \left(\int_0^\infty K_s^2\, d[Y,Y]_s\right)^{1/2}$$

Def X semimartingale, $[X,X]^c$ the path-by-path continuous part of $[X,X]$

$$[X,X]_t = [X,X]_t^c + X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2$$

(hence $[X,X]_0^c = 0$)

X is quadratic pure jump if $[X,X]^c = 0$

$$[X,X]_t = X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2$$

Examples 1) (N_t) Poisson proc is quadratic pure jump

$[N,N]_t = N_t$. 2) If X Lévy process

$$X_t = B_t + Y_t \quad \text{with } Y_t = \int_{|z|<1} z\tilde{N}(t,dz) + bt + \int_{|z|\geq 1} zN(t,dz)$$

then Y is quadratic pure jump semimartingale

Propo 7 X adapted cadlag process, f.v. on compact
then X is quadratic pure jump semimartingale

Idea: $X^2 = \int X_- dX + \underbrace{\int X dX}_{\int (X_- + \Delta X) dX} = 2\int X_- dX + [X,X]$

$= \int X_- dX + \sum_{s\leq t} (\Delta X_s)^2$

$\Rightarrow [X,X]_t = \sum_{s\leq t} (\Delta X_s)^2$

$\square$

Rq If X adapted with conti. paths of f.v. then
$[X,X]_t = X_0^2, \forall t \geq 0$

Propo 8 X local martingale with cont paths not everywhere constant then $[X,X]$ is not constant X_0^2 and $X^2 - [X,X]$ is a continuous local martingale. If $[X,X]_t = 0, \forall t \geq 0$ then $X_t = 0, \forall t \geq 0$

Coro M local martingale is martingale with $E(M_t^2) < \infty \; \forall t$ iff $E([M,M]_t) < \infty, \forall t \geq 0$.
If $E([M,M]_t) < \infty$, then $E(M_t^2) = E([M,M]_t)$
If $E([M,M]_\infty) < \infty$ then M square integrable

Propo 9 Let X quadratic pure jump semimartingale and Y any semimartingale. Then
$$[X,Y]_t = X_0 Y_0 + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s$$

Propo 10 X, Y two semimartingales, $H, K \in \mathcal{L}$
Then $[H \cdot X, K \cdot Y]_t = \int_0^t H_s K_s \, d[X,Y]_s$
$$[H \cdot X, H \cdot X]_t = \int_0^t H_s^2 \, d[X,X]_s$$

<u>Thm</u> (Itô's formula) Let X be a semimartingale and let f be a C^2-real function. Then $f(X)$ is a semimartingale and

$$f(X_t) - f(X_0) = \int_{[0,t]} f'(X_{s-})\, dX_s + \frac{1}{2}\int_{[0,t]} f''(X_{s-})\, d[X,X]^c_s$$

$$+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s-}) - f'(X_{s-})\, \Delta X_s \right\}$$

<u>Proof</u> (Protter pp 71-74)

$$Y_c(t) = G_t\,dt + F_t\,dB_t \;,\; Y_d(t) = \int_{|z|<1} H(t,z)\tilde{N}(dt,dz) + \int_{|z|\geq 1} K(t,z)N(dt,dz)$$

$$Y_t = Y_0 + Y_c(t) + Y_d(t)$$

<u>Thm</u> For every $f \in C^2(\mathbb{R})$, with probab 1

$$f(Y_t) - f(Y_0) = \int_0^t f'(Y_{s-})\,dY_c(s) + \frac{1}{2}\int_0^t f''(Y_{s-})\,d[Y_c, Y_c](s)$$

$$+ \int_0^t \int_{|z|\geq 1} \left[f(Y_{s-} + K(s,z)) - f(Y_{s-})\right] N(ds,dz)$$

$$+ \int_0^t \int_{|z|<1} \left[f(Y_{s-} + H(s,z)) - f(Y_{s-})\right] \tilde{N}(ds,dz)$$

$$+ \int_0^t \int_{|z|<1} \left[f(Y_{s-} + H(s,z)) - f(Y_{s-}) - H(s,z) f'(Y_{s-})\right] \nu(dz)ds$$

<u>Proof</u> We use again interlacing

$$Y_n(t) = \int_0^t G(s)ds + \int_0^t F(s)\,dB_s + \int_0^t \int_{A_n} H(s,z)\tilde{N}(ds,dz)$$

$$+ \int_0^t \int_{|z|\geq 1} K(s,z)\,N(ds,dz)$$

and then the preceding lemma

$$f(Y_n(t)) - f(Y_n(0)) = \underbrace{\int_0^t f'(Y_{s-})\,dY_c(s)}_{①} + \underbrace{\frac{1}{2}\int_0^t f''(Y_{s-})\,d[Y_c, Y_c]_s}_{②}$$

$$+\int_0^t \int_{|z|\geq 1} \left[f(Y_{s-} + K(s,z)) - f(Y_{s-}) \right] N(ds,dz) \quad (3)$$

$$+\int_0^t \int_{A_n} \left[f(Y_{s-} + H(s,z)) - f(Y_{s-}) \right] N(ds,dz)$$

$$-\int_0^t \int_{A_n} H(s,z) f'(Y_{s-}) \nu(dz) ds$$

$$= (1) + (2) + (3) + \int_0^t \int_{A_n} \left[f(Y_{s-} + H(s,z)) - f(Y_{s-}) \right] \tilde{N}(ds,dz)$$

$$+ \int_0^t \int_{A_n} \left[f(Y_{s-} + H(s,z)) - f(Y_{s-}) - H(s,z) f'(Y_{s-}) \right] \nu(dz) ds$$

But by Coro p 94 $Y_n(t) \xrightarrow[n\to\infty]{\text{proba}} Y_t$ so as by using a subsequence and then we get the result by taking limit on this subsequence ■

Rk 1) Terms (3) + (4) can be written as

$$\sum_{0<s\leq t} \left[f(Y_s) - f(Y_{s-}) - \Delta Y_s f'(Y_{s-}) \right]$$

(see Applebaum p. 229)

2) $$[Y,Y]_t = [Y_c, Y_c]_t + \sum_{0\leq s\leq t} \Delta Y_s \Delta Y_s$$

3) Y^1, Y^2 two real-valued Levy integrals

$$Y^1_t Y^2_t = Y^1_0 Y^2_0 + \int_0^t Y^1_{s-} dY^2_s + \int_0^t Y^2_{s-} dY^1_s + [Y^1, Y^2]_t$$

where

$$[Y^1, Y^2]_t = \lim_{n\to\infty} \sum_{j=0}^{k_n} \left(Y^1_{t^n_{j+1}} - Y^1_{t^n_j}\right)\left(Y^2_{t^n_{j+1}} - Y^2_{t^n_j}\right) \text{ with proba } 1$$

4) $$[Y^1, Y^2]_t = \langle Y^1_c, Y^2_c \rangle_t + \sum_{s \leq t} \Delta Y^1_s \, \Delta Y^2_s$$

(see Sem 1)