$$\underset{C\text{-}S\ \text{ineq}}{\le} \frac{|u|^2}{2}\underbrace{\int_{B_1}|y|^2\,\nu(dy)}_{<\infty \text{ by hypo } (\nu \text{ Lévy meas})} + \int_{B_1^c}\underbrace{|e^{i\langle u,y\rangle}-1|}_{\xrightarrow{u\to 0}\ 0 \text{ and dominated by } 2}\nu(dy)$$

$u \to 0$: $\frac{|u|^2}{2} \to 0$

Hence the r.h.s. tends to 0 as $u \to 0$ by dominated conv.
Then one can conclude by Exe 2 in HW.

Before proving the uniqueness of b, Γ and ν we state two lemmas

<u>Lemma 1</u> Let μ be a inf. div distribution and denote its c.f. φ. Then $\varphi(u) \neq 0$, $\forall u \in \mathbb{R}^d$

<u>Preuve</u> Let μ_n be s.t $\mu = \mu_n^{*n}$ so $\varphi = \varphi_n^n$ hence $|\varphi_n(u)|^2 = |\varphi(u)|^{2/n}$ is a c.f. Define

$$\tilde{\varphi}(u) = \lim_{n\to\infty}|\varphi_n(u)|^2 = \begin{cases} 1 \text{ if } \varphi(u) \neq 0 \\ 0 \text{ if } \varphi(u) = 0 \end{cases}$$

Since $\varphi(0) = 1$ and φ is continuous we get that $\tilde{\varphi}(u) = 1$ in a neighborhood of 0. We then deduce that $\tilde{\varphi}$ is a c.f hence continuous on $\mathbb{R}^d$, $\equiv 1$ and we can conclude $\square$

<u>Lemma 2</u> Let $\varphi : \mathbb{R}^d \to \mathbb{C}$ be such that $\varphi(0) = 1$ and $\varphi(u) \neq 0$, $\forall u \in \mathbb{R}^d$. Then there exists a unique $\psi : \mathbb{R}^d \to \mathbb{C}$ continuous such that $\psi(0) = 0$, $e^{\psi(u)} = \varphi(u)$. There exists a unique $\varphi_n : \mathbb{R}^d \to \mathbb{C}$ continuous s.t. $\varphi_n(0) = 1$ and $\varphi_n(u)^n = \varphi(u)$. Moreover $\varphi_n(u) = e^{\psi(u)/n}$. We write

$$\psi(u) = \log \varphi(u) \text{ and } \varphi_n(u) = \varphi(u)^{1/n}.$$

(accepted)

Let us prove uniqueness of b, Γ and ν. First note that

$$|e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle \mathbb{1}_{B_1}(y)| \le \frac{1}{2}|u|^2|y|^2\mathbb{1}_{B_1}(y) + 2\mathbb{1}_{B_1^c}(y)$$

We deduce by dominated convergence that the argument of exp in (LK) is continuous with respect to u. Thus

$$\log \varphi(su) = is\langle b,u\rangle - \frac{1}{2}s^2\langle u,\Gamma u\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle su,y\rangle} - 1 - i\langle su,y\rangle \mathbb{1}_{B_1}(y)\right)\nu(dy)$$

for $s \in \mathbb{R}$. Again by dominated convergence

$$\frac{1}{s^2}\log\varphi(su) \underset{s\to\infty}{\longrightarrow} -\frac{1}{2}\langle u,\Gamma u\rangle$$

hence Γ is uniquely determined by μ. Let now $\chi(u) = \log\varphi(u) + \frac{1}{2}\langle u,\Gamma u\rangle$

Then it can be proved (exe: see Sato p. 40) that

$$\int_{[-1,1]^d}\left(\chi(u) - \chi(u+v)\right)dv = 2^d\int_{\mathbb{R}^d} e^{i\langle u,y\rangle}\left(1 - \prod_{j=1}^{d}\frac{\sin y_j}{y_j}\right)\nu(dy)$$

The r.h.s is the Fourier transform of $\rho(dy) = 2^d\left(1 - \prod\frac{\sin y_j}{y_j}\right)\nu(dy)$ (a finite measure)

Hence ρ is uniquely determined by χ (so by μ). Since $\nu(\{0\}) = 0$, ν is uniquely determined by μ. Finally b is uniquely determined from μ, Γ and ν.

The difficult part of the theorem will be accepted (see Sato pp 41-45 (uses tightness - Ex. 3 of HW, limits of infi div distributions). We will see later another way to get this part. $\blacksquare$