22) Small noise asymptotics of Langevin type equations driven by α-stable processes

$$v_t^\varepsilon = v_0 + \varepsilon l_t - \delta \int_0^t |v_s^\varepsilon|^\beta \operatorname{sgn}(v_s^\varepsilon)\, ds \qquad \text{velocity}$$

$\delta > 0$, $\beta > -1$, (l_t) α-stable Levy process $\alpha \in (0,2]$

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon\, ds$$

For $\beta = 1$ we get the linear Langevin equation

$$\ddot{x}_t^\varepsilon = \underbrace{-\delta \dot{x}^\varepsilon}_{\text{friction force}} + \underbrace{\varepsilon \dot{l}}_{\text{small random noise}}$$

(and v^ε is the Ornstein-Uhlenbeck process)

What is the behaviour of the position process x_t^ε as $\varepsilon \to 0$?

We look here only the linear case: $\beta = 1$

$$v_t^\varepsilon = v_0 - \delta \int_0^t v_s^\varepsilon\, ds + \varepsilon l_t \implies$$ (see computation p. 113)

$$v_t^\varepsilon = v_0 e^{-\delta t} + \varepsilon \int_0^t e^{-\delta(t-s)}\, dl_s \qquad \text{and by integ by parts}$$

$$\left(= v_0 e^{-\delta t} + \varepsilon l_t - \varepsilon \delta \int_0^t e^{-\delta(t-s)} l_s\, ds \right)$$

Hence the position can be computed explicitely

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon\, ds = x_0 + \int_0^t \left(v_0 e^{-\delta s} + \varepsilon \int_0^s e^{-\delta(s-u)}\, dl_u \right) ds$$

$$= x_0 + \frac{v_0}{\delta}\left(1 - e^{-\delta t}\right) + \frac{\varepsilon}{\delta} \int_0^t \left(1 - e^{-\delta(t-u)}\right) dl_u$$

Set $x_0 = v_0 = 0$

Thm $\left(\delta x^\varepsilon_{t/\varepsilon^\alpha}\right)_{t\geq 0} \xrightarrow[\varepsilon\to 0]{\mathcal{D}} \left(l^{(\alpha)}_t\right)_{t\geq 0}$ in $D([0,T];\mathbb{R})$

($\alpha = 2$ in $C([0,T];\mathbb{R})$)

Let us sketch the proof for $\alpha \in (0,2)$. We use the self-similarity of l : $\left(\varepsilon\, l_{t/\varepsilon^\alpha}\right)_{t\geq 0} \overset{law}{=} \left(l_t^{(\alpha)}\right)_{t\geq 0}$

$$\delta x^\varepsilon_t = \varepsilon \int_0^t \left(1-e^{-\delta(t-u)}\right) dl_u .$$

Using the change of variable $t \mapsto t/\varepsilon^\alpha$ and using the self-similarity we get that $\left(\delta x^\varepsilon_{t/\varepsilon^\alpha}\right)$ coincides in law with

$$\delta X^\varepsilon_t = \int_0^t \left(1-e^{-\delta(t-u)/\varepsilon^\alpha}\right) dL_u$$

where (L_u) is another α-stable Lévy process. At this level we can assume without loss of generality that $\delta = 1$. By integration we get that

$$X^\varepsilon_t = L_t - \int_0^t e^{-(t-u)/\varepsilon^\alpha}\, dL_u$$

$$= \frac{1}{\varepsilon^\alpha}\int_0^t e^{-(t-u)/\varepsilon^\alpha} L_u\, du$$

It suffices to prove that $Y^\varepsilon_t = X^\varepsilon_t - L_t = -\int_0^t e^{-(t-u)/\varepsilon^\alpha} dL_u$ converges to zero in distribution $= \frac{1}{\varepsilon^\alpha}\int_0^t e^{-(t-u)} L_u du$

Let us compute the characteristic function of Y^ε_t

$$\mathbb{E}\exp(i\xi Y_t^\varepsilon) = \mathbb{E}\exp\left(-i\xi \int_0^t e^{-(t-u)/\varepsilon^\alpha} dL_u\right)$$

But this is a classical computation for Lévy-Itô integrals (see also Sato Lemma 17.1)

$$\ln \mathbb{E}\exp(i\xi Y_t^\varepsilon) = \int_0^t \psi_\alpha\left(-\xi e^{-(t-u)/\varepsilon^\alpha} du\right)$$

where ψ_α is given in p.26 (symbol of a α-stable r.v)

Since $\lim_{\varepsilon\to 0} \psi_\alpha\left(-\xi e^{-(t-u)/\varepsilon^\alpha} du\right) = \psi_\alpha(0) = 0$

We get by dominated convergence that the characteristic function of Y_t^ε tends to 1 hence $X_t^\varepsilon \xrightarrow[\varepsilon\to 0]{prob} L_t$.

The same type of argument work for finite dimensional marginals (see p.96-97). To conclude we have to

? [prove tightness of the family (X^ε) by using the criterion of Aldous. (see Hintze - Pavlyukevich '14 for details)] $\boxtimes$

Rk 1/ Non linear case : $\left\{\varepsilon^{\frac{\alpha(\beta+\alpha/2-2)}{\alpha+\beta-1}} x^\varepsilon_{t/\varepsilon^\alpha}\right\}_{t\geq 0} \xrightarrow[\varepsilon\to 0]{\mathcal{D}} (K_{\alpha,\beta} B_t)_{t\geq 0}$

The proof is based on Itô's formula, solving Poisson PDE and a martingale central limit theorem.

2/ Time inhomogeneous case : research of E.L & M.G