ARTICLE TEMPLATE

Locally Feller processes converging towards diffusion processes in singular potentials

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ABSTRACT
We characterise the convergence of a certain class of discrete time Markov processes towards locally Feller processes in terms of convergence of martingale problems. We apply our results of approximation to get results of convergence towards diffusions behaving into singular potentials. As a consequence we deduce the convergence of random walks in random medium towards diffusions in random potential.

KEYWORDS
Feller processes, martingale problem, random walks and diffusions in random and non-random environment, weak convergence of probability measures, Skorokhod topology

1. Introduction

During the last two decades a lot of interest has been shown in the study of diffusions in random environment. A well known model is the dynamic of a Brownian particle β in a potential. It is often given by the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\beta_t - \frac{1}{2}V'(X_t)\mathrm{d}t,$$

where $V : \mathbb{R} \to \mathbb{R}$. Thanks to the regularising property of the Brownian motion one can consider very general potentials, for example cadlag functions (see Mandle [10]). In particular, it can be supposed that the potential is a Brownian path (see Brox [1]), a Lévy path (see Carmona [2]) or other random path (Gaussian and/or fractional process ...).

The study of the convergence of sequences of general Markov processes is one of usual questions. The present paper consider this question in the setting of the preceding model. A usual way to obtain convergence results is the use of the theory of Feller processes. In this context there exist two corresponding results of convergence (see, for instance Kallenberg [7], Theorems 19.25, p. 385 and 19.27, p. 387). However, on one hand, when one needs to consider unbounded coefficients, technical difficulties could appear in the framework of Feller processes. On the other hand the cited results of

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convergence impose the knowledge of a core of the generator. This could not be the case in some probabilistic constructions. Detailed overviews on these topics and many other references on the subject can be found in [6], [8].

Our method to tackle these difficulties is to consider the context of the martingale local problems and of locally Feller processes, introduced in [4]. In this general framework we have already analysed the question of convergence of sequences of locally Feller processes, employing the setting of the local Skorokhod topology on the space of cadlag processes (see [3]). In the present paper we add the study of the convergence for processes indexed by a discrete time parameter towards processes indexed by a continuous time parameter. We obtain the characterisation of the convergence in terms of convergence of associated operators, by using the uniform convergence on compact sets, and hence operators with unbounded coefficients could be considered. Likewise, we do not impose that the operator is a generator, but we assume only the well-posed feature of the associated martingale local problem. Indeed, it could be more easy to verify the well-posed feature (see for instance, Stroock [13] for Lévy-type processes, Stroock and Varadhan [14] for diffusion processes, Kurtz [9] for Lévy-driven stochastic differential equations and forward equations...).

When studying a Brownian particle in a potential, we prove the continuous dependence of the diffusion with respect to the potential, using our abstract results. We point out that it can be possible to consider potentials with very few constraints. In particular we consider diffusions in random potentials as limits of random walks in random mediums, as an application of an approximation of the diffusion by random walks on $\mathbb{Z}$. An important example is the convergence of Sinai's random walk [12] towards the diffusion corresponding to a Brownian movement in a Poisson potential (recovering Thm. 2 from Seignourel [11], p. 296), or towards the diffusion corresponding to a Brownian movement in a Brownian potential, also called Brox's diffusion (improving Thm. 1 from Seignourel [11], p. 295) and, more generally, towards the diffusion corresponding to a Brownian movement in a Lévy potential.

The considerations on locally Feller processes are also applied to Lévy-type processes in order to get (or to improve) sharp results of convergence for discrete and continuous time sequences of processes towards Lévy-type process, in terms of Lévy parameters, but also simulation methods and Euler schemes (see for instance [5], §4.4).

Let us describe the organisation of the paper. The next section contains notations and statements from our previous paper [4], which are very useful for an easy reading of the present paper. In particular, we recall the necessary end sufficient conditions for the existence of solutions for martingale local problems and also for the convergence of continuous time locally Feller processes. Our main results are given in Sections 3 and 4. Section 3 is devoted to the study of sequences of discrete time locally Feller processes, while Section 4 contains its applications to the diffusions evolving in a potential. The appendix collect the statements of auxiliary results already proved in [4].

2. Martingale local problem setting and related results

Let S be a locally compact Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). We will denote by $A \Subset U$ the fact that a subset A is compactly embedded in an open subset $U \subset S$. If $x \in (S^\Delta)^{\mathbb{R}_+}$ we denote the explosion time by

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s\leq t} \not\Subset S\}.$$

The set of exploding cadlag paths is defined by

$$\mathbb{D}_{\text{loc}}(S) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\; x_t = \Delta, \\ \forall t \geq 0,\; x_t = \lim_{s\downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S,\; x_{t-} := \lim_{s\uparrow t} x_s \text{ exists} \end{array} \right\},$$

and it is endowed with the *local Skorokhod topology* which is also Polish (see Theorem 2.4 in [3], p. 1187). A sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{D}_{\text{loc}}(S)$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ satisfying

$$\forall t \geq 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0.$$

The local Skorokhod topology does not depend on the arbitrary metric d on S^Δ, but only on the topology on S.

Denote by $\mathrm{C}(S) := \mathrm{C}(S,\mathbb{R})$, respectively by $\mathrm{C}(S^\Delta) := \mathrm{C}(S^\Delta,\mathbb{R})$, the set of real continuous functions on S, respectively on S^Δ, and by $\mathrm{C}_0(S)$ the set of functions $f \in \mathrm{C}(S)$ vanishing in Δ. We endow the set $\mathrm{C}(S)$ with the topology of uniform convergence on compact sets and $\mathrm{C}_0(S)$ with the topology of uniform convergence. An operator L from $\mathrm{C}_0(S)$ to $\mathrm{C}(S)$, will be denoted as a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$.

We proceed by recalling the *notion of martingale local problem* (not to be confused with the local martingale problem, see Definition 3.2 in [4], p. 135). The canonical stochastic process on $\mathbb{D}_{\text{loc}}(S)$ will be always denoted by X. We endow $\mathbb{D}_{\text{loc}}(S)$ with the Borel σ-algebra $\mathcal{F} := \sigma(X_s,\ 0 \leq s < \infty)$ and the filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \leq s \leq t)$. The set $\mathcal{M}(L)$ of solutions of the *martingale local problem* associated to L is the set of probabilities $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ such that for all $(f,g) \in L$ and open subset $U \Subset S$:

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalent, to the filtration $(\mathcal{F}_{t+})_t$. Here τ^U is the stopping time given by

$$\tau^U := \inf\{t \geq 0 \mid X_t \notin U \text{ or } X_{t-} \notin U\}. \tag{1}$$

Theorem 3.10 from [4], p. 139, provides a result of existence of solutions for martingale local problem. We recall its statement since it will be one of our main tools in Section 4:

Theorem 2.1. *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that its domain $D(L) := \{f \in \mathrm{C}_0(S) \mid \exists g \in \mathrm{C}(S),\ (f,g) \in L\}$ is dense in $\mathrm{C}_0(S)$. Then, there is equivalence between*

i) existence of a solution for the martingale local problem: for any $a \in S$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$;

ii) L satisfies the positive maximum principle: for all $(f,g) \in L$ and $a_0 \in S$, if $f(a_0) = \sup_{a\in S} f(a) \geq 0$ then $g(a_0) \leq 0$.

The *martingale local problem is said well-posed* if there is existence and uniqueness of the solution, which means that for any $a \in S$ there exists an unique element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

A family of probabilities $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is called *locally Feller* if there exists $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and

$$\forall a \in S: \qquad \mathbf{P} \in \mathcal{M}(L) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a.$$

(see also Definition 4.5 in [4], p. 144). The *$\big(\mathrm{C}_0 \times \mathrm{C}\big)$-generator of a locally Feller family* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is the set of functions $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that, for any $a \in S$ and any open subset $U \Subset S$,

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}_a\text{-martingale.}$$

It was noticed in Remark 4.6(ii) from [4], p. 144, that if $h \in \mathrm{C}(S, \mathbb{R}_+^*)$ and if L is the $\mathrm{C}_0 \times \mathrm{C}$-generator of a locally Feller family, then

$$hL := \{(f, hg) \,|\, (f,g) \in L\} \text{ is the } \mathrm{C}_0 \times \mathrm{C}\text{-generator of a locally Feller family.} \tag{2}$$

A family of probability measures associated to a Feller semi-group constitutes a natural example of locally Feller family (see Theorem 4.10 from [4], p. 147). We recall that a Feller semi-group $(T_t)_{t\in\mathbb{R}_+}$ is a strongly continuous semi-group of positive linear contractions on $\mathrm{C}_0(S)$. Its $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator is the set L_0 of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that, for all $a \in S$

$$\lim_{t\to 0} \frac{1}{t}\big(T_t f(a) - f(a)\big) = g(a).$$

It can be proved that the martingale associated to L_0 admits a unique solution (consequence of Proposition 4.2 in [4], p. 142), and if L denotes the $\mathrm{C}_0(S) \times \mathrm{C}(S)$-generator of the associated Feller family, then taking the closure in $\mathrm{C}_0(S) \times \mathrm{C}(S)$, we have

$$L_0 = L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big) \quad \text{and} \quad L = \overline{L_0} \tag{3}$$

(see Proposition 4.16 in [4], p. 151).

The following result of convergence is essential for our further development in Section 4, and it was stated in Theorem 4.17 from [4], p. 151. As was already pointed out in the introduction, an improvement with respect to the classical result of convergence (see for instance Theorem 19.25, p.385, in [7]), is that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed.

Theorem 2.2 (Convergence of locally Feller family)**.** *For $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathbf{P}_a^n)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. Suppose that for any $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}_a^n)_a$, suppose also that $\mathrm{D}(L_\infty)$ is dense in*

$C_0(S)$ *and*

$$\forall a \in S: \qquad \mathbf{P} \in \mathcal{M}(L_\infty) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}^\infty_a.$$

Then we have equivalence between:

a) *the mapping*

$$\begin{array}{ccc} \left(\mathbb{N} \cup \{\infty\}\right) \times \mathcal{P}(S^\Delta) & \to & \mathcal{P}\left(\mathbb{D}_{loc}(S)\right) \\ (n, \mu) & \mapsto & \mathbf{P}^n_\mu \end{array}$$

is weakly continuous for the local Skorokhod topology, where $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(\mathrm{d}a)$ *and* $\mathbf{P}_\Delta(X_0 = \Delta) = 1$*;*

b) *for any* $a_n, a \in S$ *s.t.* $a_n \to a$, $\mathbf{P}^n_{a_n}$ *converges weakly for the local Skorokhod topology to* $\mathbf{P}^\infty_a$*, as* $n \to \infty$*;*

c) *for any* $f \in \mathrm{D}(L_\infty)$*, for each* n $\exists f_n \in \mathrm{D}(L_n)$ *s.t.* $f_n \underset{n\to\infty}{\overset{\mathrm{C}_0}{\longrightarrow}} f$, $L_n f_n \underset{n\to\infty}{\overset{\mathrm{C}}{\longrightarrow}} L_\infty f$.

3. Convergence of discrete time locally Feller families

We start our study by introducing a discrete time version of the notion of locally Feller family.

Definition 3.1 (Discrete time locally Feller family)**.** Denote by Y the discrete time canonical process on $(S^\Delta)^{\mathbb{N}}$ and endow $(S^\Delta)^{\mathbb{N}}$ with the canonical σ-algebra. A family $(\mathbf{P}_a)_a \in \mathcal{P}\left((S^\Delta)^{\mathbb{N}}\right)^S$ is said to be a discrete time locally Feller family if there exists an operator $T : \mathrm{C}_0(S) \to \mathrm{C}_b(S)$, called transition operator, such that for any $a \in S$: $\mathbf{P}_a(Y_0 = a) = 1$ and

$$\forall n \in \mathbb{N},\ \forall f \in \mathrm{C}_0(S), \quad \mathbf{E}_a\left(f(Y_{n+1}) \mid Y_0, \ldots, Y_n\right) = \mathbb{1}_{\{Y_n \neq \Delta\}} Tf(Y_n) \quad \mathbf{P}_a\text{-a.s.} \tag{4}$$

If $\mathbf{P}_\Delta$ denotes the probability defined by $\mathbf{P}_\Delta(\forall n \in \mathbb{N},\ Y_n = \Delta) = 1$, then for $\mu \in \mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(\mathrm{d}a)$ satisfies also (4).

The following theorem contains a result of convergence of a discrete time locally Feller family towards a continuous time locally Feller family. The main difference with respect to Theorem 19.27, p.387, in [7], is that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed (hence from this point of view it could be considered as a slightly improvement). In the following $\lfloor r \rfloor$ will denote the integer part of a real number r.

Theorem 3.2 (Convergence)**.** *Let* $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ *be an operator with* $\mathrm{D}(L)$ *a dense subset of* $\mathrm{C}_0(S)$*, such that the martingale local problem associated to L is well-posed. Let* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ *be the associated continuous time locally Feller family. For each* $n \in \mathbb{N}$ *we introduce* $(\mathbf{P}^n_a)_a \in \mathcal{P}((S^\Delta)^{\mathbb{N}})^S$ *a discrete time locally Feller family having its transition operator* T_n*. We denote the operator* $L_n := (T_n - \mathrm{id})/\varepsilon_n$*, where* $(\varepsilon_n)_n$ *is a sequence of positive numbers converging to* 0*, as* $n \to \infty$*. There is equivalence between:*

a) *for any* $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ *s.t.* $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ *weakly,* $\mathscr{L}_{\mathbf{P}^n_{\mu_n}}\left((Y_{\lfloor t/\varepsilon_n \rfloor})_t\right) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_\mu$ *;*

b) for any $a_n, a \in S$ s.t. $a_n \underset{n\to\infty}{\longrightarrow} a$, $\quad \mathscr{L}_{\mathbf{P}^n_{a_n}}\big((Y_{\lfloor t/\varepsilon_n\rfloor})_t\big) \overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_a$;

c) for any $f \in \mathrm{D}(L)$, there exists $(f_n)_n \in \mathrm{C}_0(S)^{\mathbb{N}}$ s.t. $f_n \overset{\mathrm{C}_0(S)}{\underset{n\to\infty}{\longrightarrow}} f$, $L_n f_n \overset{\mathrm{C}(S)}{\underset{n\to\infty}{\longrightarrow}} Lf$.

Proof. Set $\Omega := (S^\Delta)^{\mathbb{N}} \times \mathbb{R}_+^{\mathbb{N}}$ and $\mathcal{G} := \mathcal{B}(S^\Delta)^{\otimes\mathbb{N}} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes\mathbb{N}}$. For any $\mu \in \mathcal{P}(S^\Delta)$ and $n \in \mathbb{N}$, we denote

$$\mathbb{P}^n_\mu := \mathbf{P}^n_\mu \otimes \mathcal{E}(1)^{\otimes\mathbb{N}}, \tag{5}$$

where $\mathcal{E}(1)$ is the exponential distribution with expectation 1. We also set

$$Y_n : \begin{array}{ccc} \Omega & \to & S \\ \big((y_k)_k, (s_k)_k\big) & \mapsto & y_n \end{array} \qquad \text{and} \qquad E_n : \begin{array}{ccc} \Omega & \to & \mathbb{R}_+ \\ \big((y_k)_k, (s_k)_k\big) & \mapsto & s_n, \end{array} \tag{6}$$

and introduce the standard Poisson process, $N_t := \inf\big\{n \in \mathbb{N} \;\big|\; \sum_{k=1}^{n+1} E_k > t\big\}$, $t \geq 0$.

Step 1) For each $n \in \mathbb{N}$ we set

$$Z^n_t := Y_{N_{t/\varepsilon_n}}. \tag{7}$$

Consider the following slightly modified assertions concerning the processes Z^n:

$a')$ for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ s.t. $\mu_n \to \mu$, $\quad \mathscr{L}_{\mathbb{P}^n_{\mu_n}}(Z^n) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_\mu$;

$b')$ for any $a_n, a \in S$ s.t. $a_n \to a$, $\quad \mathscr{L}_{\mathbb{P}^n_{a_n}}(Z^n) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_a$.

We claim that $a') \Leftrightarrow b') \Leftrightarrow c)$.
We will prove that for all $\mu \in \mathcal{P}(S^\Delta)$, $\mathscr{L}_{\mathbb{P}^n_\mu}(Z^n) \in \mathcal{M}(L_n)$. Setting $\mathcal{G}^n_t := \sigma(N_{s/\varepsilon_n}, Z^n_s,\ s \leq t)$, it is enough to prove that, for each $f \in \mathrm{C}_0(S)$ and $0 \leq s \leq t$,

$$\mathbb{E}^n_\mu\left[f(Z^n_t) - f(Z^n_s) - \int_s^t L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_s\right] = 0. \tag{8}$$

Let us introduce the $(\mathcal{G}^n_t)_t$-stopping times $\tau^n_k := \inf\big\{u \geq 0 \,\big|\, N_{u/\varepsilon_n} = k\big\}$. Then, for all $k \in \mathbb{N}$, we split

$$\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] = A_1 + A_2, \tag{9}$$

where

$$A_1 := \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}\leq t\}} \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right],$$
$$A_2 := \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}-\tau^n_k\vee s\leq t-\tau^n_k\vee s\}} \;\middle|\; \mathcal{G}^n_{\tau^n_k\vee s}\right].$$

By using the definition of the transition operator T_n and the fact that $(N_{u/\varepsilon_n})_u$ is a

Poisson process, we get for all $k \in \mathbb{N}$,

$$
\begin{aligned}
A_1 &= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}}(T_n f(Y_k) - f(Y_k))\Big(1 - e^{-(t-\tau_k^n \vee s)/\varepsilon_n}\Big), \\
A_2 &= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}} L_n f(Z^n_{\tau_k^n \vee s})\varepsilon_n \Big(1 - e^{-(t-\tau_k^n \vee s)/\varepsilon_n}\Big). \qquad (10)
\end{aligned}
$$

Similarly, we also can split, for all $k \in \mathbb{N}$,

$$
\mathbb{E}^n_\mu \left[\int_{t\wedge(\tau_k^n \vee s)}^{t\wedge(\tau_{k+1}^n \vee s)} L_n f(Z_u^n) \mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau_k^n\vee s)} \right] = B_1 + B_2, \qquad (11)
$$

with

$$
\begin{aligned}
B_1 &:= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}} L_n f(Z^n_{\tau_k^n\vee s}) \mathbb{E}^n_\mu \left[t\wedge \tau_{k+1}^n - \tau_k^n \vee s \;\middle|\; \mathcal{G}^n_{t\wedge(\tau_k^n\vee s)} \right], \\
B_2 &:= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}} L_n f(Z^n_{\tau_k^n\vee s}) \mathbb{E}^n_\mu \left[(t - \tau_k^n \vee s) \wedge (\tau_{k+1}^n - \tau_k^n \vee s) \;\middle|\; \mathcal{G}^n_{\tau_k^n\vee s} \right].
\end{aligned}
$$

Once again, since the distribution of $\tau_{k+1}^n - \tau_k^n$ is exponential we get, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
B_1 &= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}} L_n f(Z^n_{\tau_k^n\vee s}) \int_0^\infty (1/\varepsilon_n) e^{-u/\varepsilon_n}((t - \tau_k^n \vee s)\wedge u)\mathrm{d}u \\
B_2 &= \mathbb{1}_{\{t>\tau_k^n, s<\tau_{k+1}^n\}} L_n f(Z^n_{(10)\tau_k^n\vee s})\varepsilon_n \Big(1 - e^{-(t-\tau_k^n \vee s)/\varepsilon_n}\Big). \qquad (12)
\end{aligned}
$$

Gathering (10) in (9), respectively (12) in (11) and then subtracting (11) from (9), we get, for all $k \in \mathbb{N}$,

$$
\mathbb{E}^n_\mu \left[f(Z^n_{t\wedge(\tau_{k+1}^n\vee s)}) - f(Z^n_{t\wedge(\tau_k^n\vee s)}) - \int_{t\wedge(\tau_k^n\vee s)}^{t\wedge(\tau_{k+1}^n\vee s)} L_n f(Z_u^n)\mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau_k^n\vee s)} \right] = 0. \qquad (13)
$$

Recalling the definition of the stopping times τ_k^n and by summing on $k \in \mathbb{N}$, we also get

$$
\begin{aligned}
&\mathbb{E}^n_\mu \left[f(Z_t^n) - f(Z_s^n) - \int_s^t L_n f(Z_u^n)\mathrm{d}u \;\middle|\; \mathcal{G}_s^n \right] \\
&= \sum_{k\geq 0} \mathbb{E}^n_\mu \left[\mathbb{E}^n_\mu \left[f(Z^n_{t\wedge(\tau_{k+1}^n\vee s)}) - f(Z^n_{t\wedge(\tau_k^n\vee s)}) - \int_{t\wedge(\tau_k^n\vee s)}^{t\wedge(\tau_{k+1}^n\vee s)} L_n f(Z_u^n)\mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau_k^n\vee s)} \right] \;\middle|\; \mathcal{G}_s^n \right].
\end{aligned}
$$

By using (13) we end up with (8). As a consequence, for each $n \in \mathbb{N}$, $\mathscr{L}_{\mathbb{P}^n_\mu}(Z^n) \in \mathcal{M}(L_n)$. Invoking Theorem 2.2 applied to L_n and L, our claim $a') \Leftrightarrow b') \Leftrightarrow c)$ is achieved.

Step 2. To carry out the proof we need to establish the following result.

Lemma 3.3. *For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space, let $Z^n : \Omega^n \to \mathbb{D}_{loc}(S)$ and $\Gamma^n : \Omega^n \to \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ be a increasing random bijection. Define $\widetilde{Z}^n := Z^n \circ \Gamma^n$. If,* *for each $\varepsilon > 0$ and $t \in \mathbb{R}_+$,*

$$\mathbb{P}^n\Big(\sup_{s\le t}|\Gamma^n_s - s| \ge \varepsilon\Big) \underset{n\to\infty}{\longrightarrow} 0,$$

then for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \Leftrightarrow \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P},$$

where the limits hold for the weak topology associated to the local Skorokhod topology.

We postpone the proof of this result and we finish the proof of the theorem. Recalling (5) and (6), and setting for all $t \ge 0$ and $n \in \mathbb{N}$,

$$\Gamma^n_t := \varepsilon_n \left(\sum_{k=1}^{\lfloor t/\varepsilon_n \rfloor} E_k + (t/\varepsilon_n - \lfloor t/\varepsilon_n \rfloor) E_{\lfloor t/\varepsilon_n \rfloor + 1} \right), \tag{14}$$

it is readily seen, by (7), that for any $t \ge 0$ and $n \in \mathbb{N}$, $Y_{\lfloor t/\varepsilon_n \rfloor} = Z^n_{\Gamma^n_t}$. By showing that

$$\forall t \ge 0,\ \forall \varepsilon > 0, \qquad \sup_{\mu\in\mathcal{P}(S^\Delta)} \mathbb{P}^n_\mu\left(\sup_{s\le t}|\Gamma^n_s - s| \ge \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0, \tag{15}$$

and employing the latter lemma, we can conclude that $a) \Leftrightarrow a')$ and $b \Leftrightarrow b')$, so we ends up with $a) \Leftrightarrow b) \Leftrightarrow c)$.

Step 3. It remains to verify our claim (15). This is quite classical but for the sake of completeness we sketch its proof. Denote by $\lceil r \rceil$ the smallest integer larger or equal than the real number r. Fix $t \ge 0$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}(S^\Delta)$. Since Γ^n is a continuous piecewise affine function, we have

$$\sup_{s\le t}\left|\Gamma^n_s - s\right| \le \sup_{\substack{k\in\mathbb{N}\\ k\le\lceil t/\varepsilon_n\rceil}} \left|\Gamma^n_{k\varepsilon_n} - k\varepsilon_n\right| = \sup_{\substack{k\in\mathbb{N}\\ k\le\lceil t/\varepsilon_n\rceil}} \left|\varepsilon_n \sum_{i=1}^k E_i - k\varepsilon_n\right| = \varepsilon_n \sup_{\substack{k\in\mathbb{N}\\ k\le\lceil t/\varepsilon_n\rceil}} \left|M_k\right|$$

with $M_k := \sum_{i=1}^k E_i - k$. Owing again (5) and (6), we see that the discrete martingale $(M_k)_k$ satisfies $\mathbb{E}^n_\mu[M_k^2] = k\mathbb{E}^n_\mu[(E_1-1)^2] = k$. Hence, applying Markov's and maximal Doob's inequalities, we get

$$\begin{aligned}
\mathbb{P}^n_\mu\Big(\sup_{s\le t}|\Gamma^n_s - s| \ge \varepsilon\Big) &\le \mathbb{P}^n_\mu\Big(\varepsilon_n \sup_{k\le\lceil t/\varepsilon_n\rceil}|M_k| \ge \varepsilon\Big) \le \frac{\varepsilon_n^2}{\varepsilon^2}\mathbb{E}^n_\mu\left[\sup_{k\le\lceil t/\varepsilon_n\rceil} M_k^2\right] \\
&\le \frac{4\varepsilon_n^2}{\varepsilon^2}\mathbb{E}^n_\mu\left[M^2_{\lceil t/\varepsilon_n\rceil}\right] = \frac{4\lceil t/\varepsilon_n\rceil\varepsilon_n^2}{\varepsilon^2} \le \frac{4(t+\varepsilon_n)\varepsilon_n}{\varepsilon^2} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

The proof of Theorem 3.2 is complete except for Lemma 3.3. □

Lemma 3.3 is obtained as a consequence of a more general result stated and proved below:

Lemma 3.4. *Let E be a Polish topological space, for $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider random variables $Z^n, \widetilde{Z}^n : \Omega^n \to E$. Suppose that for each compact subset $K \subset E$ and each open subset $U \subset E \times E$ containing the diagonal $\{(z,z) \mid z \in E\}$,*

$$\mathbb{P}^n \Big(Z^n \in K,\ (Z^n, \widetilde{Z}^n) \notin U \Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{16}$$

Then, for any $\mathbf{P} \in \mathcal{P}(E)$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \text{implies} \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}, \tag{17}$$

where the limits hold for the weak topology on $\mathcal{P}(E)$.

Proof. Suppose that $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$, so for any bounded continuous function $f : E \to \mathbb{R}$, $\mathbb{E}^n[f(Z^n)] \underset{n\to\infty}{\longrightarrow} \int f \mathrm{d}\mathbf{P}$. E being a Polish space, the sequence $(\mathscr{L}_{\mathbb{P}^n}(Z^n))_n$ is tight. Pick an arbitrary $\varepsilon > 0$ and let K be a compact subset of E such that

$$\forall n \in \mathbb{N}, \quad \mathbb{P}^n(Z^n \notin K) \leq \varepsilon. \tag{18}$$

By (16) applied to K and $U := \{(z, \widetilde{z}) \mid |f(\widetilde{z}) - f(z)| < \varepsilon\}$, we obtain

$$\mathbb{P}^n \Big(Z^n \in K,\ |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon \Big) \underset{n\to\infty}{\longrightarrow} 0.$$

We split successively

$$\begin{aligned}
\Big|\mathbb{E}^n[f(\widetilde{Z}^n)] - \int f \mathrm{d}\mathbf{P}\Big| &\leq \Big|\mathbb{E}^n[f(Z^n)] - \int f \mathrm{d}\mathbf{P}\Big| + \mathbb{E}^n\Big|f(\widetilde{Z}^n) - f(Z^n)\Big| \\
&\leq \big|\mathbb{E}^n[f(Z^n)] - \int f \mathrm{d}\mathbf{P}\big| + \mathbb{E}^n\Big[\big|f(\widetilde{Z}^n) - f(Z^n)\big| \mathbb{1}_{\{Z^n \in K, |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon\}}\Big] \\
&\quad + \mathbb{E}^n\Big[\big|f(\widetilde{Z}^n) - f(Z^n)\big| 1_{\{Z^n \in K, |f(\widetilde{Z}^n) - f(Z^n)| < \varepsilon\}}\Big] + \mathbb{E}^n\Big[\big|f(\widetilde{Z}^n) - f(Z^n)\big| \mathbb{1}_{\{Z^n \notin K\}}\Big].
\end{aligned}$$

Hence, by using (18), we endup with

$$\begin{aligned}
&\Big|\mathbb{E}^n[f(\widetilde{Z}^n)] - \int f \mathrm{d}\mathbf{P}\Big| \\
&\leq \Big|\mathbb{E}^n[f(Z^n)] - \int f \mathrm{d}\mathbf{P}\Big| + 2\|f\|\, \mathbb{P}^n\Big(Z^n \in K,\ |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon\Big) + \varepsilon\big(1 + 2\|f\|\big).
\end{aligned}$$

Letting successively $n \to \infty$ and $\varepsilon \to 0$, we conclude that $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. □

Proof of Lemma 3.3. We denote by $\widetilde{\Lambda}$ the space of increasing bijections λ from $\mathbb{R}_+$ to $\mathbb{R}_+$. For $t \in \mathbb{R}_+$, we set $\|\lambda - \mathrm{id}\|_t := \sup_{s \leq t} |\lambda_s - s|$. Since

$$\forall \lambda \in \widetilde{\Lambda},\ \forall t \in \mathbb{R}_+,\ \forall \varepsilon > 0, \quad \|\lambda - \mathrm{id}\|_{t+\varepsilon} < \varepsilon \Rightarrow \|\lambda^{-1} - \mathrm{id}\|_t < \varepsilon,$$

the hypotheses of Lemma 3.3 are symmetric with respect to Z and $\widetilde{Z}$, so it suffices to prove only one implication. Suppose $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$ and we prove, by applying Lemma 3.4, that $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. Let K be a compact subset of $\mathbb{D}_{\mathrm{loc}}(S)$ and U be an open subset of $\mathbb{D}_{\mathrm{loc}}(S) \times \mathbb{D}_{\mathrm{loc}}(S)$ containing the diagonal $\{(z,z) \,|\, z \in \mathbb{D}_{\mathrm{loc}}(S)\}$. It will be sufficient to prove the following assertion

$$\exists t \geq 0,\ \exists \varepsilon > 0,\ \forall z \in K,\ \forall \lambda \in \widetilde{\Lambda}, \quad \|\lambda - \mathrm{id}\|_t < \varepsilon \Rightarrow (z, z\circ\lambda) \in U. \tag{19}$$

Indeed, if we pick t and ε given by (19), then

$$\mathbb{P}^n\left(Z^n \in K,\ (Z^n, \widetilde{Z}^n) \notin U\right) \leq \mathbb{P}^n\left(\|\Gamma^n - \mathrm{id}\|_t \geq \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0,$$

and we employ Lemma 3.4 to conclude that $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$ as desired.

To verify (19) we assume that it is false, so we can find two sequences $(z^n)_n \in K^{\mathbb{N}}$ and $(\lambda^n)_n \in \widetilde{\Lambda}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(z^n, z^n \circ \lambda^n) \notin U$ and for all $t \geq 0$, $\lim\limits_{n\to\infty} \|\lambda_n - \mathrm{id}\|_t \to 0$. By compactness of K, possibly by taking a subsequence, there exists $z \in K$ such that $z^n \to z$, as $n \to \infty$. It is then straightforward to obtain

$$U \not\ni (z^n, z^n \circ \lambda^n) \underset{n\to\infty}{\longrightarrow} (z,z) \in U.$$

This is a contradiction with the fact that U is open, so (19) is verified. □

4. Convergence towards diffusions evolving in a potential

Let us recall that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ denotes the space of locally integrable functions, and a continuous real function f is called locally absolutely continuous if its distributional derivative f' belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. We introduce the set of potentials

$$\mathscr{V} := \left\{ V : \mathbb{R} \to \mathbb{R} \text{ measurable} \,\middle|\, \mathrm{e}^{|V|} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) \right\}.$$

It is straightforward to prove that there exists a unique Polish topology on $\mathscr{V}$ such that a sequence $(V_n)_n$ in $\mathscr{V}$ converges to $V \in \mathscr{V}$ if and only if

$$\forall M \in \mathbb{R}_+, \quad \lim_{n\to\infty} \int_{-M}^{M} \left|\mathrm{e}^{V(a)} - \mathrm{e}^{V_n(a)}\right| \vee \left|\mathrm{e}^{-V(a)} - \mathrm{e}^{-V_n(a)}\right| \mathrm{d}a = 0.$$

Notation 4.1. For a potential $V \in \mathscr{V}$, we introduce the operator

$$L^V := \frac{1}{2}\mathrm{e}^V \frac{\mathrm{d}}{\mathrm{d}a} \mathrm{e}^{-V} \frac{\mathrm{d}}{\mathrm{d}a} \tag{20}$$

as the set of couples $(f,g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R})$ such that f and $\mathrm{e}^{-V} f'$ are locally absolutely continuous and $g = \frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V} f')'$.

Remark 4.2. Let us notice that it is a particular case of the operator $\boldsymbol{D}_m \boldsymbol{D}_p^+$ described in [10], pp. 21-22. Heuristically, the solutions of the martingale local problem associated to L^V are solutions of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\beta_t - \frac{1}{2}V'(X_t)\mathrm{d}t,$$

where β is a standard Brownian motion.

We can state now the main results of this section. The first theorem contains some properties of the operator L^V and will be obtained as an application of Theorems 2.1 and 2.2 (or Theorems 3.10 and 4.17 in [4]).

Theorem 4.3 (Diffusions in a potential)**.**

(1) For any potential $V \in \mathscr{V}$, the operator L^V is the generator of a locally Feller family.
(2) For any sequence of potentials $(V_n)_n$ in $\mathscr{V}$ converging to $V \in \mathscr{V}$ for the topology of $\mathscr{V}$, the sequence of operators L^{V_n} converges to L^V, in the sense of the third statement of the convergence Theorem 2.2.

The second theorem gives an approximation result of a diffusion in a potential by using a sequence of random walks. Its will be based on the result Theorem 3.2 in the preceding section.

Theorem 4.4 (Approximation by random walks on $\mathbb{Z}$)**.** *For $(n,k) \in \mathbb{N}\times\mathbb{Z}$, let $q_{n,k} \in \mathbb{R}$ and $\varepsilon_n > 0$ be. For all $n \in \mathbb{N}$, in accordance with Definition 3.1, let $(\mathbf{P}^n_k)_k \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})^{\mathbb{Z}}$ be the unique discrete time locally Feller family such that*

$$\mathbf{P}^n_k(Y_1 = k+1) = 1 - \mathbf{P}^n_k(Y_1 = k-1) = \frac{1}{\mathrm{e}^{q_{n,k}} + 1}.$$

We introduce the sequence of potentials in $\mathscr{V}$ given by

$$V_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a<0}\,,$$

such that V_n converges for the topology of $\mathscr{V}$ to a potential of $\mathscr{V}$, say V. Let $(\mathbf{P}_a)_a$ be the locally Feller family associated with L^V. If $\varepsilon_n \to 0$, then, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that their pushforwards with respect to the mappings $k \mapsto \varepsilon_n k$ converge to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}}\left((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t\right) \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{D}_{loc}(S))} \mathbf{P}_\mu.$$

Before proving these two theorems, we state and prove an important consequence concerning the connection between a random walk and a diffusion in random environment. Several examples of application of the following result will be then discussed.

Corollary 4.5. *For each $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider the random variables*

$$(q_{n,k})_k : \Omega^n \to \mathbb{R}^{\mathbb{Z}}, \quad (Z^n_k)_k : \Omega^n \to \mathbb{Z}^{\mathbb{N}} \quad \textit{and} \quad \varepsilon_n : \Omega^n \to \mathbb{R}^*_+\,.$$

Suppose that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $\mathbb{P}^n$-almost surely,

$$\mathbb{P}^n\left(Z^n_{k+1} = Z^n_k + 1 \;\middle|\; \varepsilon_n, (q_{n,\ell})_{\ell\in\mathbb{Z}}, (Z^n_\ell)_{0\le\ell\le k}\right) = \frac{1}{\mathrm{e}^{q_{n,Z_k}}+1}\,,$$
$$\mathbb{P}^n\left(Z^n_{k+1} = Z^n_k - 1 \;\middle|\; \varepsilon_n, (q_{n,\ell})_{\ell\in\mathbb{Z}}, (Z^n_\ell)_{0\le\ell\le k}\right) = \frac{1}{\mathrm{e}^{-q_{n,Z_k}}+1} = 1 - \frac{1}{\mathrm{e}^{q_{n,Z_k}}+1}. \quad (21)$$

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, introduce a random potential belonging to $\mathscr{V}$,

$$W_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n\rfloor} q_{n,k}\mathbb{1}_{a\ge\varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n\rfloor-1} q_{n,-k}\mathbb{1}_{a<0}\,. \quad (22)$$

Furthermore, on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, consider two random variables $W : \Omega \to \mathscr{V}$ and $Z : \Omega \to \mathbb{D}_{loc}(\mathbb{R})$, such that the conditional distribution of Z with respect to W satisfies, $\mathbb{P}$-a.s.

$$\mathscr{L}_{\mathbb{P}}(Z \mid W) \in \mathcal{M}(L^W). \quad (23)$$

Assuming that ε_n converges in distribution to 0, that $\varepsilon_n Z^n_0$ converges in distribution to Z_0 and that W_n converges in distribution to W for the topology of $\mathscr{V}$, then $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2\rfloor})_t$ converges in distribution to Z for the local Skorokhod topology.

Proof of Corollary 4.5. Let $F : \mathbb{D}_{\mathrm{loc}}(\mathbb{R}) \to \mathbb{R}$ be a bounded continuous function. For any $a \in \mathbb{R}$, $V \in \mathscr{V}$ and $\varepsilon \in \mathbb{R}^*_+$, let $\mathbf{P}^{a,V,\varepsilon} \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})$ be the unique probability measure such that $\mathbf{P}^{a,V,\varepsilon}(Y_0 = \lfloor a/\varepsilon\rfloor) = 1$, and such that $\mathbf{P}^{a,V,\varepsilon}$-almost surely, for all $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{P}^{a,V,\varepsilon}(Y_{k+1} = Y_k + 1 \mid Y_0, \dots, Y_k) &= 1 - \mathbf{P}^{a,V,\varepsilon}(Y_{k+1} = Y_k - 1 \mid Y_0, \dots, Y_k)\\ &= \int_{\varepsilon Y_k-\varepsilon}^{\varepsilon Y_k} \mathrm{e}^{V(a)}\mathrm{d}a \Big/ \int_{\varepsilon Y_k-\varepsilon}^{\varepsilon Y_k+\varepsilon} \mathrm{e}^{V(a)}\mathrm{d}a.\end{aligned}$$

Furthermore, let $\mathbf{P}^{a,V,0} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(\mathbb{R}))$ be the unique probability measure belonging to $\mathcal{M}(L^V)$ and starting from a. Define the bounded mapping $G : \mathbb{R} \times \mathscr{V} \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$G(a, V, \varepsilon) := \mathbf{E}^{a,V,\varepsilon}\left[F\left(\varepsilon Y_{\lfloor\bullet/\varepsilon^2\rfloor}\right)\right] \quad \text{and} \quad G(a, V, 0) := \mathbf{E}^{a,V,0}\left[F(X)\right]. \quad (24)$$

An application of Theorem 4.3, shows that the mapping G is continuous at every point of $\mathbb{R} \times \mathscr{V} \times \{0\}$. Thus,

$$\mathbb{E}^n\left[G(\varepsilon_n Z^n_0, W_n, \varepsilon_n)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[G(Z_0, W, 0)\right]. \quad (25)$$

Combining the definitions (21) and (24) we can write

$$\mathbb{E}^n\Big[F\big(\varepsilon_n Z^n_{\lfloor\bullet/\varepsilon_n^2\rfloor}\big)\Big] = \mathbb{E}^n\Big[\mathbb{E}^n\big[F\big(\varepsilon_n Z^n_{\lfloor\bullet/\varepsilon_n^2\rfloor}\big)\big|\varepsilon_n,\ Z^n_0,\ (q_{n,\ell})_{\ell\in\mathbb{Z}}\big]\Big] = \mathbb{E}^n\left[G(\varepsilon_n Z^n_0, W_n, \varepsilon_n)\right]. \quad (26)$$

Gathering (25) on the right hand side of (26), and invoking (23)-(24), we obtain

$$\mathbb{E}^n\Big[F\big(\varepsilon_n Z^n_{\lfloor \bullet/\varepsilon_n^2\rfloor}\big)\Big] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[G(Z_0,W,0)\right] = \mathbb{E}\Big[\mathbb{E}\big[F(Z)\big|Z_0,\ W\big]\Big] = \mathbb{E}\left[F(Z)\right] .$$

We conclude that $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2\rfloor})_t$ converges in distribution to Z. □

Example 4.6. Let us describe three examples of application.

1) Let $(q_k)_k$ be an i.i.d sequence of centred real random variables with finite variance σ^2 and suppose that $q_{n,k} = \sqrt{\varepsilon_n} q_k$. Suppose also that W is a Brownian motion with variance σ^2. Then, by Donsker's theorem, (W_n) given by (22) converges in distribution to W, so we can apply Corollary 4.5 to deduce the convergence of a random walk in a random i.i.d. medium (introduced by Sinai in [12]) to the diffusion corresponding to a Brownian movement in a Brownian potential (introduced by Brox in [1]). We recover in this manner Theorem 1 from [11], p. 295, without a technical hypothesis imposing that the common distribution of the random variables q_k is compactly supported.

2) Fix this time a deterministic $q \in \mathbb{R}^*$ and also $\lambda > 0$. Suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables such that $\mathbb{P}^n(q_{n,k} = q) = 1 - \mathbb{P}^n(q_{n,k} = 0) = \lambda\varepsilon_n$. Suppose also that $W(a) = qN_{\lambda a}$, where N stands for a standard Poisson process on $\mathbb{R}$. Then, it is classical (see for instance [2]), that (W_n) given by (22) converges in distribution to W. So we can apply Corollary 4.5 to deduce the convergence of Sinai's random walk to the diffusion corresponding to a Brownian movement in a Poisson potential. We recover now Theorem 2 from [11], p. 296.

3) More generally, suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables. Likewise, suppose that (W_n) given again by (22), converges in distribution to some Lévy process W. We can apply Corollary 4.5 to deduce the convergence of Sinai's random walk to the diffusion corresponding to a Brownian movement in a Lévy potential (introduced in [2]).

We go further and detail the proofs of Theorems 4.3 and 4.4. To achieve this, we need to state two more auxiliary results contained in Lemma 4.7 and Remark 4.8. The proof of the lemma is essentially an application of the second chapter of [10] and it will postponed at the end of this section.

Lemma 4.7. *Let V be a potential in $\mathscr{V}$ and let $h \in \mathrm{C}(\mathbb{R},\mathbb{R}_+^*)$ be a function such that, for all $n \in \mathbb{N}$,*

$$\inf_{n\le|a|\le n+1} h(a) \le \frac{1}{n}\left[\int_n^{n+1}\int_0^a \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \wedge \int_{n+1}^{n+2}\int_n^{n+1}\mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a \right.$$
$$\left. \wedge \int_{-(n+1)}^{-n}\int_a^0 \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \wedge \int_{-(n+2)}^{-(n+1)}\int_{-(n+1)}^{-n}\mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a\right] \quad (27)$$

Then, with the notations (20) *and* (2)*, the operator* $(hL^V) \cap \big(\mathrm{C}_0(\mathbb{R})\times\mathrm{C}_0(\mathbb{R})\big)$ *is the* $(\mathrm{C}_0\times\mathrm{C}_0)$*-generator of a Feller semi-group.*

Remark 4.8. Consider $a_1, a_2 \in \mathbb{R}$ and let $V : [a_1\wedge a_2, a_1\vee a_2] \to \mathbb{R}$ be a measurable function such that $\mathrm{e}^{|V|} \in \mathrm{L}^1([a_1\wedge a_2, a_1\vee a_2])$. For any absolutely continuous function $f \in \mathrm{C}([a_1\wedge a_2, a_1\vee a_2],\mathbb{R})$ such that $\mathrm{e}^{-V}f'$ is absolutely continuous and $g := \frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V}f')'$ is continuous, we have two elementary but useful representations.

Firstly, we can write

$$f(a_2) = f(a_1) + \int_{a_1}^{a_2} f'(b)\mathrm{d}b,$$

and we deduce

$$f(a_2) = f(a_1) + \int_{a_1}^{a_2} \mathrm{e}^{V(b)} \left((\mathrm{e}^{-V} f')(a_1) + 2 \int_{a_1}^{b} \mathrm{e}^{-V(c)} g(c)\mathrm{d}c \right) \mathrm{d}b\,, \tag{28}$$

Furthermore, we can also develop

$$f(a_2) = f(a_1) + (\mathrm{e}^{-V} f')(a_1) \int_{a_1}^{a_2} \mathrm{e}^{V(b)}\mathrm{d}b + 2g(a_1) \int_{a_1}^{a_2} \int_{a_1}^{b} \mathrm{e}^{V(b)-V(c)}\mathrm{d}c\,\mathrm{d}b \\ + 2 \int_{a_1}^{a_2} \int_{a_1}^{b} \mathrm{e}^{V(b)-V(c)}(g(c) - g(a_1))\mathrm{d}c\,\mathrm{d}b. \tag{29}$$

This last equality will be useful to show that some operators satisfy the positive maximum principle.

Proof of Theorem 4.3. We are now ready to give the proof of the first part of theorem as an application of Theorem 2.1. Firstly, by using the result of Lemma 4.7 and, by quoting (2) and (3), we deduce that the operator

$$\widetilde{L} := \frac{1}{h}\overline{(hL^V) \cap \big(\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})\big)}$$

is the generator of a locally Feller family. Here the closure is taken in $\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R})$, and it is straightforward that $\widetilde{L} \subset \overline{L^V}$. Secondly, thanks to the representation (28), it is also straightforward to obtain $L^V = \overline{L^V}$. Invoking (29), we can deduce that L^V satisfies the positive maximum principle. Finally, using Theorem 2.1 we deduce the existence result for the martingale local problem associated to L^V. We conclude that $L^V = \widetilde{L}$ is the generator of a locally Feller family.

We proceed with the proof of the second part of Theorem 4.3. Let us denote by $(\mathbf{P}^n_a)_a$ and $(\mathbf{P}^\infty_a)_a$ the locally Feller families associated, respectively, to L^{V_n} and L^V. Thanks to Theorem 2.2, it is enough to prove that for each sequence of real numbers $(a_n)_n$ converging to $a_\infty \in \mathbb{R}$, $\mathbf{P}^n_{a_n}$ converges weakly to $\mathbf{P}^\infty_{a_\infty}$ for the local Skorokhod topology. According to Lemma A.2 in the Appendix (see also Lemma 4.22 from [4], p. 154), for $M \in \mathbb{N}^*$, there exists $h_M \in \mathrm{C}(\mathbb{R}, [0,1])$ such that

$$\{h_M \neq 0\} = (-2M, 2M), \quad \{h_M = 1\} = [-M, M],$$

and, for all $n \in \mathbb{N}$, the martingale local problems associated to $h_M L^V$ and to $h_M L^{V_n}$ are well-posed. For $n \in \mathbb{N}$ and $M \in \mathbb{N}^*$, we denote by $(\mathbf{P}^{n,M}_a)_a$ and $(\mathbf{P}^{\infty,M}_a)_a$ the locally Feller families associated, respectively to $h_M L^{V_n}$ and $h_M L^V$. For $n \in \mathbb{N}$, define the extension of $h_M L^{V_n}$:

$$\widetilde{L_{n,M}} := \Big\{ (f,g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R}) \,\Big|\, g = \frac{1}{2} h_M \mathrm{e}^{V_n} (\mathrm{e}^{-V_n} f')' \mathbb{1}_{(-2M,2M)} \Big\},$$

where f and $\mathrm{e}^{-V_n} f'$ are supposed to be locally absolutely continuous only on $(-2M, 2M)$. By (29) it is straightforward to obtain that $\widetilde{L_{n,M}}$ satisfies the positive maximum principle, so using Theorem 2.1, $\widetilde{L_{n,M}}$ is a linear subspace of the generator of the family $(\mathbf{P}_a^{n,M})_a$. We will prove that the sequence of operators $(\widetilde{L_{n,M}})$ converges to the operator $h_M L^V$ in the sense of the third statement of Theorem 2.2. Pick $f \in \mathrm{D}(L)$ and define $f_n \in \mathrm{C}_0(\mathbb{R})$ by

$$f_n(a) := \begin{cases} f(a), \quad a \notin (-2M - n^{-1}, 2M + n^{-1}), \\ f(0) + \displaystyle\int_0^a \mathrm{e}^{V_n(b)} \Big[(\mathrm{e}^{-V} f')(0) + 2 \int_0^b \mathrm{e}^{-V_n(c)} L^V f(c) \mathrm{d}c \Big] \mathrm{d}b, \quad a \in [-2M, 2M], \end{cases}$$

with f_n affine function on $[-2M - n^{-1}, -2M]$ and on $[2M, 2M + n^{-1}]$. Hence $f_n \in \mathrm{D}(\widetilde{L_{n,M}})$ and $\widetilde{L_{n,M}} f_n = h_M L^V f$. We can deduce the upper bound

$$\|f_n - f\| \leq \sup_{a \in [-2M, 2M]} |f_n(a) - f(a)| + \sup_{\substack{2M \leq |a_1|, |a_2| \leq 2M + n^{-1} \\ 0 \leq a_1 a_2}} |f(a_2) - f(a_1)|.$$

Since f is continuous, the second supremum in the latter equation tends to 0. By using the expression of f_n and the convergence $V_n \to V$, it is straightforward to deduce from (28) that

$$\sup_{a \in [-2M, 2M]} |f_n(a) - f(a)| \underset{n \to \infty}{\longrightarrow} 0.$$

Hence $\lim_{n \to \infty} \|f_n - f\| = 0$, so according to Theorem 2.2,

$$\mathbf{P}_{a_n}^{n,M} \underset{n \to \infty}{\longrightarrow} \mathbf{P}_{a_\infty}^{\infty,M}. \tag{30}$$

At this level we need to employ Lemma A.1 in the Appendix (see also Proposition 4.15 from [4], p. 153): for all $M \in \mathbb{N}^*$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathscr{L}_{\mathbf{P}_{a_n}^{n,M}} \left(X^{\tau^{(-M,M)}} \right) = \mathscr{L}_{\mathbf{P}_{a_n}^{n}} \left(X^{\tau^{(-M,M)}} \right). \tag{31}$$

Finally, we use a result of localisation of the continuity contained in Lemma A.3 in the Appendix (see also Lemma A1 from [4], p. 159) combining (30) and (31) and also letting $M \to \infty$, we conclude that $\mathbf{P}_{a_n}^n \underset{n \to \infty}{\longrightarrow} \mathbf{P}_{a_\infty}^\infty$. □

Proof of Theorem 4.4. For $n \in \mathbb{N}$, define the continuous function $\varphi_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ given by

$$\varphi_n(a, h) := 2 \int_a^{a+h} \int_a^b \mathrm{e}^{V_n(b) - V_n(c)} \mathrm{d}c \, \mathrm{d}b.$$

For each $a \in \mathbb{R}$, it is clear that $\varphi_n(a, \cdot)$ is strictly increasing on $\mathbb{R}_+$ and $\varphi_n(a, 0) = 0$.

Furthermore, since V_n is constant on the interval $\big[\varepsilon_n\lceil a/\varepsilon_n\rceil, \varepsilon_n(\lceil a/\varepsilon_n\rceil + 1)\big)$,

$$\varphi_n(a, 2\varepsilon_n) \geq 2\int_{\varepsilon_n\lceil a/\varepsilon_n\rceil}^{\varepsilon_n(\lceil a/\varepsilon_n\rceil+1)}\int_{\varepsilon_n\lceil a/\varepsilon_n\rceil}^{b} \mathrm{e}^{V_n(b)-V_n(c)}\mathrm{d}c\,\mathrm{d}b = \varepsilon_n^2.$$

Hence, there exists a unique $\psi_{1,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, \psi_{1,n}(a)) = \varepsilon_n^2. \tag{32}$$

Using the continuity of φ_n and the compactness of $[0, 2\varepsilon_n]$, it is straightforward to obtain that $\psi_{1,n}$ is continuous. In the same manner, we can prove that, for each $a \in \mathbb{R}$, there exists a unique $\psi_{2,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, -\psi_{2,n}(a)) = \varepsilon_n^2, \tag{33}$$

and that $\psi_{2,n}$ is continuous. Introduce the continuous function $p_n : \mathbb{R} \to (0, 1)$ given by

$$p_n(a) := \int_{a-\psi_{2,n}(a)}^{a} \mathrm{e}^{V_n(b)}\mathrm{d}b \Bigg/ \int_{a-\psi_{2,n}(a)}^{a+\psi_{1,n}(a)} \mathrm{e}^{V_n(b)}\mathrm{d}b. \tag{34}$$

Also define a transition operator $T_n : \mathrm{C}_0(\mathbb{R}) \to \mathrm{C}_0(\mathbb{R})$ by

$$T_n f(a) := p_n(a) f(a + \psi_{1,n}(a)) + (1 - p_n(a)) f(a - \psi_{2,n}(a)).$$

According to Definition 3.1, we can denote by $(\widetilde{\mathbf{P}}_a^n)_a \in \mathcal{P}\left(\mathbb{R}^{\mathbb{N}}\right)^{\mathbb{R}}$ the discrete time locally Feller family with T_n as a transition operator. For any $k \in \mathbb{Z}$, since V_n is constant on $[\varepsilon_n k, \varepsilon_n(k+1))$ and on $[\varepsilon_n(k-1), \varepsilon_n k)$, we have

$$\varphi_n(\varepsilon_n k, \pm\varepsilon_n) = 2\int_{\varepsilon_n k}^{\varepsilon_n(k\pm1)}\int_{\varepsilon_n k}^{b} \mathrm{d}c\,\mathrm{d}b = \varepsilon_n^2$$

and therefore $\psi_{1,n}(\varepsilon_n k) = \psi_{2,n}(\varepsilon_n k) = \varepsilon_n$. Furthermore

$$p_n(\varepsilon_n k) := \frac{\int_{\varepsilon_n(k-1)}^{\varepsilon_n k} \mathrm{e}^{V_n(b)}\mathrm{d}b}{\int_{\varepsilon_n(k-1)}^{\varepsilon_n(k+1)} \mathrm{e}^{V_n(b)}\mathrm{d}b} = \frac{\varepsilon_n \mathrm{e}^{V_n(\varepsilon_n(k-1))}}{\varepsilon_n \mathrm{e}^{V_n(\varepsilon_n(k-1))} + \varepsilon_n \mathrm{e}^{V_n(\varepsilon_n k)}} = \frac{1}{1 + \mathrm{e}^{q_{n,k}}},$$

hence for any $f \in \mathrm{C}_0(\mathbb{R})$,

$$T_n f(\varepsilon_n k) := \frac{1}{1 + \mathrm{e}^{q_{n,k}}} f(\varepsilon_n(k+1)) + \frac{1}{1 + \mathrm{e}^{-q_{n,k}}} f(\varepsilon_n(k-1)).$$

We deduce that for any $\mu \in \mathcal{P}(\mathbb{Z})$ and $n \in \mathbb{N}$, $\mathscr{L}_{\mathbf{P}_\mu^n}(\varepsilon_n Y) = \widetilde{\mathbf{P}}_{\widetilde{\mu}}^n$, where $\widetilde{\mu}$ is the pushforward measure of μ with respect to the mapping $k \mapsto \varepsilon_n k$.

We shall now use Theorem 3.2 of convergence of discrete time Markov families. If $f \in \mathrm{D}(L^V)$, we need to prove that there exists a sequence of continuous functions $f_n \in \mathrm{C}_0(\mathbb{R})$ converging to f such that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. By the second part of Proposition 4.3, there exists a sequence of continuous functions $f_n \in \mathrm{D}(L^{V_n})$

such that f_n converges to f and $L^{V_n} f_n$ converges to $L^V f$. Applying (29) to f_n and V_n and recalling (32) and (33), we can write, for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} f(a + \psi_{1,n}(a)) =& f(a) + (\mathrm{e}^{-V} f')(a) \int_a^{a+\psi_{1,n}(a)} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\ &+ 2 \int_a^{a+\psi_{1,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b, \end{aligned}$$

and

$$\begin{aligned} f(a - \psi_{2,n}(a)) =& f(a) - (\mathrm{e}^{-V} f')(a) \int_{a-\psi_{2,n}(a)}^{a} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\ &+ 2 \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b. \end{aligned}$$

Hence by (34), for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\left| \frac{T_n f_n(a) - f_n(a)}{\varepsilon_n^2} - L^{V_n} f_n(a) \right| \\ &\qquad \le \frac{2p_n(a)}{\varepsilon_n^2} \Big| \int_a^{a+\psi_{1,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big| \\ &\qquad + \frac{2(1-p_n(a))}{\varepsilon_n^2} \Big| \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big| \\ &\qquad \le \sup_{|h| \le 2\varepsilon_n} |L^{V_n} f_n(a+h) - L^{V_n} f_n(a)|. \end{aligned}$$

It is not difficult to deduce that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. Therefore we can apply Theorem 3.2 of convergence of discrete time Markov families, so for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that $\widetilde{\mu}_n$ converges to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}} \big((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) = \mathscr{L}_{\widetilde{\mathbf{P}}^n_{\widetilde{\mu}_n}} \big((Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_\mu,$$

where $\widetilde{\mu}_n$ are the pushforwards of μ_n with respect to the mappings $k \mapsto \varepsilon_n k$. □

Proof of Lemma 4.7. As was already announced this proof is essentially an application of the second chapter of [10]. For the sake of completeness we give here few details.

As was quoted in Remark 4.2, the operator hL^V coincides on $\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})$ with the operator $D_m D_p^+ \subset \mathrm{C}(\overline{\mathbb{R}}) \times \mathrm{C}(\overline{\mathbb{R}})$, on the extended real line $\overline{\mathbb{R}}$, described in [10], pp. 21-22, where

$$\mathrm{d}m(a) := \frac{2\mathrm{e}^{-V(a)}}{h(a)} \mathrm{d}a \quad \text{and} \quad \mathrm{d}p(a) := \mathrm{e}^{V(a)} \mathrm{d}a.$$

Applying our hypothesis(27) we can obtain

$$\int_0^\infty \int_0^a \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} \int_{n+1}^{n+2} \int_n^{n+1} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty,$$
$$\int_0^\infty \int_0^a \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} \int_{n}^{n+1} \int_0^{a} \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty,$$
$$\int_{-\infty}^0 \int_a^0 \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} \int_{-n-2}^{-n-1} \int_{-n-1}^{-n} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty,$$
$$\int_{-\infty}^0 \int_a^0 \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} \int_{-n-1}^{-n} \int_a^{0} \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty.$$

Thus, according to the definition given in [10], pp. 24-25, the boundary points $-\infty$ and $+\infty$ are natural. Thanks to Theorem 1 and Remark 2 p. 38 in [10], $D_m D_p^+$ is the generator of a conservative Feller semi-group on $\mathrm{C}(\overline{\mathbb{R}})$. Furthermore by steps 7 and 8 from [10], pp. 31-32,

$$D_m D_p^+ f(-\infty) = D_m D_p^+ f(+\infty) = 0, \quad \forall f \in \mathrm{D}(D_m D_p^+),$$

so that the operator

$$(hL^V) \cap \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R}) = D_m D_p^+ \cap \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})$$

is the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator of a Feller semi-group. □

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Appendix A.

We recall below the statements of three results already proved in [4] and used in the proofs of Theorem 4.3. We refer the interested reader to the paper [4] for the introductory contexts and complete proofs of each lemma.

Lemma A.1 (cf. Proposition 4.20 in [4], p. 153)**.** *Let $L_1, L_2 \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L_1) = \mathrm{D}(L_2)$ is dense in $\mathrm{C}_0(S)$ and assume that the martingale local problems associated to L_1 and L_2 are well-posed. Let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ be two solutions of these problems having the same initial distribution and let $U \subset S$ be an open subset. If for all $f \in D(L_1)$, $(L_2 f)_{|U} = (L_1 f)_{|U}$, then $\mathscr{L}_{\mathbf{P}^2}\big(X^{\tau^U}\big) = \mathscr{L}_{\mathbf{P}^1}\big(X^{\tau^U}\big)$.*

Lemma A.2 (cf. Lemma 4.22 in [4], p. 154)**.** *Let U be an open subset of S and L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ is dense in $\mathrm{C}(S)$. Assume that the martingale local problem associated to L is well-posed. Then there exists a function $h_0 \in \mathrm{C}(S, \mathbb{R}_+)$ satisfying $\{h_0 \neq 0\} = U$, such that for all $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a \in U}(h/h_0)(a) < \infty$, the martingale local problem associated to hL is well-posed.*

Lemma A.3 (cf. Lemma A.1 in [4], p. 159)**.** *Let $(U_m)_{m\in\mathbb{N}}$ be an increasing sequence of open subsets such that $S = \bigcup_m U_m$. For $n, m \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{P}^{n,m} \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be such that*

i) for each $m \in \mathbb{N}$, $\mathbf{P}^{n,m} \underset{n\to\infty}{\longrightarrow} \mathbf{P}^{\infty,m}$, weakly for the local Skorokhod topology,

ii) for each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $\mathscr{L}_{\mathbf{P}^{n,m}}\left(X^{\tau^{U_m}}\right) = \mathscr{L}_{\mathbf{P}^{n,\infty}}\left(X^{\tau^{U_m}}\right)$.

Then $\mathbf{P}^{n,\infty} \underset{n\to\infty}{\longrightarrow} \mathbf{P}^{\infty,\infty}$, weakly for the local Skorokhod topology.