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THE ANNULUS AS A K-SPECTRAL SET

Pour Michel Pierre, à l'occasion de son soixantième anniversaire

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ABSTRACT. We consider the annulus $\mathcal{A}_R$ of complex numbers with modulus and inverse of modulus bounded by $R > 1$. We present some situations, in which this annulus is a K-spectral set for an operator A, and some related estimates.

1. **Introduction.** Let us consider the annulus $\mathcal{A}_R := \{z \in \mathbb{C}\,; R^{-1} \leq |z| \leq R\}$ with $R > 1$; $\mathcal{A}_R$ is the intersection of two disks of the Riemann sphere $\mathcal{A}_R = D_1 \cap D_2$, with $D_1 := \{z \in \mathbb{C}\,; |z| \leq R\}$ and $D_2 := \{z \in \mathbb{C} \cup \{\infty\}\,; |z|^{-1} \leq R\}$. Let $A \in B(H)$ be a bounded operator acting on a complex Hilbert space H. The aim of this paper is to present some assumptions on the pairs (D_1, A) and (D_2, A), ensuring that the annulus $\mathcal{A}_R$ is a (complete) K-spectral set for A.

Recall that, for a fix constant $K \geq 1$, a closed subset X of the complex plane which contains the spectrum $\sigma(A)$ is called a K-spectral set for A if the inequality

$$\|f(A)\| \leq K\,\|f\|_X, \qquad \text{with } \|f\|_X := \sup_{z\in X} |f(z)|,$$

holds for all bounded rational functions f (from $\mathbb{C}$ into $\mathbb{C}$) on X. Furthermore, if $K = 1$, the set X is said to be a spectral set for A, [5]. We also consider rational functions $F = (f_{ij})$ on X with values in the set $M_d(\mathbb{C})$ of complex $d \times d$ matrices; then $F(A) = (f_{ij}(A))$ becomes a linear operator on H^d. The set X is said to be a complete K-spectral for A if the inequality

$$\|F(A)\| \leq K\,\|F\|_X, \qquad \text{with } \|F\|_X := \sup_{z\in X} \|F(z)\|,$$

holds for all bounded rational functions F on X with values in $M_d(\mathbb{C})$, and for all values of d. In the case $K = 1$, the set X is said to be completely spectral for A.

There exists a best constant $C(R)$ (resp. $C_{cb}(R)$) such that each bounded rational function f on $\mathcal{A}_R$, with values in $\mathbb{C}$ (resp. in $M_d(\mathbb{C})$), may be written as $f = f_1 + f_2$ (resp. $F = F_1 + F_2$), with

$$\|f_1\|_{D_1} \leq C(R)\,\|f\|_{\mathcal{A}_R} \quad \text{and} \quad \|f_2\|_{D_2} \leq C(R)\,\|f\|_{\mathcal{A}_R}$$
$$(\text{resp.} \quad \|F_1\|_{D_1} \leq C_{cb}(R)\,\|F\|_{\mathcal{A}_R} \quad \text{and} \quad \|F_2\|_{D_2} \leq C_{cb}(R)\,\|F\|_{\mathcal{A}_R}).$$

It has been noticed, for instance in [4, 6, 7], that, if D_1 is a K_1-spectral set for A and if D_2 is a K_2-spectral for the same operator A, then $\mathcal{A}_R$ is a K-spectral set for A, with $K \leq C(R)(K_1 + K_2)$. Similarly, if D_1 is a complete K_1-spectral set for A and if D_2

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is a complete K_2-spectral set for A, then $\mathcal{A}_R$ is a complete K-spectral for A, with $K \leq C_{cb}(R)(K_1 + K_2)$. In Section 2, we obtain some estimates of $C(R)$ and of $C_{cb}(R)$ that improve the ones given in [9] and in [8]. In particular we show that $C(R) = C_{cb}(R) = 1.5$ if $R \geq 2.3919$, and $\lim_{R\to 1} C(R) = \lim_{R\to 1} C_{cb}(R) = +\infty$. We do not know whether $C(R) = C_{cb}(R)$ for all $R > 1$.

The previous result is not fully satisfactory, in particular for R closed to 1. Indeed, there exist situations in which the previous estimates may be strongly improved. For instance, it is shown in [2, Theorem 1.2] that, if D_1 is a spectral set for A and D_2 is a spectral set for A (or equivalently if $\|A\| \leq R$ and $\|A^{-1}\| \leq R$), then $\mathcal{A}_R$ is a complete $K(R)$-spectral set for A, with $K(R) \leq 2 + \frac{R+1}{\sqrt{R^2+R+1}}$. In particular we have $K(R) \leq 2 + 2/\sqrt{3}$, for all R, while the previous estimate $K(R) \leq 2\, C_{cb}(R)$ blows up as $R \to 1$. In Section 3, we consider the assumptions " $w(A) \leq R$ and $w(A^{-1}) \leq R$", where $w(A) := \sup\{|\langle Av, v\rangle|\,; v \in H, \|v\| = 1\}$ is the numerical radius of A. We will say that $\mathcal{A}_R$ is a numerical annulus for A if these assumptions are satisfied. This situation infers that the sets D_1 and D_2 are completely 2-spectral for A [1]; therefore, it follows from the previous part that the annulus $\mathcal{A}_R$ is completely $K(R)$-spectral for A with $K(R) \leq 4\, C_{cb}(R)$. Using a method similar to [2], we show that $K(R) \leq 4 + \frac{R^2-1}{\sqrt{(R-2)(R^3-1/2)}}$, for $R > 2$. More generally, if we add to the hypothesis "$\mathcal{A}_R$ is a numerical annulus for A" the assumptions $\|A\| \leq \tau^2$ and $\|A^{-1}\| \leq \tau^2$, with $\sqrt{R} < \tau < R$, we show the estimate $K(R,\tau) \leq 4 + \frac{1}{\sqrt{1-\gamma^2}}$, with $\gamma = \frac{\tau-\tau^{-1}}{R-R^{-1}}$. Note also that this estimate is still valid if $1 < \tau \leq \sqrt{R}$, but in this case the inequalities $\|A\| \leq R$ and $\|A^{-1}\| \leq R$ are satisfied, and then a better estimate $K(R) \leq 2 + \frac{R+1}{\sqrt{R^2+R+1}}$ holds.

From the well-known inequalities $w(A) \leq \|A\| \leq 2\, w(A)$ and $w(A)\, w(A^{-1}) \geq 1$, we conclude that there exists a best (i.e. minimal) function φ such that the inequality

$$\|A\| \leq w(A)\, \varphi\big(\sqrt{w(A)w(A^{-1})}\big)$$

holds for all bounded operators A with bounded inverses. The function φ is defined on the interval $[1, +\infty)$ with values in $[1, 2]$. In [10], Stampfli has shown that the equality $w(A)\, w(A^{-1}) = 1$ holds, if and only if $A = \lambda U$, with $\lambda > 0$ and U is a unitary operator; therefore $\varphi(1) = 1$. In Section 4, we prove the estimates

$$\max(1 + \sqrt{1-x^2},\, 2 - x^{-4}) \leq \varphi(x) \leq \min(1 + c_1(x-1)^{1/4},\, 2 - c_2 x^{-4}),$$

for some positive constants c_1 and c_2. In particular this shows that, if $w(A) \leq 1+\varepsilon$ and $w(A^{-1}) \leq 1+\varepsilon$, then there exists a unitary operator U such that $\|A-U\| \leq c_3\, \varepsilon^{1/4}$.

2. **Decomposition of bounded rational functions in an annulus.** Let f be a bounded rational function in the annulus $\mathcal{A}_R$. Then, f may be written as $f = f_1 + f_2$, with rational functions f_1 bounded in D_1 and f_2 bounded in D_2. Note that, if $f = \varphi_1 + \varphi_2$ is another decomposition, with φ_1 and φ_2 holomorphic in the interior of D_1 and in the interior of D_2, respectively, φ_2 being furthermore assumed bounded at infinity, then the function $\varphi_1 - f_1 = f_2 - \varphi_2$ is holomorphic in the interior of D_1 and in the interior of D_2, thus in all the complex plane ; furthermore the function $\varphi_1 - f_1$ is bounded in the unit disk while $f_2 - \varphi_2$ is bounded in the complementary of the unit disk. So, the function $\varphi_1 - f_1 = f_2 - \varphi_2$ is holomorphic and bounded in all the complex plane, therefore it is constant. This shows the uniqueness, up to an additive constant, of the decomposition $f = f_1 + f_2$.

From now on, we use the notations

$$\|f\|_{\mathcal{A}_R} = \sup_{z\in\mathcal{A}_R} |f(z)|, \quad \|f_1\|_{D_1} = \sup_{z\in D_1} |f_1(z)|, \quad \|f_2\|_{D_2} = \sup_{z\in D_2} |f_2(z)|.$$

Lemma 2.1. *There exists a best constant $C(R)$ such that all bounded rational functions in $\mathcal{A}_R$ may be written in the form $f = f_1 + f_2$, with*

$$\|f_1\|_{D_1} \le C(R)\,\|f\|_{\mathcal{A}_R} \quad \textit{and} \quad \|f_2\|_{D_2} \le C(R)\,\|f\|_{\mathcal{A}_R}.$$

Furthermore, the following estimates hold

$$(a) \quad C(R) \le \max\Big(1.5, 1 + \sum_{n\ge1} \tfrac{2}{R^{2n}-1}\Big), \qquad (b) \quad C(R) \le 1 + \frac{1}{2\pi}\int_0^{\pi} \Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}\Big|\,d\theta,$$

$$(c) \quad C(R) \ge 1.5, \qquad (d) \quad C(R) \ge \tfrac{1}{7}\log\tfrac{1}{R-1}.$$

Proof. From the Cauchy formula, we may write $f = f_1 + f_2$ with

$$f_1(z) = \frac{1}{2\pi i}\int_{\partial D_1} f(\sigma)\left(\tfrac{1}{\sigma-z} - \tfrac{1}{2\sigma}\right)d\sigma \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i}\int_{\partial D_2} f(\sigma)\left(\tfrac{1}{\sigma-z} - \tfrac{1}{2\sigma}\right)d\sigma,$$

by using a counterclockwise orientation for ∂D_1 and a clockwise for ∂D_2. The functions f_1 and f_2 are rational functions bounded in D_1 and in D_2, respectively.

a) We consider the Laurent series expansion, $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, then

$$f_1(z) = \tfrac{1}{2}a_0 + \sum_{n\ge1} a_n z^n \quad \text{and} \quad f_2(z) = \tfrac{1}{2}a_0 + \sum_{n\le-1} a_n z^n.$$

Without loss of generality, we may assume that $\|f\|_{\mathcal{A}_R} = 1$ and $a_0 \ge 0$. We note that, for $R^{-1} \le r \le R$,

$$\begin{aligned} a_n r^n + \overline{a_{-n}}\, r^{-n} &= -\frac{1}{2\pi}\int_0^{2\pi} \big(1 - f(re^{i\theta})\big)e^{-ni\theta}d\theta - \frac{1}{2\pi}\int_0^{2\pi}\big(1 - \overline{f(re^{i\theta})}\big)e^{-ni\theta}d\theta \\ &= -\frac{1}{\pi}\int_0^{2\pi} e^{-ni\theta}\,\mathrm{Re}\,\big(1 - f(re^{i\theta})\big)\,d\theta. \end{aligned}$$

Using the fact that $\mathrm{Re}\,\big(1 - f(re^{i\theta})\big) \ge 0$, which follows from $\|f\|_{\mathcal{A}_R} = 1$, we get

$$|a_n r^n + \overline{a_{-n}}\, r^{-n}| \le \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\,\big(1 - f(re^{i\theta})\big)d\theta = 2(1 - a_0),$$

and then, by taking $r = R$ and $r = R^{-1}$,

$$|a_n R^n + \overline{a_{-n}}\, R^{-n}| \le 2(1 - a_0), \quad |a_n R^{-n} + \overline{a_{-n}}\, R^n| \le 2(1 - a_0);$$

thus

$$|a_n| R^n \le 2(1 - a_0) + |a_{-n}|\, R^{-n} \le 2(1 - a_0)(1 + R^{-2n}) + |a_n| R^{-3n},$$

and

$$|a_n| R^{-n} \le \frac{2(1 - a_0)}{R^{2n} - 1}.$$

We note that, on the boundary ∂D_2,

$$\|f_1\|_{L^\infty(\partial D_2)} \le \frac{a_0}{2} + \sum_{n\ge1} |a_n| R^{-n} \le \frac{a_0}{2} + 2(1 - a_0)\sum_{n\ge1} \frac{1}{R^{2n} - 1};$$

consequently, since $0 \le a_0 \le 1$, we have $\|f_1\|_{L^\infty(\partial D_2)} \le \max(\frac12, \sum_{n\ge1} \frac{2}{R^{2n}-1})$. Then, using the maximum principle, we obtain

$$\|f_2\|_{D_2} = \|f_2\|_{L^\infty(\partial D_2)} = \|f - f_1\|_{L^\infty(\partial D_2)} \le 1 + \max(\frac{1}{2}, \sum_{n\ge1} \frac{2}{R^{2n}-1}).$$

The same estimate for $\|f_1\|_{D_1}$ may be proved in a similar way; this infers the inequality (a).

b) For $z = R^{-1}e^{i\varphi} \in \partial D_2$, we have

$$f_1(z) = \frac{1}{2\pi i}\int_{\partial D_1} f(\sigma)\Big(\frac{1}{\sigma - z} - \frac{1}{2\sigma}\Big)\, d\sigma = \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\theta})\Big(\frac{Re^{i\theta} + R^{-1}e^{i\varphi}}{Re^{i\theta} - R^{-1}e^{i\varphi}}\Big)\, d\theta.$$

It then follows that

$$\|f_1\|_{L^\infty(\partial D_2)} \le \frac{1}{2\pi}\int_0^{\pi}\Big|\frac{R^2 + e^{i\theta}}{R^2 - e^{i\theta}}\Big|\, d\theta;$$

thus

$$\|f_2\|_{D_2} = \|f - f_1\|_{L^\infty(\partial D_2)} \le 1 + \frac{1}{2\pi}\int_0^{\pi}\Big|\frac{R^2 + e^{i\theta}}{R^2 - e^{i\theta}}\Big|\, d\theta,$$

which shows the estimate (b).

c) We now consider the function $f = f_1 + f_2$, defined by

$$f_1(z) = \frac{1}{2} + \frac{z/R - 1 + \varepsilon}{1 - (1-\varepsilon)z/R}, \quad 0 < \varepsilon < 1, \qquad f_2(z) = f_1(1/z).$$

The image of D_1 by f_1, as well as the image of D_2 by f_2, is the disk of radius 1 centered in $1/2$. This infers

$$\min_{c\in\mathbb{C}}\big(\max\{\|f_1 - c\|_{D_1}, \|f_2 + c\|_{D_2}\}\big) = \|f_1\|_{D_1} = 1.5,$$

and then $1.5 \le C(R)\,\|f\|_{\mathcal{A}_R}$. Using the symmetry $f(z) = f(1/z)$, we note that

$$\begin{aligned}\|f\|_{\mathcal{A}_R} = \max_\theta |f(Re^{i\theta})| &\le \|f_1 - \tfrac12\|_{D_1} + \max_\theta |f_2(Re^{i\theta}) + \tfrac12| \\ &\le 1 + \max_\theta\Big|\frac{\varepsilon(1 + R^{-2}e^{-i\theta})}{1 - (1-\varepsilon)R^{-2}e^{-i\theta}}\Big| \\ &\le 1 + \frac{\varepsilon(1 + R^{-2})}{1 - (1-\varepsilon)R^{-2}}.\end{aligned}$$

We obtain the inequality $1.5 \le C(R)$ by letting ε tend to zero in the estimate

$$1.5 \le C(R)\Big(1 + \frac{\varepsilon(1 + R^{-2})}{1 - (1-\varepsilon)R^{-2}}\Big).$$

d) Up to now, we have considered rational functions f, but the results may be easily extended to bounded holomorphic functions in the annulus. Here we consider the function $f = f_1 + f_2$, defined by

$$f_1(z) = \log(R(1+\varepsilon) - z), \quad f_2(z) = -f_1(z^{-1}) = -\log(R(1+\varepsilon) - z^{-1}),$$

with $\varepsilon > 0$. The logarithmic functions are chosen in such a way that the functions f_1 and f_2 be continuous in D_1 and D_2, respectively, and that $f_1(1) = -f_2(1) \in \mathbb{R}$. We note that, for all complex numbers c, it holds $\|f_1 - c\|_{D_1} = \|f_2 + c\|_{D_2}$; thus

$$\inf_{c\in\mathbb{C}} \|f_2 + c\|_{D_1} = \inf_{c\in\mathbb{C}} \|f_1 - c\|_{D_2} \ge \tfrac12(f_1(R) - f_1(-R)) = \tfrac12 \log\tfrac{2+\varepsilon}{\varepsilon}.$$

This yields

$$C(R) \geq \tfrac{1}{2\|f\|_{\mathcal{A}_R}}\ \log \tfrac{2+\varepsilon}{\varepsilon}.$$

From the maximum principle and the symmetries $f(z) = -f(z^{-1})$, $f(\bar z) = \overline{f(z)}$, we have

$$\|f\|_{\mathcal{A}_R} = \max_{0\leq\theta\leq\pi} \big|f(Re^{i\theta})\big| = \max_{0\leq\theta\leq\pi} \big|\log \tfrac{g_1(\theta)}{g_2(-\theta)}\big|,$$

$$\text{with} \quad g_1(\theta) = 1+\varepsilon - e^{i\theta}, \quad g_2(\theta) = 1+\varepsilon - R^{-2}e^{i\theta}.$$

From one hand, for $0 \leq \theta \leq \pi$, we have the estimates $-\frac{\pi}{2} \leq \arg g_1(\theta) \leq 0$ and $0 \leq \arg g_2(-\theta) \leq \frac{\pi}{2}$; thus $\big|\operatorname{Im}\big(\log \frac{g_1(\theta)}{g_2(-\theta)}\big)\big| \leq \pi$. From the other hand, the quantity

$$\Big|\frac{g_1(\theta)}{g_2(-\theta)}\Big|^2 = \frac{(1+\varepsilon)^2 + 1 - 2(1+\varepsilon)\cos\theta}{(1+\varepsilon)^2 + R^{-4} - 2R^{-2}(1+\varepsilon)\cos\theta}$$

is an increasing function of θ on $[0, \pi]$; this yields

$$\big|\operatorname{Re}\big(\log \tfrac{g_1(\theta)}{g_2(-\theta)}\big)\big| \leq \max\Big(\log \tfrac{1-R^{-2}+\varepsilon}{\varepsilon}, \log \tfrac{2+\varepsilon}{1+\varepsilon+R^{-2}}\Big) = \log \tfrac{1-R^{-2}+\varepsilon}{\varepsilon}.$$

Choosing $\varepsilon = 1 - R^{-2}$, we obtain $\big|\operatorname{Re}\big(\log \frac{g_1(\theta)}{g_2(-\theta)}\big)\big| \leq \log 2$; thus $\|f\|_{\mathcal{A}_R} \leq \sqrt{\pi^2 + \log^2 2}$ ≤ 3.5, and finally

$$C(R) \geq \tfrac{1}{7}\ \log \tfrac{3-R^{-2}}{1-R^{-2}} \geq \tfrac{1}{7}\ \log \tfrac{1}{R-1}. \qquad \square$$

Remark 2.2. The rational functions f considered in this lemma take their values in $\mathbb{C}$. But the estimates would be exactly the same for functions with values in $M_d(\mathbb{C})$, independently of the value of d. Therefore the bounds for $C(R)$ given in this lemma are still valid for $C_{cb}(R)$. It is clear that $C(R) \leq C_{cb}(R)$, but we do not know whether $C(R) = C_{cb}(R)$ for all $R > 1$.

Remark 2.3. In our choice, the functions f_1 and f_2 play symmetric roles with respect to the change of variables $z \to 1/z$. This is not the case for the decomposition considered by Shields [9], which is slightly different. Translated in our context, his estimates would be

$$C_{cb}(R) \leq 1 + \tfrac{1}{2}\sqrt{\frac{R^2+1}{R^2-1}}.$$

The estimate (a) is essentially a variant of one obtained by Paulsen and Singh [8, Theorem 4.2], it improves Shields' estimate if $R \geq 2.2227\ldots$. The estimate (b) improves Shields' estimate for all values of R.

Remark 2.4. Choosing the best established estimate in each case, we obtain, with $\varepsilon \simeq 2.753\,10^{-5}$,

$$\begin{aligned}
C(R) = {}&C_{cb}(R) = 1.5, \quad \text{if } R \geq 2.3919,\\
1.5 \leq C(R) \leq {}&C_{cb}(R) \leq 1 + \sum_{n\geq 1} \frac{2}{R^{2n}-1}, \quad \text{if } 2.3634 \leq R \leq 2.3919,\\
1.5 \leq C(R) \leq {}&C_{cb}(R) \leq 1 + \frac{1}{2\pi}\int_0^{\pi} \Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}\Big|\, d\theta, \quad \text{if } 1+\varepsilon < R \leq 2.3634,\\
\tfrac{1}{7}\ \log \tfrac{1}{R-1} \leq C(R) \leq {}&C_{cb}(R) \leq 1 + \frac{1}{2\pi}\int_0^{\pi} \Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}\Big|\, d\theta, \quad \text{if } 1 < R \leq 1+\varepsilon.
\end{aligned}$$

Remark 2.5. It is easily verified that

$$\sup\{\Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}-\frac{2}{2(R-1)-i\theta}\Big|\,; R>1, 0\leq\theta\leq\pi\}<+\infty.$$

Therefore, in a neighborhood of $R=1$,

$$1+\frac{1}{2\pi}\int_0^\pi\Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}\Big|\,d\theta=\frac{1}{2\pi}\int_0^\pi\Big|\frac{2}{2(R-1)-i\theta}\Big|\,d\theta+O(1)=\tfrac{1}{\pi}\,\log\tfrac{1}{R-1}+O(1).$$

This shows that the estimates (b) and (d) provide a good control of the behaviour of $C_{cb}(R)$ in this neighborhood .

3. **Numerical annulus.** In this section, we consider an operator A which satisfies the assumptions $w(A)\leq R$, $w(A^{-1})\leq R$, and $\max(\|A\|,\|A^{-1}\|)\leq\tau^2$, with $1<\tau<R$. We will show the estimate

$$\|f(A)\|\leq(4+\frac{1}{\sqrt{1-\gamma^2}})\,\|f\|_{\mathcal{A}_R},\quad\text{with }\ \gamma=\frac{\tau-\tau^{-1}}{R-R^{-1}},\tag{1}$$

for all bounded rational functions f in the annulus $\mathcal{A}_R$.

Proof of (1). It suffices to do it under the hypotheses $w(A)<R$ and $w(A^{-1})<R$. Then we can write (using the appropriate orientations of ∂D_1 and of ∂D_2)

$$f(A)=\frac{1}{2\pi i}\int_{\partial D_1}f(\sigma)(\sigma-A)^{-1}d\sigma+\frac{1}{2\pi i}\int_{\partial D_2}f(\sigma)(\sigma-A)^{-1}d\sigma=F_1+F_2+F_3,$$

with

$$F_1=\frac{1}{2\pi i}\int_{\partial D_1}f(\sigma)\big((\sigma-A)^{-1}d\sigma-(\bar\sigma-A^*)^{-1}d\bar\sigma\big)$$
$$F_2=\frac{1}{2\pi i}\int_{\partial D_2}f(\sigma)\big((\sigma-A)^{-1}d\sigma-(\bar\sigma-A^*)^{-1}d\bar\sigma\big)$$
$$F_3=\frac{1}{2\pi i}\int_{\partial D_1}f(\sigma)(\bar\sigma-A^*)^{-1}d\bar\sigma+\frac{1}{2\pi i}\int_{\partial D_2}f(\sigma)(\bar\sigma-A^*)^{-1}d\bar\sigma.$$

Setting $\sigma=Re^{i\theta}$, we note that

$$\frac{1}{2\pi i}\big((\sigma-A)^{-1}d\sigma-(\bar\sigma-A^*)^{-1}d\bar\sigma\big)=\frac{R}{2\pi}\big((R-e^{i\theta}A)^{-1}+(R-e^{-i\theta}A^*)^{-1}\big)\,d\theta.$$

The assumption $w(A)\leq R$ implies $(R-e^{i\theta}A)^{-1}+(R-e^{-i\theta}A^*)^{-1}\geq 0$. Therefore (see [2, Lemma 2.1])

$$\|F_1\|\leq\Big\|\frac{1}{2\pi i}\int_{\partial D_1}\big((\sigma-A)^{-1}d\sigma-(\bar\sigma-A^*)^{-1}d\bar\sigma\big)\Big\|\,\|f\|_{\mathcal{A}_R}=2\,\|f\|_{\mathcal{A}_R}.$$

Similarly, from $w(A^{-1})\leq R$, we get $\|F_2\|\leq 2\,\|f\|_{\mathcal{A}_R}$.

It remains to show that $\|F_3\|\leq(1-\gamma^2)^{-1/2}$. For this, we note that $\bar\sigma=R^2/\sigma$ on ∂D_1, while $\bar\sigma=R^{-2}/\sigma$ on ∂D_2. Thus

$$F_3=-\frac{1}{2\pi i}\int_{\partial D_1}f(\sigma)R^2(R^2-\sigma A^*)^{-1}\frac{d\sigma}{\sigma}+\frac{1}{2\pi i}\int_{\partial D_2}f(\sigma)R^{-2}(R^{-2}-\sigma A^*)^{-1}\frac{d\sigma}{\sigma}.$$

The integrands being holomorphic with respect to σ in the annulus $\mathcal{A}_R$, we can move the integration paths ∂D_1 and ∂D_2 into the unit circle. Taking into account the different orientations of the paths, this gives

$$\begin{aligned} F_3 &= -\frac{1}{2\pi i}\int_{|\sigma|=1} f(\sigma)\big(R^2(R^2-\sigma A^*)^{-1}- R^{-2}(R^{-2}-\sigma A^*)^{-1}\big)\frac{d\sigma}{\sigma}\\ &= -\frac{R^2-R^{-2}}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})(M(\theta,A^*))^{-1}d\theta,\\ \text{with } & M(\theta,A^*) := R^2+R^{-2}-e^{i\theta}A^*-(e^{i\theta}A^*)^{-1}. \end{aligned}$$

We now write $A^* = UG$, with a unitary operator U and a positive self-adjoint operator G. The assumptions $\max(\|A\|,\|A^{-1}\|)\le\tau$ read $\tau^{-1}\le G\le\tau$. Setting $\rho=\frac{1}{2}(\tau+\tau^{-1})$, we have

$$\|G+G^{-1}-(\rho+1)I\|\le \max\{|x+x^{-1}-\rho-1|\,;\tau^{-1}\le x\le\tau\}=\rho-1.$$

This yields, for the self-adjoint part of $M(\theta,A^*)$,

$$\begin{aligned} \operatorname{Re} M(\theta,A^*) &= R^2+R^{-2}-(\rho+1)\operatorname{Re}(e^{i\theta}U)+\operatorname{Re}(e^{i\theta}U(G+G^{-1}-\rho-1))\\ &\ge R^2+R^{-2}-(\rho+1)\operatorname{Re}(e^{i\theta}U)-\rho+1\ge R^2+R^{-2}-2\rho>0. \end{aligned}$$

We then have the estimate (see [2, Lemma 2.2])

$$\|F_3\|\le\Big\|\frac{R^2-R^{-2}}{2\pi}\int_0^{2\pi}\big(R^2+R^{-2}-(\rho+1)\operatorname{Re}(e^{i\theta}U)-\rho+1\big)^{-1}d\theta\Big\| = \|h(U)\|,$$

where we have introduced the holomorphic function

$$h(z)=\frac{R^2-R^{-2}}{2\pi}\int_0^{2\pi}\frac{d\theta}{R^2+R^{-2}-\rho+1-(\rho+1)(e^{i\theta}z+e^{-i\theta}z^{-1})/2}.$$

Note that

$$\begin{aligned} h(e^{i\varphi}) &= \frac{R^2-R^{-2}}{2\pi}\int_0^{2\pi}\frac{d\theta}{R^2+R^{-2}-\rho+1-(\rho+1)\cos(\theta+\varphi)}\\ &= \frac{R^2-R^{-2}}{2\pi}\,\frac{2\pi}{(R+R^{-1})\sqrt{R^2+R^{-2}-2\rho}}=\frac{1}{\sqrt{1-\gamma^2}}=h(1). \end{aligned}$$

This shows that $h(U)=h(1)$ and gives the estimate

$$\|F_3\|\le h(1)=\frac{1}{\sqrt{1-\gamma^2}}.$$ □

Now, we only assume $w(A)\le R$ and $w(A^{-1})\le R$. In the case $R\ge 2$, the inequality $\max(\|A\|,\|A^{-1}\|)\le\tau^2$ is automatically satisfied with $\tau=\sqrt{2R}$, since $\|A\|\le 2w(A)$ and $\|A^{-1}\|\le 2w(A^{-1})$. The inequality (1) provides the existence of the best constant $K(R)$ such that

$$\|f(A)\|\le K(R)\,\|f\|_{\mathcal{A}_R},\quad \text{with}\quad K(R)\le 4+\frac{R^2-1}{\sqrt{(R-2)(R^3-\frac{1}{2})}},$$

for all bounded rational functions f in the annulus $\mathcal{A}_R$ and for all operators A satisfying $w(A)\le R$ and $w(A^{-1})\le R$.

Remark 3.1. We also have the estimate $K(R)\le 4\,C(R)$, since D_1 and D_2 are 2-spectral sets for A. Choosing the best known estimate in each case, we obtain

$$\begin{aligned}
&K(R)\le 4+\tfrac{R^2-1}{\sqrt{(R-2)(R^3-\frac12)}}, && \text{if } R\ge 2.43618,\\
&K(R)\le 6, && \text{if } 2.3919\le R\le 2.43618,\\
&K(R)\le 4+\sum_{n\ge1}\frac{8}{R^{2n}-1}, && \text{if } 2.3634\le R\le 2.3919,\\
&K(R)\le 4+\frac{2}{\pi}\int_0^\pi\Big|\frac{R^2+e^{i\theta}}{R^2-e^{i\theta}}\Big|\,d\theta, && \text{if } 1\le R\le 2.3634.
\end{aligned}$$

Remark 3.2. These estimates blows up as $R\to1$, but we do not know whether the best constant $K(R)$ is bounded as $R\to1$.

Remark 3.3. In this section, we only have considered scalar functions, but all the estimates are still valid, with the same constants, in completely bounded form.

4. **Norm of operators and numerical radius.** From the classical inequalities $w(A)\le\|A\|\le 2\,w(A)$ and $w(A)\,w(A^{-1})\ge1$, it follows that there exists a minimal function φ such that the inequality

$$\|A\|\le w(A)\,\varphi\big(\sqrt{w(A)w(A^{-1})}\big) \tag{2}$$

holds for all bounded operators A on a Hilbert space H with bounded inverses, and for all Hilbert spaces H. The function φ is defined on the interval $[1,+\infty)$ with values in $[1,2]$ and satisfies $\varphi(1)=1$. In this section, we will show that φ is an increasing function that satisfies the following estimates

$$\varphi(x)\ge 1+\sqrt{1-x^{-2}},\quad \forall x\ge1, \tag{3}$$

$$\varphi(x)\ge 2-x^{-4},\quad \forall x\ge1, \tag{4}$$

$$\varphi(x)\le 2-c_2x^{-4},\quad \forall x\ge1,\quad \text{with a constant } c_2,\ 0<c_2<1, \tag{5}$$

$$\varphi(x)\le 1+c_1(x-1)^{1/4},\quad \forall x\ge1,\quad \text{with a constant } c_1>0. \tag{6}$$

Proof that φ is increasing. Let $A\in B(H)$ be an invertible operator. We set $B=A\oplus\alpha$, with $\alpha=(t^2w(A^{-1}))^{-1}$, $t\ge1$. Then, we have $0<\alpha\le\frac{1}{w(A^{-1})}\le w(A)\le\|A\|$; therefore $\|B\|=\|A\|$, $w(B)=w(A)$ and $w(B^{-1})=t^2w(A^{-1})$. Replacing A by B in inequality (2), we obtain

$$\|A\|\le w(A)\,\varphi(t\sqrt{w(A)w(A^{-1})}),\quad \forall t\ge1,\forall A \text{ and } A^{-1}\in B(H).$$

From the minimality of φ, we deduce $\varphi(t\sqrt{w(A)w(A^{-1})})\ge\varphi(\sqrt{w(A)w(A^{-1})})$ for all $t\ge1$. This shows that φ is increasing. □

Proof of the lower bound (3). We use

$$A=\begin{pmatrix}1 & 2y\\ 0 & -1\end{pmatrix}\quad \text{with } y=\sqrt{x^2-1},\ x\ge1.$$

Then, we have $w(A)=w(A^{-1})=x$ and $\|A\|=y+\sqrt{1+y^2}=x+\sqrt{x^2-1}$. We obtain (3) by using the matrix A in (2). □

Proof of the lower bound (4). We will show a more precise inequality

$$\varphi(x) \geq 2-y, \quad \text{with } y = \frac{4x^4 - x^2 + 1 - \sqrt{(4x^4 - x^2 + 1)^2 - 16x^4}}{4x^4}.$$

The lower bound (4) then follows by noticing that $0 < y \leq x^{-4}$. To this end, we take

$$A = \begin{pmatrix} 0 & 0 & \sqrt{y} \\ 2-y & 0 & 0 \\ 0 & \sqrt{y} & 0 \end{pmatrix}.$$

Using the formulae

$$w\begin{pmatrix} 0 & 0 & b \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} = w\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{pmatrix} = \frac{a + \sqrt{a^2 + 8b^2}}{4},$$

it is easy to verify that $\|A\| = 2-y$, $w(A) = 1$, and $w(A^{-1}) = x^2$. The inequality $\varphi(x) \geq 2-y$ then follows by putting the matrix A in (2). □

Proof of the upper bound (5). It suffices to show that if the operator A satisfies $\|A\| = (2-\varepsilon)w(A)$ with $0 < \varepsilon < 1$, then it holds

$$w(A)w(A^{-1}) \geq \frac{1}{6\sqrt{5\varepsilon}}.$$

For this, we can assume that $w(A) = 1$. Then, there exists a unit normed vector e_1 such that $\|Ae_1\| \geq 2\sqrt{1-\varepsilon}$. Replacing A by $e^{i\theta}A$ if needed, we can assume that $\alpha = \langle Ae_1, e_1\rangle \geq 0$. This allows to write $Ae_1 = \alpha e_1 + \beta e_2$, $Ae_2 = \gamma e_1 + \delta e_2 + ue_3$, with $\beta \geq 0$, $u \geq 0$, and e_1, e_2, e_3 being three orthonormal vectors in H. We note that

$$w(A^{-1}) \geq \tfrac{1}{2}\,\|A^{-1}\| \geq \frac{1}{2\,\|Ae_2\|} = \frac{1}{2\,\sqrt{|\gamma|^2 + |\delta|^2 + u^2}}.$$

Thus, it suffices to show that $|\gamma|^2 + |\delta|^2 + u^2 \leq 45\,\varepsilon$. Let us now consider the orthogonal projector P from H onto the subspace spanned by e_1, e_2 and e_3, and let us set $A' = PAP^*$. Clearly $2-\varepsilon \geq \|A'\| \geq \|A'e_1\| = \sqrt{\alpha^2 + \beta^2} \geq 2\sqrt{1-\varepsilon}$ and $w(A') \leq w(A) = 1$. We identify A' with its corresponding matrix in the basis $\{e_1, e_2, e_3\}$,

$$A' = \begin{pmatrix} \alpha & \gamma & v \\ \beta & \delta & w \\ 0 & u & z \end{pmatrix} = B + C, \quad \text{with } B = \mathrm{Re}(A') = \tfrac{1}{2}(A' + A'^*), \;\; C = \tfrac{1}{2}(A' - A'^*).$$

The condition $w(A') \leq 1$ also reads, for all $\theta \in \mathbb{R}$, $\|\,\mathrm{Re}(e^{i\theta}A')\| \leq 1$, and, in particular, induces $\|B\| \leq 1$ and $\|C\| \leq 1$. It follows that

$$\tfrac{1}{2}\,|\beta e^{i\theta} + \bar{\gamma}e^{-i\theta}| = |\langle \mathrm{Re}(e^{i\theta}A')e_1, e_2\rangle| \leq 1,$$

and then $\beta + |\gamma| \leq 2$, by a judicious choice of θ. We use

$$4\,\mathrm{Re}\langle Be_1, Ce_1\rangle = 2\|A'e_1\|^2 - 2\|Be_1\|^2 - 2\|Ce_1\|^2 \geq 8(1-\varepsilon) - 2 - 2,$$

that reads

$$\beta^2 - |\gamma|^2 - |v|^2 \geq 4 - 8\varepsilon.$$

We also have

$$\beta^2 + |\delta|^2 + |w|^2 = \|A'^*e_2\|^2 \leq (2-\varepsilon)^2;$$

together with the previous inequality, this gives

$$|\delta|^2 + |w|^2 + |\gamma|^2 + |v|^2 \leq 4\varepsilon + \varepsilon^2.$$

In particular, this shows $|w| \le \sqrt{4\varepsilon+\varepsilon^2} \le (2+\varepsilon)\sqrt{\varepsilon}$. Taking now the vectors $x_1^* = (1,1,t)$ and $x_2^* = (1,-1,t)$, $t \in \mathbb{R}$, in the inequality

$$\mathrm{Re}\left(\tfrac12\langle A'x_1,x_1\rangle - \tfrac12\langle A'x_2,x_2\rangle\right) \le \tfrac12(\|x_1\|^2+\|x_2\|^2) = 2+t^2,$$

we get

$$\beta + \mathrm{Re}\,\gamma + t(u+\mathrm{Re}\,w)| \le 2+t^2, \quad \forall t \in \mathbb{R};$$

thus, choosing $t = \frac12(u{+}\mathrm{Re}\,w)$ and using the inequalities $\beta{+}|\gamma| \le 2$ and $\beta^2{-}|\gamma|^2 \ge 4{-}8\varepsilon$,

$$\frac{|u+\mathrm{Re}\,w|^2}{4} \le 2-\beta-\mathrm{Re}\,\gamma \le 2-\frac{\beta^2-|\gamma|^2}{\beta+|\gamma|} \le 4\varepsilon.$$

This yields $u \le |w| + 4\sqrt{\varepsilon}$, and we finally obtain

$$\begin{aligned}|\gamma|^2+|\delta|^2+u^2 &\le |\delta|^2+|w|^2+|\gamma|^2+|v|^2+u^2-|w|^2 \le 4\,\varepsilon+\varepsilon^2+u^2-|w|^2\\ &\le 4\,\varepsilon+\varepsilon^2+8|w|\sqrt{\varepsilon}+16\,\varepsilon \le 4\,\varepsilon+\varepsilon^2+16\,\varepsilon+8\,\varepsilon^2+16\,\varepsilon\\ &\le 36\,\varepsilon+9\,\varepsilon^2 \le 45\,\varepsilon.\end{aligned}$$

□

Proof of the upper bound (6). The work of Stampfli [10] has been an inspiration for this proof. We have to show that there exists a constant c_1 such that

$$\varphi(1{+}\varepsilon) \le 1 + c_1\varepsilon^{1/4}, \qquad \forall \varepsilon > 0.$$

We shall obtain a constant $c_1 > 4$. Since $\varphi(1{+}\varepsilon) \le 2$, the inequality will automatically be satisfied for $\varepsilon \ge \frac{1}{256}$. Thus, we only have to consider, from now on, the case $0 < \varepsilon < \frac{1}{256}$. Then, there exists an integer $n \ge 35$ such that

$$\frac{1}{\cos\frac{\pi}{n+1}} < 1+\varepsilon \le \frac{1}{\cos\frac{\pi}{n}}$$

We set $t = \tan\frac{\pi}{n}$, and note that $t = \sqrt{2\varepsilon} + O(\varepsilon^{3/2})$ and $t \le \frac{1}{11}$. In order to prove (6), it suffices to show that

$$\varphi(1{+}\varepsilon) \le 1 + c\sqrt{t} + O(t) \qquad \text{in a neighborhood of } t = 0.$$

To this end, we consider an operator A satisfying $w(A) = w(A^{-1}) \le 1+\varepsilon$, and write it as $A = BU$, with B self-adjoint positive and U unitary. We introduce a partition of the unit circle in n arcs

$$C_k = \{e^{i\theta}\,;\theta \in I_k\}, \quad I_k = [(2k{-}1)\pi/n, (2k{+}1)\pi/n),\ k = 1,\dots,n.$$

We consider the spectral decomposition of U and the orthogonal projector P_k onto the invariant subspace corresponding to the arc C_k:

$$U = \int_0^{2\pi} e^{it}\,dE(t), \quad P_k = E(I_k).$$

We admit, for the time being, the following result

Lemma 4.1. *Let $x \in P_kH$ be a unit element in the invariant subspace corresponding to C_k. Let us write $Bx = \lambda x + \beta\, t\, w$, with $\|x\| = \|w\| = 1$, $\langle x, w\rangle = 0$ and $\beta \ge 0$. Then, the following estimates hold*

$$\frac{1}{1+3\,t^2} \le \lambda \le 1{+}8t^2, \quad 0 \le \beta \le 7.$$

For an arbitrary unit element $x \in H$, $\|x\| = 1$, we write

$$x = \sum_{0\le k<n} \xi_k x_k \quad \text{with } \ x_k \in P_k H,\ \|x_k\| = 1,\ \textstyle\sum_k |\xi_k|^2 = 1.$$

It follows from the lemma that $Bx_k = \lambda_k x_k + \beta_k t\, w_k$, with $\|w_k\| = 1$, $0 < \lambda_k \le 1{+}8\,t^2$ and $0 \le \beta_k \le 7$. Thus,

$$Bx = \textstyle\sum_k \xi_k \lambda_k x_k + t \sum_k \xi_k \beta_k w_k.$$

Using the orthonormality of the elements $\{x_k\}$ and the Cauchy-Schwarz inequality, we get

$$\|Bx\| \le (\textstyle\sum_k \lambda_k^2 |\xi_k|^2)^{1/2} + t\,(\sum_k |\xi_k|^2)^{1/2}(\sum_k |\beta_k|^2)^{1/2} \le 1{+}8\,t^2 + 7\,t\sqrt{n}.$$

This shows that $\|A\| = \|B\| \le 1 + 7\sqrt{\pi}\,\sqrt{t} + O(t)$, consequently

$$\varphi(1{+}\varepsilon) \le 1 + 7\sqrt{\pi}\,\sqrt{t} + O(t),$$

which infers the inequality (6). □

Proof of Lemma 4.1. Starting from $x \in P_k H$, a unit element in the subspace corresponding to C_k, we can write

$$Ux = e^{i\psi}\cos\theta\,(x + \tan\theta\, y), \quad \text{with } \ \|x\| = \|y\| = 1, \langle x, y\rangle = 0.\ \psi \in \mathbb{R},\ \theta \in [0, \pi/2].$$

As noticed by Donoghue [3], the complex number

$$\cos\theta\, e^{i\psi} = \langle Ux, x\rangle = \int_{I_k} e^{it} d\|E(t)x\|^2$$

belongs to the convex hull of C_k. This infers that $\cos\frac{\pi}{n} \le \cos\theta \le 1$, i.e., $0 \le \theta \le \pi/n$; thus $|\tan\theta| \le t$. Recall that $Bx = \lambda\, x + t\,\beta\, w$, with $\|w\| = 1$, $\langle x, w\rangle = 0$ and $\beta \ge 0$. Thus $\lambda = \langle Bx, x\rangle \in \mathbb{R}^+$. Using

$$\begin{aligned}\langle Ax, x\rangle = \langle Ux, Bx\rangle &= \cos\theta\, e^{i\psi}\langle x{+}\tan\theta y, \lambda x{+}t\,\beta w\rangle \\ &= \cos\theta\, e^{i\psi}(\lambda + \beta\, t\, \tan\theta\langle y, w\rangle)\end{aligned}$$

together with the inequality $w(A) \le 1{+}\varepsilon \le 1/\cos\frac{\pi}{n}$, we obtain

$$|\lambda + \beta\, t\, \tan\theta\langle y, w\rangle| \le \frac{1+\varepsilon}{\cos\theta}; \quad \text{thus} \quad \lambda \le 1{+}t^2 + \beta\, t^2|\langle y, w\rangle|.$$

In particular, there holds

$$\lambda \le 1 + (1{+}\beta)t^2. \tag{7}$$

Starting now from the relation $\lambda\, B^{-1}x = x - t\,\beta\, B^{-1}w$, we have

$$\begin{aligned}\lambda\langle A^{-1}x, x\rangle = \langle \lambda B^{-1}x, Ux\rangle &= \cos\theta\, e^{-i\psi}\langle x{-}t\beta B^{-1}w, x{+}\tan\theta y\rangle \\ &= \cos\theta\, e^{-i\psi}(1 + \tfrac{t^2\beta^2}{\lambda}\langle B^{-1}w, w\rangle - t\,\beta\tan\theta\langle B^{-1}w, y\rangle).\end{aligned}$$

We now use the assumption $\lambda w(A^{-1}) \le \lambda(1{+}\varepsilon)$, to get

$$|1 + \tfrac{t^2\beta^2}{\lambda}\langle B^{-1}w, w\rangle - t\,\beta\tan\theta\langle B^{-1}w, y\rangle| \le \lambda\frac{1{+}\varepsilon}{\cos\frac{\pi}{n}} \le \lambda(1 + t^2).$$

We also have

$$\begin{aligned}\langle B^{-1}w, w\rangle \ge 1/\|B\| = 1/\|A\| \ge \frac{1}{2w(A)} \ge \frac{1}{2(1{+}\varepsilon)} \ge \frac{128}{257}, \\ |\tan\theta\,\langle B^{-1}w, y\rangle| \le \tan\tfrac{\pi}{n}\|B^{-1}\| = t\,\|A^{-1}\| \le 2t\, w(A^{-1}) \le \frac{257\,t}{128};\end{aligned}$$

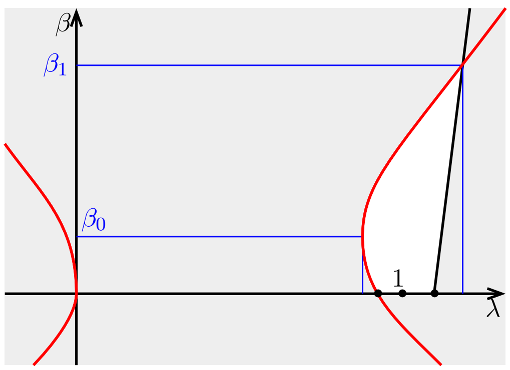
this yields

$$1 + \tfrac{128\,\beta^2 t^2}{257\,\lambda} - \beta t^2 \tfrac{257}{128} \le \lambda(1+t^2),$$

or equivalently

$$\beta^2 - a^2\beta\,\lambda - a\left(\lambda^2 + \frac{\lambda^2-\lambda}{t^2}\right) \le 0, \quad \text{with } a = \tfrac{257}{128}. \tag{8}$$

The set of (λ, β) satisfying (8) is the union of two convex parts delimited by a hyperbola $\mathcal{H}$, while the inequality (7) is corresponding to a half-plane. Recall that the inequalities $\lambda > 0$ and $\beta \ge 0$ also hold.



The hyperbola $\mathcal{H}$ is tangent to the axis $\{\lambda = 0\}$ at the origin, and admits another vertical tangent at the point

$$\Big(\frac{4}{4(1+t^2)+a^3t^2}, \frac{2a^2}{4(1+t^2)+a^3t^2}\Big).$$

This yields the estimate $\lambda \ge \frac{4}{4(1+t^2)+a^3t^2} \ge \frac{1}{1+3t^2}$. The hyperbola $\mathcal{H}$ crosses the straight line $\lambda = 1 + (1+\beta)t^2$ through the points

$(1+t^2(1+\beta_1), \beta_1)$ and $(1+t^2(1+\beta_2), \beta_2)$, with $\beta_1 > 0$ and $\beta_2 < 0$ being the roots of

$$E_t(\beta) := \beta^2 - a\,\beta\,\frac{1+a+4t^2+at^2+2t^4}{1-at^2-a^2t^2-at^4} - \frac{(1+t^2)(2+t^2)}{1-at^2-a^2t^2-at^4} = 0.$$

Recall that $t < \frac{1}{11}$, and then $E_t(7) \ge E_{1/11}(7) > 1.6308 > 0$. This shows the inequality $\beta < 7$ and completes the proof of the lemma. ☐

Remark 4.2. The estimates (4) and (5) give the fork

$$2 - x^{-4} \le \varphi(x) \le 2 - c_2 x^{-4};$$

this gives a good control on the behaviour of φ for large x, while the estimates (3) and (6) give a fork

$$1 + (1-x^{-2})^{1/2} \le \varphi(x) \le 1 + c_1(x-1)^{1/4},$$

which gives a control in a neighborhood of $x = 1$. We think that the exponent $1/4$ in this estimate effectively corresponds to the behavior of φ for x close to 1. This intuition is confirmed by numerical tests, that we have realized with the family of $n \times n$ matrices, $A = BD$, defined by, with $n = 4(2k+1)$,

$$B = I + \frac{1}{2n^{3/2}}E, \quad \text{with} \quad \begin{array}{l} e_{ij} = 1 \ \text{ if } 3k+2 \le |i-j| \le 5k+3, \\ e_{ij} = 0 \ \text{ otherwise,} \end{array}$$

$$D = \operatorname{diag}(e^{2i\pi/n}, \dots, e^{2\ell i\pi/n}, \dots, e^{2ni\pi/n}).$$

The points, with coordinates $\big(\log\big(\frac{\|A\|}{w(A)} - 1\big), \log\big(\sqrt{w(A)w(A^{-1})}\big)\big)$, computed for $k = 1, 2, \dots, 12$, are close to a straight line with a slope 0.2506.

Remark 4.3. We think that the function φ is continuous, but have not succeeded to prove it.

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