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Models of Curves

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Abstract. The main aim of these lectures is to present the stable reduction theorem with the point of view of Deligne and Mumford. We introduce the basic material needed to manipulate models of curves, including intersection theory on regular arithmetic surfaces, blow-ups and blow-downs, and the structure of the jacobian of a singular curve. The proof of stable reduction in characteristic 0 is given, while the proof in the general case is explained and important parts are proved. We give applications to the moduli of curves and covers of curves.

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1. Introduction

The problem of resolution of singularities over a field has a cousin of more arithmetic flavor known as *semistable reduction*. Given a field K, complete with respect to a discrete valuation v, and a proper smooth K-variety X, its concern is to find a regular scheme $\mathcal{X}$, proper and flat over the ring of integers of v, with generic fibre isomorphic to X and with special fibre a reduced normal crossings divisor in $\mathcal{X}$. Such a scheme $\mathcal{X}$ is called a *semistable model*. In general, one can not expect K-varieties to have smooth models, and semistable models are a very nice substitute; they are in fact certainly the best one can hope. Their occurrence in arithmetic geometry is ubiquitous for the study of ℓ-adic or p-adic cohomology, and of Galois representations. They are useful for the study of general models $\mathcal{X}'$, but also if one is interested in X in the first place. Let us give just one example showing some of the geometry of X revealed by its semistable models. If X is a curve, then Berkovich proved that the dual graph Γ of the special fibre of any semistable model has a natural embedding in the analytic space X^{an} (in the sense of Berkovich) associated to X and that this analytic space deformation retracts to Γ. (See [Be], Chapter 4.) In other words, the homotopy type of the analytic space

X^{an}, which is just a transcendental incarnation of X, is encoded in the special fibres of semistable models.

It is believed that semistable reduction is always possible after a finite extension of K. It is known only in the case of curves, where a refinement called *stable reduction* leads to the construction of a smooth compactification of the moduli stack of curves. The objective of the present text is to give a quick introduction to the original proof of these facts, following Deligne and Mumford's paper [DM]. Other subsequent proofs from Artin and Winters [AW], Bosch and Lütkebohmert [BL] or Saito [Sa] are not at all mentioned. (Note that apart from the original papers, some nice expositions such as [Ra2], [De], [Ab] are available.)

The exposition follows quite faithfully the plan of the lectures given by the author at the GAMSC summer school held in Istanbul in June 2008. Here is now a more detailed description of the contents of the article. When the residue characteristic is 0, the theorem is a simple computation of normalisation. Otherwise, the proof uses more material than could reasonably be covered within the lectures. I took for granted the semistable reduction theorem for abelian varieties proven by Grothendieck, as well as Raynaud's results on the Picard functor; this is consistent with the development in [DM]. Section 2 focuses on the manipulations on models: blow-ups and contractions, existence of (minimal) regular models. In Section 3, the description of the Picard functor of a singular curve is explained, and it is then used to make the link between semistable reduction of a curve and semistable reduction of its jacobian. This is the path to the proof of Deligne and Mumford. Finally, in Section 4, we translate these results to prove that moduli spaces (or moduli stacks) of stable curves, or covers of stables curves, are proper.

The main references are Deligne and Mumford [DM], Lichtenbaum [Lic], Liu's book [Liu] together with other sources which the reader will find in the bibliography in the end of this paper. I wish to thank the students and colleagues who attended the Istanbul summer school for their questions and comments during, and after, the lectures. Also, I wish to thank the referee for valuable comments leading to several clarifications.

2. Models of curves

In all the text, a *curve* over a base field is a proper scheme over that field, of pure dimension 1. Starting in Subsection 2.2, we fix a complete discrete valuation ring R with fraction field K and algebraically closed residue field k.

2.1. Definitions: normal, regular, semistable models

If K is a field equipped with a discrete valuation v and C is a smooth curve over K, then a natural question in arithmetic is to ask about the reduction of C modulo v. This implies looking for flat models of C over the ring of v-integers $R \subset K$ with the mildest possible singularities. If there exists a model with smooth special fibre over the residue field k of R, we say that C has *good reduction* at v (and otherwise we say that C has *bad reduction* at v).

It is known that there exist curves which do *not* have good reduction, and there are at least two reasons for this deficiency. The first reason is arithmetic: sometimes, the smooth special fibre (if it existed) must have rational points and this imposes some constraints on C. For example, consider the smooth projective conic C over the field $K = \mathbb{Q}_2$ of 2-adic numbers given by the equation $x^2+y^2+z^2 = 0$. If C had a smooth model X over $R = \mathbb{Z}_2$, then the special fibre X_k would have a rational point by the Chevalley-Warning theorem (as in [Se], Chap. 1) and hence X would have a $\mathbb{Z}_2$-integral point by the henselian property of $\mathbb{Z}_2$. However, it is easy to see by looking modulo 4 that C has no $\mathbb{Q}_2$-rational point. (One can easily cook up similar examples with curves of higher genus over a field K with algebraically closed residue field.) The second reason is geometric. Assuming a little familiarity with the moduli space of curves $\mathcal{M}_g$, it can be explained as follows: the "direction" in the *nonproper* space $\mathcal{M}_g$ determined by the path $\mathrm{Spec}(R)\backslash\{\text{closed point}\} \to \mathcal{M}_g$ corresponding to the curve C points to the boundary at infinity. For a simple example of this, consider the field of Laurent series $K = k((\lambda))$ which is complete for the λ-adic topology, and the Legendre elliptic curve E/K with equation $y^2 = x(x-1)(x-\lambda)$. Its j-invariant $j(\lambda) = 2^8(\lambda^2-\lambda+1)^3/(\lambda^2(\lambda-1)^2)$ determines the point corresponding to E in the moduli space of elliptic curves. Since $j(\lambda) \notin R = k[[\lambda]]$, the curve E has bad reduction (see [Si], Chap. VII, Prop. 5.5).

The arithmetic problem is not so serious, and we usually allow a finite extension K'/K before testing if the curve admits good reduction. However, the geometric problem is more considerable.

So, we have to consider other kinds of models. The mildest curve singularity is a *node*, also called *ordinary double point*, that is to say a rational point $x \in C$ such that the completed local ring $\widehat{\mathcal{O}}_{C,x}$ is isomorphic to $k[[u,v]]/(uv)$.

This leads to:

Definition 2.1.1. A *stable* (resp. *semistable*) curve over an algebraically closed field k is a curve which is reduced, connected, has only nodal singularities, all of whose irreducible components isomorphic to $\mathbb{P}^1_k$ meet the other components in at least 3 points (resp. 2 points).

A proper flat morphism of schemes $X \to S$ is called a *stable* (resp. *semistable*) curve if it has *stable* (resp. *semi-stable*) geometric fibres. In particular, given a smooth curve C over a discretely valued field K, a stable (resp. semistable) curve $X \to S = \mathrm{Spec}(R)$ with a specified isomorphism $X_K \simeq C$ is called a *stable* (resp. *semi-stable*) *model* of C over R.

One can also understand the expression *the mildest possible singularities* in an absolute meaning. For example, one can look for *normal* or *regular* models of the K-curve C, by which we mean a curve $X \to S = \mathrm{Spec}(R)$ whose total space is normal, or regular. By normalization, one may always find normal models. Regular models will be extremely important, firstly because they are somehow easier to produce than stable models, secondly because it is possible to do intersection theory on them, and thirdly because they are essential to the construction of stable models. We emphasize that in contrast with the notions of stable and semistable models, the notions of normal and regular models are not relative over S, in particular such models have in general singular, possibly nonreduced, special fibres.

For simplicity we shall call *arithmetic surface* a proper, flat scheme relatively of pure dimension 1 over R with smooth geometrically connected generic fibre. We will specify each time if we speak about a normal arithmetic surface, or a regular arithmetic surface, etc.

2.2. Existence of regular models

From this point until the end of the notes, we consider a complete discrete valuation ring R with fraction field K and algebraically closed residue field k.

For two-dimensional schemes, the problem of resolution of singularities has a satisfactory solution, with a strong form. Before we state the result, recall that a divisor D in a regular scheme X has *normal crossings* if for every point $x \in D$ there is an étale morphism of pointed schemes $p : (U, u) \to (X, x)$ such that p^*D is defined by an equation $a_1 \dots a_n = 0$ where $a_1, \dots, a_n$ are part of a regular system of parameters at u.

Theorem 2.2.1. *For every excellent, reduced, noetherian two-dimensional scheme X, there exists a proper birational morphism $X' \to X$ where X' is a regular scheme. Furthermore, we may choose X' such that its reduced special fibre is a normal crossings divisor.*

In fact, following Lipman [Lip2], one may successively blow up the singular locus and normalize, producing a sequence

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 = X$$

that is eventually stationary at some regular X^*. Then one can find a composition of a finite number of blow-ups $X' \to X^*$ so that the reduced special fibre of X' is a normal crossings divisor. For details on this point, see [Liu], Section 9.2.4 (note that in *loc. cit.* the definition of a normal crossings divisor is different from ours, since it allows the divisor to be nonreduced).

2.3. Intersection theory on regular arithmetic surfaces

The intersection theory on an arithmetic surface, provided it can be defined, is determined by the intersection numbers of 1-cycles or Weil divisors. The prime cycles fall into two types: *horizontal* divisors are finite flat over R, and *vertical* divisors are curves over the residue field k of R. Let $\mathrm{Div}(X)$ be the free abelian group generated by all prime divisors of X, and $\mathrm{Div}_k(X)$ be the subgroup generated by vertical divisors.

In classical intersection theory, as exposed for example in Fulton's book [Ful], the possibility to define an intersection product $E \cdot F$ for *arbitrary* cycles E, F in a variety V requires the assumption that V is smooth. It would be too strong an assumption to require our surfaces to be smooth over R, but as we saw in the previous subsection, we can work with regular models. As it turns out, for them one can define at least a bilinear map $\mathrm{Div}_k(X) \times \mathrm{Div}(X) \to \mathbb{Z}$.

More precisely, let X be a regular arithmetic surface over R, let $i : E \hookrightarrow X$ be a prime vertical divisor and $j : F \hookrightarrow X$ an arbitrary effective divisor. By regularity, Weil divisors are the same as Cartier divisors, so the ideal sheaf $\mathcal{I}$ of F is invertible. Since E is a curve over the residue field k there is a usual notion of *degree* for line bundles, and we may define an intersection number by the formula

$$E \cdot F := \deg_E(i^*\mathcal{I}^{-1}) \ .$$

It follows from this definition that if $E \neq F$, then $E \cdot F$ is at least equal to the number of points in the support of $E \cap F$, in particular it is nonnegative. It is easy to see also that if E and F intersect transversally at all points, then $E \cdot F$ is exactly the number of points in the support of $E \cap F$ (the assumption that k is algebraically closed allows not to care about the degrees of the residue fields extensions). The intersection product extends by bilinearity to a map $\mathrm{Div}_k(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ satisfying the following properties:

Proposition 2.3.1. *Let E, F be divisors on a regular arithmetic surface X with E vertical. Then one has:*

(1) *if F is a vertical divisor then $E \cdot F = F \cdot E$,*
(2) *if E is prime then $E \cdot F = \deg_E(\mathcal{O}(F) \otimes \mathcal{O}_E)$,*
(3) *if F is principal then $E \cdot F = 0$.*

Proof. Cf. [Lic], Part I, Section 1. □

Here are the most important consequences concerning intersection with vertical divisors.

Theorem 2.3.2. *Let X be a regular arithmetic surface and let $E_1, \dots, E_r$ be the irreducible components of X_k. Then:*

(1) *$X_k \cdot F = 0$ for all vertical divisors F,*
(2) *$E_i \cdot E_j \geq 0$ if $i \neq j$ and $E_i^2 < 0$,*
(3) *the bilinear form given by the intersection product on $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is negative semi-definite, with isotropic cone equal to the line generated by X_k.*

Proof. (1) The special fibre X_k is the pullback of the closed point of $\mathrm{Spec}(R)$, a principal Cartier divisor, so it is a principal Cartier divisor in X. Hence $X_k \cdot F = 0$ for all vertical divisors F, by 2.3.1(3).

(2) If $i \neq j$, we have $E_i \cdot E_j \geq \#|E_i \cap E_j| \geq 0$. From this together with point (1) and the fact that the special fibre is connected, we deduce that

$$E_i^2 = (E_i - X_k) \cdot E_i = -\sum_{j \neq i} E_j \cdot E_i < 0 \ .$$

(3) Let d_i be the multiplicity of E_i, $a_{ij} = E_i \cdot E_j$, $b_{ij} = d_i d_j a_{ij}$. Let $v = \sum v_i E_i$ be a vector in $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $w_i = v_i/d_i$. We have $\sum_i b_{ij} = X_k \cdot (d_j F_j) = 0$ by point (1), and $\sum_j b_{ij} = 0$ by symmetry, so

$$v \cdot v = \sum_{i,j} a_{ij} v_i v_j = \sum_{i,j} b_{ij} w_i w_j = -\frac{1}{2} \sum_{i \neq j} b_{ij} (w_i - w_j)^2 \leq 0 \ .$$

Hence the intersection product on $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is negative semi-definite. Finally if $v \cdot v = 0$, then $b_{ij} \neq 0$ implies $w_i = w_j$. Since X_k is connected, we obtain that all the w_i are equal and hence $v = w_1 X_s$. Thus the isotropic cone is included in the line generated by X_k, and the opposite inclusion has already been proved. $\square$

Example 2.3.3. Let X be a regular arithmetic surface whose special fibre is reduced, with nodal singularities. Let $E_1, \ldots, E_r$ be the irreducible components of X_k. Then $E_i \cdot E_j$ is the number of intersection points of E_i and E_j if $i \neq j$, and $(E_i)^2$ is the opposite of the number of points where E_i meets another component, by point (1) of the theorem. Hence X_k is stable (resp. semi-stable) if and only it does not contain a projective line with self-intersection -2 (resp. with self-intersection -1).

As far as horizontal divisors are concerned, the most interesting one to intersect with is the *canonical divisor* associated to the canonical sheaf, whose definition we recall below. If E is an effective vertical divisor in X, the adjunction formula gives a relation between the canonical sheaves of X/R and that of E/k. The main reason why the canonical divisor is interesting is that on a regular arithmetic surface, the canonical sheaf is a dualizing sheaf in the sense of the Grothendieck-Serre duality theory, therefore the adjunction formula translates, via the Riemann-Roch theorem, into an expression of the intersection of E with the canonical divisor of X in terms of the Euler-Poincaré characteristic χ of E. We will now explain this.

Let us first recall briefly the definition of the canonical sheaf of a regular arithmetic surface X, assuming that X is projective (it can be shown that this is always the case, see [Lic]). We choose a projective embedding $i : X \hookrightarrow P := \mathbb{P}^n_R$ and note that since X and P are regular, then i is a regular immersion. It follows that the conormal sheaf $\mathcal{C}_{X/P} = i^*(\mathcal{I}/\mathcal{I}^2)$ is locally free over X, where $\mathcal{I}$ denotes the ideal sheaf of X in P. Also since P is smooth over R, the sheaf of differential 1-forms $\Omega^1_{P/R}$ is locally free over R. Thus the maximal exterior powers of the sheaves $\mathcal{C}_{X/P}$ and $i^*\Omega^1_{P/R}$, also called their *determinant*, are invertible sheaves on X. The canonical sheaf is defined to be the invertible sheaf

$$\omega_{X/R} := \det(\mathcal{C}_{X/P})^\vee \otimes \det(i^*\Omega^1_{P/R})$$

where $(\cdot)^\vee = \mathcal{H}om(\cdot, \mathcal{O}_X)$ is the linear dual. It can be proved that $\omega_{X/R}$ is independent of the choice of a projective embedding for X, and that it is a dualizing sheaf. Any divisor K on X such that $\mathcal{O}_X(K) \simeq \omega_{X/R}$ is called a *canonical divisor.*

Theorem 2.3.4. *Let X be a regular arithmetic surface over R, E a vertical positive Cartier divisor with $0 < E \leq X_k$, and $K_{X/R}$ a canonical divisor. Then we have the adjunction formula*

$$-2\chi(E) = E \cdot (E + K_{X/R}) \ .$$

Proof. In fact, the definition of $\omega_{X/R}$ is valid as such for an arbitrary local complete intersection (lci) morphism. Moreover, for a composition of two lci morphisms $f : X \to Y$ and $g : Y \to Z$ we have the general adjunction formula $\omega_{X/Z} \simeq \omega_{X/Y} \otimes_{\mathcal{O}_X} f^*\omega_{Y/Z}$, see [Liu], Section 6.4.2. In particular we have $\omega_{E/R} \simeq \omega_{E/k} \otimes$

$f^*\omega_{k/R} \simeq \omega_{E/k}$ where $f : E \to \mathrm{Spec}(k)$ is the structure morphism. A useful particular case of computation of the canonical sheaf is $\omega_{D/X} = \mathcal{O}_X(D)|_D$ for an effective Cartier divisor D in a locally noetherian scheme X (this is left as an exercise). Using this particular case and the general adjunction formula for the composition $E \to X \to \mathrm{Spec}(R)$, we have

$$\omega_{E/k} \simeq \omega_{E/R} \simeq \omega_{E/X} \otimes \omega_{X/R}|_E \simeq (\mathcal{O}_X(E) \otimes \omega_{X/R})|_E \ .$$

By the Riemann-Roch theorem, we have $\deg(\omega_{E/k}) = -2\chi(E)$ and the asserted formula follows, by taking degrees. □

2.4. Blow-up, blow-down, contraction

We assume that the reader has some familiarity with blow-ups, and we recall only the features that will be useful to us. Let X be a noetherian scheme and $i : Z \hookrightarrow X$ a closed subscheme with sheaf of ideals $\mathcal{I}$. The *blow-up of X along Z* is the morphism $\pi : \widetilde{X} \to X$ with $\widetilde{X} = \mathrm{Proj}(\oplus_{d\ge 0}\mathcal{I}^d)$. The *exceptional divisor* is $E := V(\mathcal{I}\mathcal{O}_{\widetilde{X}})$; it is a Cartier divisor. If i is a regular immersion, then the conormal sheaf $\mathcal{C}_{Z/X} = i^*(\mathcal{I}/\mathcal{I}^2)$ is locally free and $E \simeq \mathbb{P}(i^*(\mathcal{I}/\mathcal{I}^2))$ as a projective fibre bundle over Z; it carries a sheaf $\mathcal{O}_E(1)$. In this case, one can see that the sheaf $\mathcal{O}_{\widetilde{X}}(E)|_E$ is naturally isomorphic to $\mathcal{O}_E(-1)$, because $\mathcal{O}_{\widetilde{X}}(E) \simeq (\mathcal{I}\mathcal{O}_{\widetilde{X}})^{-1}$.

Example 2.4.1. Let X be a regular arithmetic surface and $Z = \{x\}$ a regular closed point of the special fibre. Then $\widetilde{X}$ is again a regular arithmetic surface and the exceptional divisor is a projective line over k, with self-intersection -1.

Example 2.4.2. Let x be a nodal singularity in the special fibre of a normal arithmetic surface. The completed local ring is isomorphic to $\mathcal{O} = R[[a,b]]/(ab-\pi^n)$ for some $n \ge 1$. We call the integer n the *thickness* of the node. We blow up $\{x\}$ inside $X = \mathrm{Spec}(\mathcal{O})$. If $n = 1$, the point x is regular so we are in the situation of the preceding example. If $n \ge 2$, the point x is a singular normal point and it is an exercise to compute that the blow-up of X at this point is

$$\widetilde{X} = \mathrm{Proj}(\mathcal{O}[[u,v,w]]/(uv-\pi^{n-2}w^2, av-bu, bw-\pi v, aw-\pi u)) \ .$$

If $n = 2$, the exceptional divisor is a smooth conic over k with self-intersection -2. If $n \ge 3$, the exceptional divisor is composed of two projective lines intersecting in a nodal singularity of thickness $n-2$, each meeting the rest of the special fibre in one point.

Remark 2.4.3. We saw that among the nodal singularities $ab - \pi^n$, the regular one for $n = 1$ shows a different behaviour. Here is one more illustration of this fact. Let X be a regular arithmetic surface and assume that X_K has a rational point $\mathrm{Spec}(K) \to X$. By the valuative criterion of properness, this point extends to a section $\mathrm{Spec}(R) \to X$, and we denote by $x : \mathrm{Spec}(k) \to X$ the reduction. Let $\mathcal{O} = \mathcal{O}_{X,x}$, $i : R \to \mathcal{O}$ the structure morphism, m the maximal ideal of R, n the maximal ideal of $\mathcal{O}$. Thus we have a map $s : \mathcal{O} \to R$ such that $s \circ i = \mathrm{id}$, and one checks that this forces to have an injection of cotangent k-vector spaces $m/m^2 \subset n/n^2$. Therefore we can choose a basis of n/n^2 containing the image of

π, in other words we can choose a system of parameters for $\mathcal{O}$ containing π. This proves that $\mathcal{O}/\pi = \mathcal{O}_{X_k,x}$ is regular. To sum up, the reduction of a K-rational point on a regular surface X is a regular point of X_k. Of course, this is false as soon as $n \geq 2$, since the point with coordinates $a = \pi$, $b = \pi^{n-1}$ reduces to the node.

The process of blowing-up is a prominent tool in the birational study of regular surfaces. For obvious reasons, it is also very desirable to reverse this operation and examine the possibility to *blow down*, that is to say to characterize those divisors $E \subset X$ in regular surfaces that are exceptional divisors of some blow-up of a *regular* scheme. Note that if $f : X \to Y$ is the blow-up of a point y, then π is also the blow-down of $E := f^{-1}(y)$ and the terminology is just a way to put emphasis on (Y, y) or on (X, E).

As a first step, it is a general fact that one can *contract* the component E, and the actual difficult question is the nature of the singularity that one gets. We choose to present contractions in their natural setting, and then we will state without proof the classical results of Castelnuovo, Artin and Lipman on the control of the singularities.

Definition 2.4.4. Let X be a normal arithmetic surface. Let $\mathcal{E}$ be a set of irreducible components of the special fibre X_k. A *contraction* is a morphism $f : X \to Y$ such that Y is a normal arithmetic surface, $f(E)$ is a point for all $E \in \mathcal{E}$, and f induces an isomorphism

$$X \setminus \bigcup_{E\in\mathcal{E}} E \longrightarrow Y \setminus \bigcup_{E\in\mathcal{E}} f(E) \ .$$

Using the Stein factorization, it is relatively easy to see that f is unique if it exists, and that its fibres are connected. Under our assumption that R is complete with algebraically closed residue field, one can always construct an effective relative (i.e., R-flat) Cartier divisor D of X meeting exactly the components of X_k not belonging to $\mathcal{E}$. Indeed, for example if X_k is reduced, one can choose one smooth point in each component not in $\mathcal{E}$. Since R is henselian these points lift to sections of X over R, and we can take D to be the sum of these sections. If X_k is not reduced, a similar argument using Cohen-Macaulay points instead of smooth points does the job, cf. [BLR], Proposition 6.7/4. Thus, existence of contractions follows from the following result:

Theorem 2.4.5. *Let X be a normal arithmetic surface. Let $\mathcal{E}$ be a strict subset of the set of irreducible components of the special fibre X_k, and D an effective relative Cartier divisor of X over R meeting exactly the components of X_k not belonging to $\mathcal{E}$. Then the morphism*

$$f : X \to Y := \mathrm{Proj}\big(\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))\big)$$

is a contraction of the components of $\mathcal{E}$.

Proof. We first explain what is f. Let us write $H^0(X, \mathcal{O}_X(nD))^\sim$ for the associated constant sheaf on X. Note that $\mathrm{Proj}(\oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))^\sim) \simeq Y \times_R X$, and $\mathrm{Proj}(\oplus_{n\geq 0}\mathcal{O}_X(nD))\simeq X$ canonically (see [Ha], Chap. II, Lemma 7.9). The restriction of sections gives a natural map of graded $\mathcal{O}_X$-algebras

$$\bigoplus_{n\geq 0} H^0(X,\mathcal{O}_X(nD))^{\sim}\to\bigoplus_{n\geq 0}\mathcal{O}_X(nD)\ .$$

We obtain f by taking Proj and composing with the projection $Y\times_R X\to Y$.

Since D_K has positive degree on X_K, it is ample and it follows that the restriction of f to the generic fibre is an isomorphism. Also, after some more work this implies that $\mathcal{O}_X(nD)$ is generated by its global sections if n is large enough; we will admit this point, and refer to [BLR], p. 168 for the details. Therefore the ring $A=\oplus_{n\geq 0}H^0(X,\mathcal{O}_X(nD))$ is of finite type over R by [EGA2], 3.3.1, and so Y is a projective R-scheme. Moreover X is covered by the open sets X_ℓ where ℓ does not vanish, for all global sections $\ell\in H^0(X,\mathcal{O}_X(nD))$, and f induces an isomorphism

$$A_{(\ell)}\xrightarrow{\sim}H^0(X_\ell,\mathcal{O}_X)\ .$$

If follows that $A_{(\ell)}$, and hence Y, is normal and flat over R. Moreover we see that $f_*\mathcal{O}_X\simeq\mathcal{O}_Y$, so by Zariski's connectedness principle (cf. [Liu], 5.3.15) it follows that the fibres of f are connected.

It remains to prove that f is a contraction of the components of $\mathcal{E}$. If $E\in\mathcal{E}$, then $\mathcal{O}_X(nD)|_E\simeq\mathcal{O}_E$ and hence any global section of $\mathcal{O}_X(nD)$ induces a constant function on E, since E is proper. It follows that the image $f(E)$ is a point. If $E\notin\mathcal{E}$, we may choose a point $x\in E\cap\mathrm{Supp}(D)$. Let ℓ be a global section that generates $\mathcal{O}_X(nD)$ on a neighbourhood U of x, for some n large enough. Then $1/\ell$ is a function on X_ℓ that, by definition, vanishes on $U\cap\mathrm{Supp}(D)$ (with order n) and is non-zero on $U-\mathrm{Supp}(D)$. Thus $f|_E$ is not constant, so it is quasi-finite. Since its fibres are connected, in fact $f|_E$ is birational, and since Y is normal we deduce that $f|_E$ is an isomorphism onto its image, by Zariski's main theorem (cf. [Liu], 4.4.6). $\square$

The numerical information that we have collected about exceptional divisors in Subsection 2.3 is crucial to control the singularity at the image points of the components that are contracted, as in the following two results which we will use without proof. The first is Castelnuovo's criterion about blow-downs.

Theorem 2.4.6. *Let X be a regular arithmetic surface and E a vertical prime divisor. Then there exists a blow-down of E if and only if $E\simeq\mathbb{P}^1_k$ and $E^2=-1$.*

Proof. See [Lic], Theorem 3.9, or [Liu], Theorem 9.3.8. $\square$

The second result which we want to mention is an improvement by Lipman [Lip1] of previous results of Artin [Ar] on contractions for algebraic surfaces. The statement uses the following fact, which we quote without proof (see [Liu], Lemma 9.4.12): for a regular arithmetic surface X and distinct vertical prime divisors $E_1,\dots,E_r$ such that the intersection matrix $(E_i\cdot E_j)$ is negative semi-definite, there exists a smallest effective divisor $C=\sum a_iE_i$ such that $C\geq\sum_i E_i$ and $C\cdot E_i\geq 0$ for all i. We call C the *fundamental divisor* for $\{E_i\}_i$.

Theorem 2.4.7. *Let X be a regular arithmetic surface and let $E_1, \ldots, E_r$ be distinct reduced vertical prime divisors with negative semi-definite intersection matrix. Assume that the Euler-Poincaré characteristic of the fundamental divisor C associated to the E_i is positive. Then the contraction of $E_1, \ldots, E_r$ is a normal arithmetic surface, and the resulting singularity is a regular point if and only if $-C^2 = H^0(C, \mathcal{O}_C)$.*

Proof. See [Lip1], Theorem 27.1, or [Liu], Theorem 9.4.15. Note that in the terminology of [Lip1], a rational double point, (i.e., a rational singularity with multiplicity 2) is none other than a node of the special fibre. □

2.5. Minimal regular models

We can now state the main results of the birational theory of arithmetic surfaces:

Theorem 2.5.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 1$. Then C has a minimal regular model over R, unique up to a unique isomorphism.*

Proof. By Theorem 2.2.1, there exists a regular model for C. By successive blow-downs of exceptional divisors, we construct a regular model X that is relatively minimal. Let X' be another such model. Since any two regular models are dominated by a third ([Lic], Proposition 4.2) and any morphism between two models factors into a sequence of blow-ups ([Lic], Theorem 1.15), there exist sequences of blow-ups

$$Y = X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X$$

and

$$Y = X'_n \to X'_{n-1} \to \cdots \to X'_1 \to X'_0 = X'$$

terminating at the same Y. We may choose Y such that $m+n$ is minimal. If $m > 0$, there is an exceptional curve E for the morphism $Y \to X_{m-1}$. Since X' has no exceptional curve, the image of E in X' is not an exceptional curve, hence there is an r such that the image of E in X'_r is the exceptional divisor of $X'_r \to X'_{r-1}$. Also, for all $i \in \{r, \ldots, n-1\}$ the image of E in the surface X'_i does not contain the center of the blow-up $X'_{i+1} \to X'_i$. Thus, we can rearrange the blow-ups so that E is the exceptional curve of $Y \to X'_{n-1}$. Therefore $X_{m-1} \simeq X'_{n-1}$ and this contradicts the minimality of $m+n$. It follows that $m = 0$, so there is a morphism $X \to X'$, and since X is relatively minimal we obtain $X \simeq X'$. □

Theorem 2.5.2. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 1$. Then C has a minimal regular model with normal crossings over R. It is unique up to a unique isomorphism.*

Proof. In fact Theorem 2.2.1 asserts the existence of a regular model with normal crossings. Proceeding along the same lines as in the proof of the above theorem, one produces a minimal regular model with normal crossings. □

3. Stable reduction

In this section, C is a smooth geometrically connected curve over K, of genus $g \geq 2$.

3.1. Stable reduction is equivalent to semistable reduction

Proposition 3.1.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Then the following conditions are equivalent:*

(1) *C has stable reduction,*
(2) *C has semistable reduction,*
(3) *the minimal regular model of C is semistable.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): let X be a semistable model of C over R. Replacing X by the repeated blow-down of all exceptional divisors in the regular locus of X, we may assume that it has no exceptional divisor. Then, by the deformation theory of the node (cf. [Liu], 10.3.22), the completed local ring of a singular point $x \in X_k$ is $\widehat{\mathcal{O}}_{X,x} \simeq R[[a,b]]/(ab - \pi^n)$ for some $n \geq 2$. By Example 2.4.2, blowing-up $[n/2]$ times the singularity leads to a regular scheme X' whose special fibre has $n-1$ new projective lines of self-intersection -2. This is the minimal regular model of C, which is therefore semistable.

(3) $\Rightarrow$ (1): let X be the minimal regular model of C. Consider the family of all components of the special fibre that are projective lines of self-intersection -2. A connected configuration of such lines is either topologically a circle, or a segment. Since $g \geq 2$, the first possibility can not occur. It follows that such a configuration has positive Euler-Poincaré characteristic, so by Theorem 2.4.7, the contraction of these lines is a normal surface with nodal singularities. □

3.2. Proof of semistable reduction in characteristic 0

Theorem 3.2.1. *Assume that the residue field k has characteristic 0. Let X be the minimal regular model with normal crossings of C and let $n_1, \dots, n_r$ be the multiplicities of the irreducible components of X_k. Let n be a common multiple of $n_1, \dots, n_r$ and $R' = R[\rho]/(\rho^n - \pi)$. Then the normalization of $X \times_R R'$ is semistable.*

The key fact is that in residue characteristic 0, divisors with normal crossings have a particularly simple local shape. This is due to the possibility to extract nth roots.

Proof. Let $x \in X$ be a closed point of X_k and let A be the completion of its local ring in X. We will use two facts about A: firstly, since k is algebraically closed of characteristic 0 and A is complete, it follows from Hensel's lemma that one can extract nth roots in A for all integers $n \geq 1$. Note that by the same argument R contains all roots of unity. Secondly, since A is a regular noetherian local ring, it is a unique factorization domain, and each regular system of parameters (f,g) is composed of prime elements.

Since $(X_k)_{\mathrm{red}}$ is a normal crossings divisor, we have two possibilities. The first possibility is that $\sqrt{\pi}A = (f)$ for some regular system of parameters (f, g). In this case f is the only prime factor of π, so $\pi = uf^a$ for some unit $u \in A$. Since k is algebraically closed of characteristic 0 and A is complete, one sees that u is an ath power in A so that changing f if necessary we have $\pi = f^a$. Then one checks that the natural map $R[[u, v]]/(u^a - \pi) \to A$ taking u to f and v to g is an isomorphism. Here a is the multiplicity of the component of X_k containing x, so by assumption $n = am$ for some integer m. Then

$$A \otimes_R R' \simeq R'[[u, v]]/(u^a - \rho^{am}) \simeq R'[[u, v]]/(\Pi(u - \zeta\rho^m))$$

with the product ranging over the ath roots of unity ζ. The normalization of this ring is the product of the normal rings $R'[[u, v]]/(u - \zeta\rho^m) \simeq R'[[v]]$ so the normalization of $X \times_R R'$ is smooth at all points lying over x.

The second possibility is that $\sqrt{\pi}A = (fg)$ for some regular system of parameters (f, g). In this case f and g are the only prime factors of π, so $\pi = uf^ag^b$ for some unit $u \in A$ which as above may be chosen to be 1. Thus $\pi = f^ag^b$ and one checks that the natural map $R[[u, v]]/(u^av^b - \pi) \to A$ taking u to f and v to g is an isomorphism. Again a and b are the multiplicities of the two components at x. Let $d = \gcd(a, b)$, $a = d\alpha$, $b = d\beta$, $n = d\alpha\beta m$. Then as above the normalization of $A \otimes_R R'$ is the product of the normalizations of the rings $R'[[u, v]]/(u^\alpha v^\beta - \zeta\rho^{\alpha\beta m})$ for all dth roots of unity ζ. If we introduce $\xi \in R$ such that $\xi^{\alpha\beta} = \zeta$ then the normalization is the morphism

$$A = R'[[u, v]]/(u^\alpha v^\beta - \zeta\rho^{\alpha\beta m}) \to B = R'[[x, y]]/(xy - \xi\rho^m)$$

given by $u \mapsto x^\beta$ and $v \mapsto y^\alpha$. Indeed, the ring B is normal and one may realize it in the fraction field of A by choosing i, j such that $i\alpha + j\beta = 1$ and setting

$$x = u^j(\xi^\alpha\rho^{\alpha m}/v)^i \quad \text{and} \quad y = v^i(\xi^\beta\rho^{\beta m}/u)^j \ . \qquad \square$$

3.3. Generalized jacobians

Let X be an arbitrary connected projective curve over an algebraically closed field k. It can be shown that the identity component $\underline{\mathrm{Pic}}^0(X)$ of the Picard functor is representable by a smooth connected algebraic group called the *generalized jacobian* of X and denoted $\mathrm{Pic}^0(X)$. In this subsection, which serves as a preparation for the next subsection, we will give a description of $\mathrm{Pic}^0(X)$. The first feature of $\mathrm{Pic}^0(X)$ which is readily accessible is its tangent space at the identity:

Lemma 3.3.1. *The tangent space of* $\mathrm{Pic}^0(X)$ *at the identity is canonically isomorphic to* $H^1(X, \mathcal{O}_X)$.

Proof. Let $k[\epsilon]$, with $\epsilon^2 = 0$, be the ring of dual numbers and let $X[\epsilon] := X \times_k k[\epsilon]$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{x \mapsto 1+\epsilon x} \mathcal{O}_{X[\epsilon]}^\times \longrightarrow \mathcal{O}_X^\times \longrightarrow 0 \ .$$

In the associated long exact sequence, the map $H^0(\mathcal{O}_{X[\epsilon]}^\times) \to H^0(\mathcal{O}_X^\times)$ is surjective since the second group contains nothing else but the invertible constant functions.

It follows that the kernel of the morphism $H^1(\mathcal{O}_{X[\epsilon]}^\times) \to H^1(\mathcal{O}_X^\times)$ is isomorphic to $H^1(X, \mathcal{O}_X)$. Since $H^1(\mathcal{O}_X^\times) = \mathrm{Pic}(X)$ and $H^1(\mathcal{O}_{X[\epsilon]}^\times) = \mathrm{Pic}(X[\epsilon])$, the kernel is by definition the tangent space at the identity. □

In order to go further into the structure of $\mathrm{Pic}^0(X)$, we introduce an intermediary curve X' sandwiched between the reduced curve X_{red} and its normalization $\widetilde{X}$. This curve is obtained topologically as follows. Look at all points $x \in X_{\mathrm{red}}$ with $r \geq 2$ preimages $\widetilde{x}_1, \dots, \widetilde{x}_r$ in $\widetilde{X}$, and glue these preimages transversally. The curve X' may be better described by its structure sheaf as a subsheaf of $\mathcal{O}_{\widetilde{X}}$: its functions are the functions on $\widetilde{X}$ taking the same value on $\widetilde{x}_1, \dots, \widetilde{x}_r$ for all points x as above. Thus X' has only ordinary singularities, that is to say singularities that locally look like the union of the coordinate axes in some affine space $\mathbb{A}^r$. Note that the integer r, called the *multiplicity*, may be recovered as the dimension of the tangent space at the ordinary singularity. The curve X' is called the *curve with ordinary singularities associated to* X. It is also the largest curve between X_{red} and $\widetilde{X}$ which is universally homeomorphic to X_{red}. To sum up we have the picture:

$$\widetilde{X} \to X' \to X_{\mathrm{red}} \to X \ .$$

By pullback, we have morphisms $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X') \to \mathrm{Pic}^0(\widetilde{X})$.

Lemma 3.3.2. *The morphism* $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ *is surjective with unipotent kernel of dimension* $\dim H^1(X, \mathcal{O}_X) - \dim H^1(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$.

Proof. Let $\mathcal{I}$ be the ideal sheaf of X_{red} in X, i.e., the sheaf of nilpotent functions on X. Let $X_n \subset X$ be the closed subscheme defined by the sheaf of ideals $\mathcal{I}^{n+1}$. We use the filtration $\mathcal{I} \supset \mathcal{I}^2 \supset \cdots$. For each $n \geq 1$ we have an exact sequence

$$0 \to \mathcal{I}^n \to (\mathcal{O}_X/\mathcal{I}^{n+1})^\times \to (\mathcal{O}_X/\mathcal{I}^n)^\times \to 0$$

where the map $\mathcal{I}^n \to (\mathcal{O}_X/\mathcal{I}^{n+1})^\times$ takes x to $1 + x$. Since X is complete and connected the map $H^0(X, (\mathcal{O}_X/\mathcal{I}^{n+1})^\times) \to H^0(X, (\mathcal{O}_X/\mathcal{I}^n)^\times) = k^\times$ is surjective. Consequently the long exact sequence of cohomology gives a short exact sequence

$$0 \to H^1(X, \mathcal{I}^n) \to H^1(X_n, \mathcal{O}_{X_n}^\times) \to H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}^\times) \to 0 \ .$$

Since the base is a field, all schemes are flat and hence this description is valid after any base change $S \to \mathrm{Spec}(k)$. So there is an induced exact sequence of algebraic groups

$$0 \to V_n \to \mathrm{Pic}^0(X_n) \to \mathrm{Pic}^0(X_{n-1}) \to 0$$

where V_n is the algebraic group which is the vector bundle over $\mathrm{Spec}(k)$ determined by the vector space $H^1(X, \mathcal{I}^n)$. Thus V_n is unipotent; note that the fact that V_n factors through the identity component of the Picard functor comes from the fact that it is connected. Finally $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ is surjective and the kernel is a successive extension of unipotent groups, so it is a unipotent group. The dimension count for the dimension of the kernel is immediate by inspection of the exact sequences. □

Remark 3.3.3. It is not true that $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ is an isomorphism if and only if $X_{\mathrm{red}} \hookrightarrow X$ is. For example if X is generically reduced, i.e., the sheaf of nilpotent functions has finite support, then $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(X_{\mathrm{red}})$.

Recall that the *arithmetic genus* of a projective curve over a field k is defined by the equality $p_a(X) = 1 - \chi(\mathcal{O}_X)$ where χ is the Euler-Poincaré characteristic.

Lemma 3.3.4. *The morphism* $\mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X')$ *is surjective with unipotent kernel of dimension* $p_a(X_{\mathrm{red}}) - p_a(X')$*. Moreover,* $p_a(X_{\mathrm{red}}) = p_a(X')$ *if and only if* $X' \to X_{\mathrm{red}}$ *is an isomorphism.*

Proof. Recall that the morphism $h : X' \to X_{\mathrm{red}}$ is a homeomorphism. We have an exact sequence

$$0 \to (\mathcal{O}_{X_{\mathrm{red}}})^\times \to (h_*\mathcal{O}_{X'})^\times \to \mathcal{F} \to 0$$

where the cokernel $\mathcal{F}$ has finite support, hence no higher cohomology. Since h is bijective and the curves X_{red}, X' are complete and connected we have

$$H^0(X_{\mathrm{red}}, (\mathcal{O}_{X_{\mathrm{red}}})^\times) = H^0(X', (\mathcal{O}_{X'})^\times) = k^\times$$

so the long exact sequence of cohomology gives

$$0 \to H^0(X_{\mathrm{red}}, \mathcal{F}) \to H^1(X_{\mathrm{red}}, (\mathcal{O}_{X_{\mathrm{red}}})^\times) \to H^1(X', (\mathcal{O}_{X'})^\times) \to 0 \ .$$

Moreover $H^0(X_{\mathrm{red}}, \mathcal{F}) = \oplus \mathcal{O}_{X',x'}/\mathcal{O}_{X,x}$ where the direct sum runs over the non-ordinary singular points x of X_{red}, and x' is the unique point above x. Denoting by m_x the maximal ideal of the local ring of x, it is immediate to see that the inclusion $1 + m_{x'} \to \mathcal{O}_{X',x'}$ induces an isomorphism $\mathcal{O}_{X',x'}/\mathcal{O}_{X_{\mathrm{red}},x} \simeq (1 + m_{x'})/(1 + m_x)$. Using the fact that $\mathcal{O}_{X',x'}/m_x$ is an artinian ring, one may see that there is an integer $r \geq 1$ such that $(m_{x'})^r \subset m_r$. Then one introduces a filtration of $(1 + m_{x'})/(1 + m_x)$ and proves as in the proof of Lemma 3.3.2 that the algebraic group U that represents $H^0(X_{\mathrm{red}}, \mathcal{F})$ is unipotent. We refer to [Liu], Lemmas 7.5.11 and 7.5.12 for the details of these assertions. Finally the exact sequence above induces an exact sequence of algebraic groups

$$0 \to U \to \mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X') \to 0$$

with U unipotent. The proof of the final statement about the dimension of the kernel can be found in [Liu], Lemma 7.5.18. □

Lemma 3.3.5. *The morphism* $\mathrm{Pic}^0(X') \to \mathrm{Pic}^0(\widetilde{X})$ *is surjective with toric kernel of dimension* $\mu - c + 1$*, where* μ *is the sum of the excess multiplicities* $m_x - 1$ *for all ordinary multiple points* $x \in X'$ *and* c *is the number of connected components of* $\widetilde{X}$*.*

Proof. Write $\pi : \widetilde{X} \to X'$ for the normalization map. We have an exact sequence

$$0 \to (\mathcal{O}_{X'})^\times \to (\pi_*\mathcal{O}_{\widetilde{X}})^\times \to \mathcal{F} \to 0$$

where the cokernel $\mathcal{F}$ has finite support, hence no higher cohomology. Let c be the number of connected components of $\widetilde{X}$. The long exact sequence of cohomology gives

$$0 \to k^\times \to (k^\times)^c \to H^0(X, \mathcal{F}) \to H^1(X', (\mathcal{O}_{X'})^\times) \to H^1(X', (\pi_*\mathcal{O}_{\widetilde{X}})^\times) \to 0 \ .$$

One has the following supplementary information: the map $k^\times \to (k^\times)^c$ is the diagonal inclusion, the sheaf $\mathcal{F}$ is supported at all ordinary multiple points and $H^0(X, \mathcal{F})$ is the sum $\oplus_{x \in X'} (k^\times)^{m_x - 1}$ over all these points, and

$$H^1(X', (\pi_*\mathcal{O}_{\widetilde{X}})^\times) = H^1(\widetilde{X}, (\mathcal{O}_{\widetilde{X}})^\times)$$

since π is affine. As above, these statements are valid after any base change $S \to \operatorname{Spec}(k)$, so we obtain an induced exact sequence of algebraic groups

$$0 \to \mathbb{G}_m \to (\mathbb{G}_m)^c \to \Pi\, (\mathbb{G}_m)^{m_x - 1} \to \operatorname{Pic}^0(X') \to \operatorname{Pic}^0(\widetilde{X}) \to 0$$

and this proves the lemma. □

3.4. Relation with semistable reduction of abelian varieties

Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Let X be the minimal regular model of C. Its special fibre X_k may be singular, possibly nonreduced and we have seen the structure of its generalized jacobian in the previous subsection. This algebraic group turns out to be tightly linked to the reduction type of C. In fact, quite generally, classical results of Chevalley imply that any smooth connected commutative algebraic group over an algebraically closed field is an extension of an abelian variety by a product of a torus and a connected smooth unipotent group. In this section, following Deligne and Mumford, we will prove the following theorem:

Theorem 3.4.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$, with a K-rational point. Let X be the minimal regular model of C. Then C has stable reduction over R if and only if* $\operatorname{Pic}^0(X_k)$ *has no unipotent subgroup.*

Proof. Assume that C has stable reduction. Then X_k is reduced and has only nodal singularities, by Proposition 3.1.1, so it is equal to its associated curve with ordinary singularities. Since the normalization of X_k is a smooth curve, its generalized jacobian is an abelian variety. Hence it follows from Lemma 3.3.5 that $\operatorname{Pic}^0(X_k)$ is an extension of an abelian variety by a torus, so it has no unipotent subgroup.

Conversely, assume that $\operatorname{Pic}^0(X_k)$ has no unipotent subgroup.

By Lemma 3.3.2 the morphism $\operatorname{Pic}^0(X_k) \to \operatorname{Pic}^0((X_k)_{\mathrm{red}})$ is an isomorphism. Thus by Lemma 3.3.1 we have $H^1(X_k, \mathcal{O}_{X_k}) = H^1((X_k)_{\mathrm{red}}, \mathcal{O}_{(X_k)_{\mathrm{red}}})$. But since X_k has at least one reduced component (the given K-rational point of C reduces by 2.4.3 to a regular point of X_k), we have also $H^0(X_k, \mathcal{O}_{X_k}) = H^0((X_k)_{\mathrm{red}}, \mathcal{O}_{(X_k)_{\mathrm{red}}}) = k$. In other words X_k and its reduced subscheme have equal Euler-Poincaré characteristics. Let $E_1, \dots, E_r$ be the irreducible components of X_k and $d_1, \dots, d_r$ their multiplicities. By the adjunction formula of Theorem 2.3.4 we get

$$\Sigma\, d_i E_i \cdot (\Sigma\, d_i E_i + K) = \Sigma\, E_i \cdot (\Sigma\, E_i + K)$$

where K is a canonical divisor of X/R. Since $\sum d_i E_i = X_k$ is in the radical of the intersection form, we obtain

$$\Sigma\, (d_i - 1) E_i \cdot K = \Sigma\, E_i \cdot \Sigma\, E_i \ .$$

Now assume that $d_i > 1$ for some i. Then $\sum E_i \neq X_k$ and hence $\sum E_i \cdot \sum E_i < 0$, because the intersection form is negative semi-definite with isotropic cone generated by X_k. Therefore by the above equality, we must have $E_{i_0} \cdot K < 0$ for some i_0. Since also $E_{i_0} \cdot E_{i_0} < 0$, we have

$$-2 \geq E_{i_0} \cdot E_{i_0} + E_{i_0} \cdot K = E_{i_0} \cdot (E_{i_0} + K) = -2\chi(E_{i_0}) \geq -2 \ .$$

Finally $\chi(E_{i_0}) = -1$, so E_{i_0} is a projective line with self-intersection -1. This is impossible since X is the minimal regular model. It follows that $d_i = 1$ for all i, hence X_k is reduced. Again since $\mathrm{Pic}^0(X_k)$ has no unipotent subgroup, by Lemma 3.3.4 the curve X_k has ordinary multiple singularities. Since X_k lies on a regular surface, the dimension of the tangent space at all points is less than 2, hence the singular points are ordinary double points. This proves that C has stable reduction over R. □

We can now state the stable reduction theorem in full generality, and we will indicate how Deligne and Mumford deduce it from the above theorem (see [DM], Corollary 2.7).

Theorem 3.4.2. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Then there exists a finite field extension L/K such that the curve C_L has a stable model. Furthermore, this stable model is unique.*

The unicity statement means that if C_L and C_M have stable models for some finite field extensions L, M then these models become isomorphic in the ring of integers of N, for all fields N containing L and M. This fact follows directly from the proof of the implication (3) $\Rightarrow$ (1) of Proposition 3.1.1. Indeed, if C has stable reduction, the stable model is determined uniquely as the blow-down of all chains of projective lines with self-intersection -2 in the special fibre of the minimal regular model of C.

The proof of the existence part given in the article [DM] requires much more material from algebraic geometry, in particular it uses results on Néron models of abelian varieties. We give the sketch of the argument, for the readers acquainted with these notions. To prove the theorem, we may pass to a finite field extension and hence assume that C has a K-rational point. Moreover, a result of Grothendieck [SGA7] asserts that after a further finite field extension (again omitted from the notations), the Néron model $\mathcal{J}$ of the jacobian $J = \mathrm{Pic}^0(C/K)$ has a special fibre $\mathcal{J}_k$ without unipotent subgroup. Now, let X be the minimal regular model of C over the ring of integers R of K. By properness there is a section $\mathrm{Spec}(R) \to X$ that extends the rational point of C, and the corresponding k-point is regular (Remark 2.4.3). In particular, this section hits the special fibre in a component of multiplicity 1. Under these assumptions, by a theorem of Raynaud [Ra1], the Picard functor $\mathrm{Pic}^0(X/R)$ is isomorphic to $\mathcal{J}$ (in particular, it is representable). It follows that the special fibre of $\mathrm{Pic}^0(X/R)$, in other words $\mathrm{Pic}^0(X_k)$, has no unipotent subgroup. By Theorem 3.4.1, C has stable reduction.

4. Application to moduli of curves and covers

4.1. Valuative criterion for the stack of stable curves

Let $g \geq 2$ be a fixed integer and let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus g.

Once it is known that $\overline{\mathcal{M}}_g$ is separated (cf. the next subsection), the valuative criterion of properness for $\overline{\mathcal{M}}_g$ is the following statement: for all discrete valuation rings R with fraction field K, and all K-points $\mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$, there exists a finite field extension K'/K such that $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$ extends to a point $\mathrm{Spec}(R') \to \overline{\mathcal{M}}_g$ where R' is the integral closure of R in K'.

Once it is known that $\overline{\mathcal{M}}_g$ is of finite type, it is enough to verify the valuative criterion for complete valuation rings R with algebraically closed residue field.

Finally, by the well-known Lemma 4.1.1 below, it is enough to test the criterion for points $\mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$ that map into some open dense substack $U \subset \overline{\mathcal{M}}_g$. The deformation theory of stable curves proves that smooth curves are dense in $\overline{\mathcal{M}}_g$, hence we may take U to be the open substack of smooth curves. Then, the valuative criterion is just Theorem 3.4.2.

Lemma 4.1.1. *Let S be a noetherian scheme and let $\mathcal{X}$ be an algebraic stack of finite type and separated over S. Let $\mathcal{U}$ be a dense open substack. Then $\mathcal{X}$ is proper over S if and only if for all discrete valuation rings R with fraction field K and all S-morphisms $\mathrm{Spec}(K) \to \mathcal{U}$, there exists a finite extension K'/K and a morphism $\mathrm{Spec}(R') \to \mathcal{X}$, where R' is the integral closure of R in K', such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(R') \arrow[rrr] \arrow[urrr, dotted] & & & S.
\end{tikzcd}$$

Proof. For simplicity, we will prove the lemma in the case where $\mathcal{X}$ is a scheme X. The proof for an algebraic stack is exactly the same, but we want to avoid giving references to the literature on algebraic stacks for the necessary ingredients. It is enough to prove the *if* part. Since the notion of properness is local on the target, we may assume that S is affine. Then by [EGA2], 5.4.5, we may replace S by one of its reduced irreducible components Z and then X by one of the reduced irreducible components of the preimage of Z in X. Thus we may assume that X and S are integral. By Chow's lemma [EGA2], 5.6.1, there exists a scheme X' quasi-projective over S and a projective, surjective, birational morphism $X' \to X$. It is easy to see that $X \to S$ is proper if and only if $X' \to S$ is proper, thus we may replace X by X' and assume X quasiprojective. Let $j : X \to P$ be an open dense immersion into a projective S-scheme. Then $X \to S$ is proper if and only if j is surjective. Let x be a point in P. Since U is dense in X hence also in P, there exists a point $y \in U$ and a morphism $\mathrm{Spec}(R) \to P$ where R is a discrete valuation ring with fraction field K, mapping the open point to y and the closed point to x (see [EGA2], 7.1.9). By the valuative criterion which is the assumption of the lemma, the map $\mathrm{Spec}(K) \to X$ extends (maybe after a finite extension) to $\mathrm{Spec}(R) \to X$. Since X is separated, such an extension is unique and this means that $x \in X$. So j is surjective and the lemma is proved. □

4.2. Automorphisms of stable curves

As a preparation for the next subsection, we need some preliminaries concerning automorphisms of stable curves. Not just the automorphism *groups*, but also the automorphism *functors*, are interesting. Even more generally, if X, Y are stable curves over a scheme S, then by Grothendieck's theory of the Hilbert scheme and related functors, the functor of isomorphisms between X and Y is representable by a quasi-projective S-scheme denoted $\mathrm{Isom}_S(X, Y)$. It is really this scheme that we want to describe.

Lemma 4.2.1. *Let X be a stable curve over a field k. Then, the group of automorphisms of X/k is finite and the group of global vector fields* $\mathrm{Ext}^0(\Omega_{X/k}, \mathcal{O}_X)$ *is zero.*

Proof. Let S be the set of singular points of X and let $\pi : \widetilde{X} \to X$ be the normalization morphism. Let A be the group of automorphisms of X and let A_0 be the subgroup of those automorphisms φ such that for all $x \in S$, we have $\varphi(x) = x$ and φ preserves the branches at x. Since S is finite, A_0 has finite index in A and hence it is enough to prove that A_0 is finite. Then elements of A_0 are the same as automorphisms of $\widetilde{X}$ acting trivially on $\pi^{-1}(S)$. Let us call the points of $\pi^{-1}(S)$ *marked points.* Since X is connected, the components of $\widetilde{X}$ are either smooth curves of genus $g \geq 2$ with maybe some marked points, or elliptic curves with at least one marked point, or rational curves with at least three marked points. Each of these has finitely many automorphisms, hence A_0 is finite.

A global vector field D on X is the same as a global vector field $\widetilde{D}$ on $\widetilde{X}$ which vanishes at all marked points. We proceed again by inspection of the three different types of components of $\widetilde{X}$. It is known that smooth curves of genus $g \geq 2$ have no vector field, elliptic curves have no vector field vanishing in one point, and smooth rational curves ones have no vector field vanishing in three points. Hence $\widetilde{D} = 0$ and $D = 0$. □

Lemma 4.2.2. *Let X, Y be a stable curves over a scheme S. Then, the isomorphism scheme* $\mathrm{Isom}_S(X,Y)$ *is finite and unramified over S.*

Proof. The scheme $\mathrm{Isom}_S(X,Y)$ is of finite type as an open subscheme of a Hilbert scheme. It is also proper, since the valuative criterion is exactly the unicity statement in Theorem 3.4.2. Hence in order to prove the lemma we may assume that S is the spectrum of an algebraically closed field k. Then, either $\mathrm{Isom}_S(X,Y)$ is empty or it is isomorphic to $\mathrm{Aut}_k(X)$. Hence, it is finite by Lemma 4.2.1. Let $k[\epsilon]$ with $\epsilon^2 = 0$ be the ring of dual numbers. In order to prove that $\mathrm{Aut}_k(X)$ is unramified, it is enough to prove that an automorphism φ of $X \times_k k[\epsilon]$ which is the identity modulo ϵ is the identity. Such a φ stabilizes each affine open subscheme $\mathrm{Spec}(A) \subset X$ and acts there via a ring homomorphism $\varphi^\sharp(a) = a + \lambda(a)\epsilon$. Since $\varphi^\sharp$ is multiplicative we get that λ is in fact a derivation. By gluing on all open affine, the various λ's define a global vector field, which is zero by Lemma 4.2.1 again. Hence φ is the identity. □

The stable reduction theorem for Galois covers which we will prove below is valid when the order of the Galois group is prime to all residue characteristics. In the proof, we will use the following lemma:

Lemma 4.2.3. *Let X be a reduced, irreducible curve over a field k and let x be a smooth closed point. Let φ be an automorphism of X of finite order n prime to the characteristic of k, belonging to the inertia group at x. Then the action of φ on the tangent space to X at x is via a primitive nth root of unity, i.e., it is faithful.*

Proof. We can assume that $n \geq 2$ and that x is a rational point, passing to a finite extension of k if necessary. Then the completed local ring of x is isomorphic to the ring of power series $k[[t]]$. The action of φ on the tangent space to C at x is done via multiplication by some mth root of unity ζ, with $m|n$. If $m \neq n$, then replacing φ by φ^m we reduce to the case where $\zeta = 1$. Since φ is not the trivial automorphism of C, there is an integer i and a nonzero scalar $a \in k$ such that $\varphi(t) = t + at^i$ modulo t^{i+1}. Then $\varphi^n(t) = t + nat^i$ modulo t^{i+1}. Since $\varphi^n(t) = t$ and n is not zero in k, this is impossible. Therefore, $m = n$. □

4.3. Reduction of Galois covers at good characteristics

We now give the applications to stable reduction of Galois covers of curves (by *cover* we mean a finite surjective morphism). To do this, we fix a finite group G of order n and we consider a cover of smooth, geometrically connected curves $f : C \to D$ which is Galois with group G. We assume as usual that the genus of C is $g \geq 2$. The case where the order n is divisible by the residue characteristic p of k brings some more complicated pathologies, and here we will rather have a look at the case where n is prime to p. We make the following definition.

Definition 4.3.1. Let k be a field of characteristic p, and G a finite group of order n prime to p. Let X be a stable curve over k endowed with an action of G, and for all nodes $x \in X$, let $H_x \subset G$ denote the subgroup of the inertia group of x composed of elements that preserve the branches at x. We say that the action is *stable*, or that the Galois cover $X \to Y := X/G$ is *stable*, if the action of G on X is faithful and for all nodes $x \in X$, the action of H_x on the tangent space of X at x is faithful with characters on the two branches χ_1, χ_2 satisfying the relation $\chi_1\chi_2 = 1$.

Note that the stabilizer is cyclic when it preserves the branches at x, and dihedral when some elements of H permute the branches at x.

An extremely important consequence of the assumption $(n, p) = 1$ is that the formation of the quotient $X \to X/G$ commutes with base change. Consequently, the definition of a stable cover above makes sense in families, i.e., if $X \to S$ is a stable curve over a scheme S endowed with an action of G by S-automorphisms and $Y = X/G$, then we say that the cover $X \to Y$ is a stable Galois cover if and only if it is stable the fibre over each point $s \in S$. Then we arrive at the following stable reduction theorem for covers:

Theorem 4.3.2. *Let G be a finite group of order n prime to the characteristic of k, the residue field of R. Let $C \to D$ be a cover of smooth, geometrically connected curves which is Galois with group G, and assume that the genus of C is $g \geq 2$. Then after a finite extension of K, the cover $C \to D$ has a stable model $X \to Y$ over R. Furthermore, this model is unique.*

Proof. By the stable reduction theorem, there exists a finite field extension L/K such that C_L has a stable model X. Replacing K by L for notational simplicity, we reduce to the case $L = K$. Then by unicity of the stable model and by abstract nonsense, the group action extends to an action of G on X by R-automorphisms. By Lemma 4.2.2, the induced action of G on the special fibre X_k is faithful: indeed, if $\varphi \in G$ has trivial image in $\mathrm{Aut}_k(X)$, then by the property of unramification of the automorphism functor, it has trivial image in $\mathrm{Aut}_{R/m^n}(X \otimes_R R/m^n)$ for all $n \geq 1$, so since R is complete, it has trivial image in $\mathrm{Aut}_R(X)$. We define $Y = X/G$.

We now prove that the action is stable. Let $x \in X_k$ be a nodal point and let $H_x \subset G$ be the subgroup of the stabilizer of x composed of elements that preserve the branches at x. The completion of the local ring $\mathcal{O}_{X,x}$ is isomorphic to $R[[a,b]]/(ab - \pi^n)$ for some $n \geq 1$. Then the tangent action on the branches is obviously via multiplication by inverse roots of unity of order $|H_x|$. It remains to see that the kernel N of the action of H_x on the tangent space $T_{X_k,x}$ is trivial. In fact N acts trivially on the whole irreducible components containing x, as one sees by applying Lemma 4.2.3 to the normalization of X_k. Since X_k is connected, it follows at once that N acts trivially on X_k, hence $N = 1$. □

Moreover, one can prove, using deformation theory, that a stable Galois cover of curves over k can be deformed into a smooth curve over R with faithful G-action. For details about this point, we refer for example to [BR].

In the case where the order of G is divisible by the residue characteristic p, things are much more complicated. We will conclude by a simple example, which gives an idea of the local situation around a node of the special fibre. Assume that R contains a primitive pth root of unity ζ. We look at the affine R-curve

X with function ring $R[x,y]/(xy-a)$, for some fixed a in the maximal ideal of R. We consider the group $G = \mathbb{Z}/p\mathbb{Z}$, with generator σ, and the action on a neighbourhood of the node of X_k given by

$$\sigma(x) = \zeta x + a \qquad \text{and} \qquad \sigma(y) = \frac{y}{\zeta + y} .$$

Then the reduced action is given by $\sigma(x) = x$ and $\sigma(y) = y/(1+y)$, hence it is faithful on one branch but not on the other. Apparently some information on the group action is lost in reduction, but it is not clear what to do in order to recover it. At the moment, no "reasonable" stable reduction theorem for covers at "bad" characteristics is known.

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