INFINITELY MANY ARITHMETIC ALTERNATING LINKS

M. D. BAKER AND A. W. REID

Abstract. We prove the existence of infinitely many alternating links in S^3 whose complements are arithmetic.

1. Introduction

Let d be a square-free positive integer and let O_d denote the ring of integers of $\mathbb{Q}(\sqrt{-d})$. A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and the Bianchi orbifold $Q_d = \mathbb{H}^3/\mathrm{PSL}(2, O_d)$ are commensurable, that is to say they share a common finite sheeted cover. (see [MR, Chapters 8 & 9] for further details). If $X = S^3 \setminus L$, we call L *an arithmetic link*.

Since Thurston's original studies of hyperbolic structures on 3-manifolds [Th], link complements in S^3 have played a prominent role, and indeed arithmetic links were also very much at the heart of his work. Several arithmetic link complements were constructed in [Th], and, over the years, many more examples constructed; for a selection see [ALR], [AR], [Ba1], [Ba2], [Ba3], [BGR1], [BGR2], [Goe], [GH], and [Ha]. Several of these arithmetic links are alternating, and although there are infinitely many arithmetic links in S^3 (for example, those links determining certain cyclic covers of the complement of the Whitehead link), whether there were infinitely many alternating arithmetic links remained open.

By relating the spectral geometry of the complement to combinatorics of an alternating diagram, Lackenby [Lac] showed that there are only finitely many *congruence* alternating links, and motivated by this, asks in [Lac], whether there are only finitely many arithmetic alternating links. More recently, the question as to whether there were infinitely many arithmetic alternating links was asked of the second author by D. Futer in 2019. The main result of this note resolves these questions by answering Futer's question in the positive (and hence Lackenby's in the negative).

Theorem 1.1. *There are infinitely many alternating links in S^3 whose complements are arithmetic.*

Indeed, we prove something more precise. We will construct two infinite families of alternating links L_j and $\mathcal{L}_j$ whose complements are arithmetic. In more detail, the family of links L_j is built from $(j+1)$ concentric circles centered at the origin in the Euclidean plane, with a "horizontal" component (which we will denote by K) added intersecting each of the concentric circles in 4 points, and each intersection point resolved to make the diagram alternating (see Figure 1(a) where L_4 is shown). Thus L_j is an alternating link with $j+2$ components. The family of links $\mathcal{L}_j$ is constructed in a similar fashion using $(j+1)$ concentric circles centered at the origin in the Euclidean plane, with two additional components (which we will denote by K_1 and K_2) added intersecting each of the concentric circles in 2 points, and each intersection point resolved to make the diagram alternating (see Figure 1(b) where $\mathcal{L}_4$ is shown). Thus $\mathcal{L}_j$ is an alternating link with $j+3$ components

Theorem 1.2. *L_j and $\mathcal{L}_j$ are arithmetic for all $j \geq 1$ with $S^3 \setminus L_j \to Q_3$ and $S^3 \setminus \mathcal{L}_j \to Q_3$ both of degree $60j$.*

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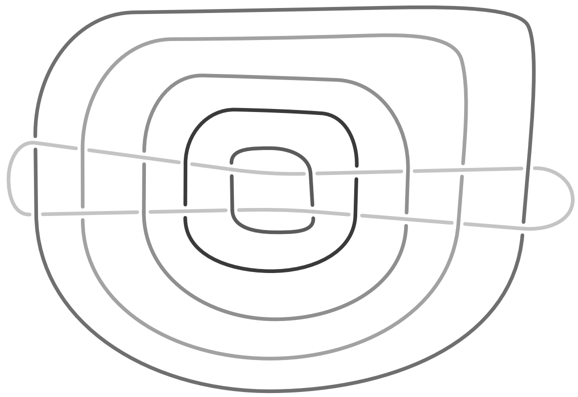

Figure 1(a)

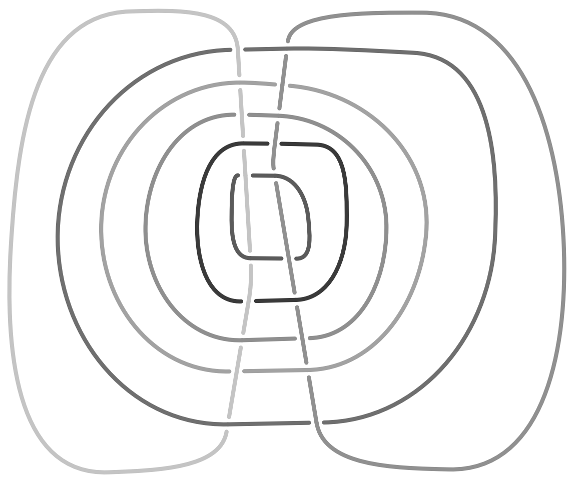

Figure 1(b)

The arithmetic nature of the link L_1 was first explicitly described in [Ha, Example 5], and we recall this briefly here. As described in [Ha], the complement of L_1 can be obtained as the union of two regular ideal hyperbolic cubes (all of whose dihedral angles are $\pi/3$), and as noted in [Ha], a regular ideal cube can be subdivided into 5 regular ideal hyperbolic simplices, from which Hatcher deduces that L_1 is arithmetic since the fundamental group of its complement arises as a subgroup of the isometry group of the tessellation of $\mathbb{H}^3$ by regular ideal hyperbolic simplices; this is the group $\mathrm{PGL}(2, O_3)$. Hence the link L_1 is arithmetic. In fact (see the discussion in the proof of Theorem 1.2 given in §2.2), the fundamental group of its complement arises as a subgroup $\mathrm{PSL}(2, O_3)$. Given the description of $S^3 \setminus L_1$ as a union of 10 regular ideal tetrahedra, its volume can be computed as $10v_0$ where v_0 is the volume of the regular ideal simplex in $\mathbb{H}^3$ (i.e. approximately $10.14941606\ldots$). Since the volume of Q_3 is $v_0/6$, $S^3 \setminus L_1$ is a 60-fold cover of Q_3. In [Ha, Example 5] Hatcher constructs a second link complement as the union of two regular ideal hyperbolic cubes, and this is homeomorphic to $S^3 \setminus \mathcal{L}_1$.

The manifolds $S^3 \setminus L_1$ and $S^3 \setminus \mathcal{L}_1$ have been reconstructed in other places in the literature. By volume considerations [AHW], $S^3 \setminus L_1$ (resp. $S^3 \setminus \mathcal{L}_1$) can be seen to be homeomorphic to the complement of the three component link 8^3_4 (resp. to the complement of 8^4_1). It can be checked (e.g. using SnapPy [CDGW]) that $S^3 \setminus L_1$ is also homeomorphic to a 5-fold irregular cover of the complement of the figure-eight knot (namely the so-called Roman link of [HLM]). The complements of L_1 and $\mathcal{L}_1$ were constructed again in [AR, Example 3] as well as being identified as the tetrahedral census manifolds otet10_{00006} and otet10_{00011} of [Tet] (see also [Goe2]).

In a different direction, neither $S^3 \setminus L_1$ or $S^3 \setminus \mathcal{L}_1$ contain a closed embedded essential surface (see [HM] for L_1 and [Oe] for $\mathcal{L}_1$). By comparison, in §3 we show that both of $S^3 \setminus L_j$ and $S^3 \setminus \mathcal{L}_j$ contain a closed embedded essential surface for all $j \geq 2$.

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2. Proof of Theorem 1.2

Our proof will be motivated by that given in [Ha], but we shall certify arithmeticity in a slightly different way.

2.1. Tessellation by regular ideal cubes.

Motivated by the description of $S^3 \setminus L_1$ as a union of two regular ideal cubes, we make the following definition (cf. [Tet]).

Definition 2.1. *Let M be a finite volume cusped hyperbolic 3-manifold. We call M cubical if it can be decomposed into regular ideal hyperbolic cubes.*

Let $M = \mathbb{H}^3/\Gamma$ be a cubical manifold. On lifting to the universal cover, we obtain a tessellation $\mathcal{T}(C)$ of $\mathbb{H}^3$ by regular ideal cubes, C, and so Γ is a subgroup of the group of isometries of $\mathcal{T}(C)$, which we denote $\mathrm{Isom}(\mathcal{T}(\mathrm{C}))$ (which is a discrete group of isometries of $\mathbb{H}^3$). We will denote by $\mathrm{Isom}^+(\mathcal{T}(\mathrm{C}))$ the subgroup of $\mathrm{Isom}(\mathcal{T}(\mathrm{C}))$ of index 2 consisting of orientation-preserving isometries.

Lemma 2.2. $\mathrm{Isom}(\mathcal{T}(\mathrm{C}))$ *is an arithmetic subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$ *commensurable with* $\mathrm{PSL}(2, O_3)$. *Hence any cubical manifold is arithmetic.*

A proof of Lemma 2.2 is implicit in [NR], but we include a proof here for completeness. Before proving Lemma 2.2, we recall some notation. Let $\Gamma_0(2) < \mathrm{PSL}(2, O_3)$ be the image of the subgroup of $\mathrm{SL}(2, O_3)$ given by:

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, O_3) |\ c \equiv 0 \mod\ < 2 >\}.$$

It is easy to check that $[\mathrm{PSL}(2, O_3) : \Gamma_0(2)] = 5$, that $\mathbb{H}^3/\Gamma_0(2)$ has two cusps (corresponding to the inequivalent parabolic fixed points 0 and ∞) and that the peripheral subgroup of $\Gamma_0(2)$ fixing ∞ coincides with that of $\mathrm{PSL}(2, O_3)$, namely the image in $\mathrm{PSL}(2, O_3)$ of the subgroup

$$< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & 1/\omega \end{pmatrix} >, \text{ where } \omega^2 + \omega + 1 = 0.$$

Let ι and τ be the elements of $\mathrm{PSL}(2, \mathbb{C})$ given by the images of the elements $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$ respectively. Note that ι and τ both have order 2, and they normalize $\Gamma_0(2)$. Hence the group $G =< \Gamma_0(2), \iota, \tau >$ is arithmetic containing $\Gamma_0(2)$ as a normal subgroup with quotient group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. To prove Lemma 2.2, it suffices to show that $\mathrm{Isom}^+(\mathcal{T}(\mathrm{C}))$ is commensurable with $\mathrm{PSL}(2, O_3)$. To that end, we will show that the orbifolds $N_1 = \mathbb{H}^3/\mathrm{Isom}^+(\mathcal{T}(\mathrm{C}))$ and $N_2 = \mathbb{H}^3/G$ are isometric and hence $\mathrm{Isom}^+(\mathcal{T}(\mathrm{C}))$ and G are conjugate by Mostow-Prasad Rigidity. Using the remarks prior to the proof, this proves commensurability.

In the notation established above, since $\tau(0) = \infty$, the orbifold N_2 has a single cusp, and since $\iota \in G$, this is a rigid cusp of type $(2, 3, 6)$ (in the notation of [NR]). Moreover, since the volume of Q_3 is $v_0/6$, the computation of indices given above shows that the volume of N_2 is $5v_0/24$.

Now consider the group $\mathrm{Isom}^+(\mathcal{T}(\mathrm{C}))$. This is generated by the extension to $\mathbb{H}^3$ of the orientation-preserving symmetries of a single cube C of $\mathcal{T}(C)$, along with rotations of $2\pi/6$ in the edges of C. As noted in §1, C can be subdivided in 5 regular ideal tetrahedra, and so the volume of C is $5v_0$. From this it now follows that N_1 has volume $5v_0/24$ and a rigid cusp of type $(2, 3, 6)$.

Finally, using Adams [Ad], we deduce that N_1 and N_2 are isometric, since it is proved there that there is a unique orientable hyperbolic 3-orbifold of volume $5v_0/24$ and a single rigid cusp of type $(2, 3, 6)$. □

Remark 2.3. *Part of the proof in* [Ad] *of the uniqueness of a hyperbolic* 3*-orbifold with a single rigid cusp of type* $(2, 3, 6)$ *was found to have a gap, but this was corrected in the recent paper* [DK].

Remark 2.4. *As noted in* [NR] *the group* $\mathrm{Isom}(\mathcal{T}(\mathrm{C}))$*, can be identified with the group generated by reflections in the faces of the tetrahedron* $T[4, 2, 2; 6, 2, 3] \subset \mathbb{H}^3$ *in the notation of* [NR].

2.2. **The link complements $S^3 \setminus L_j$ and $S^3 \setminus \mathcal{L}_j$ are cubical.** Given Lemma 2.2, we must show that $S^3 \setminus L_j$ (for $j \geq 1$) and $S^3 \setminus \mathcal{L}_j$ (for $j \geq 1$) are cubical. We will take a slightly different perspective from Hatcher's construction of a cubical structure for $S^3 \setminus L_1$ (more in keeping with [ALR] and [AR]) which we now describe. This is what we generalize for the links L_j $(j \geq 2)$ and $\mathcal{L}_j (j \geq 2)$.

Consider an alternating diagram for L_1 on some projection plane $S^2 \subset S^3$. This produces the 4-valent planar graph P_1 shown in Figure 2(a). Two-coloring the regions in checkerboard fashion and labelling these regions as $+$ and $-$ affords a decomposition of S^3 into two 3-balls, each of which is endowed with an abstract polyhedral structure. Denote these polyhedra by Π_+ and Π_-. These polyhedra are identical up to reversing all the colors and signs. Each face f_i of Π_+ is a n_i-gon (where $n_i = 2$ or 4 in this case) with a sign $\sigma_i \in \{\pm\}$, and the polyhedra Π_+ and Π_- are identified by sending f_i to the corresponding face of Π_- using a rotation of $\sigma_i 2\pi/n_i$ (with $+$ denoting clockwise). The resulting complex with vertices deleted is then homeomorphic to $S^3 \setminus L_1$ (see [ALR] for example).

Note that P_1 contains 4 bigons, and we can collapse each of these bigons to an edge in each of the polyhedra Π_+ and Π_-, and then make the identifications described above. The resulting polyhedra obtained are cubes (see Figure 2(b)), so that $S^3 \setminus L_1$ is the identification space of two cubes with vertices deleted.

This combinatorial realization can be done geometrically: namely the identifications described above can be realized as identifications of the regular ideal cube in $\mathbb{H}^3$ with six 2-cells meeting along an edge (with dihedral angle $\pi/3$).

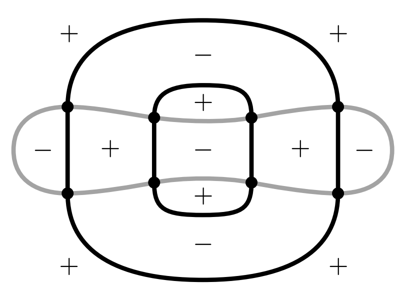

Figure 2(a)

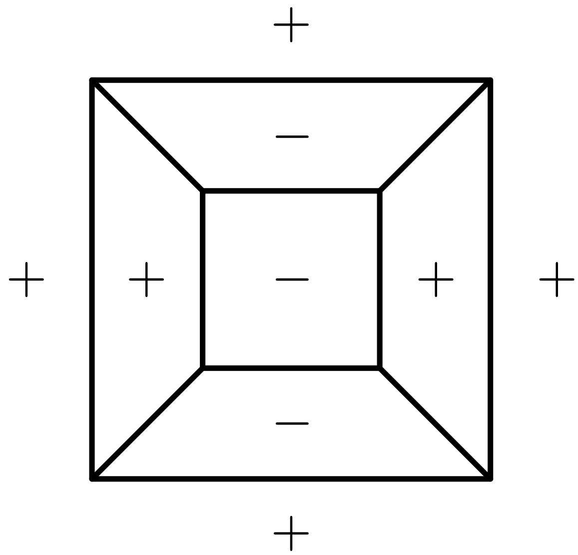

Figure 2(b)

For the general case of L_j, we refer to Figure 3(a) and 3(b) (which shows the case of L_4) and proceed as follows.

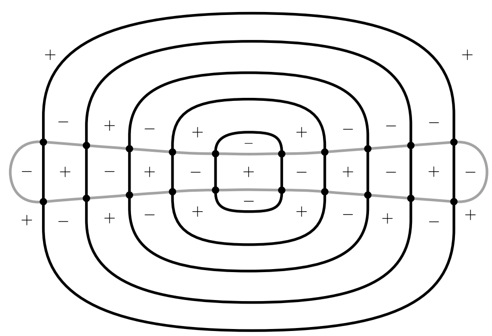

Figure 3(a)

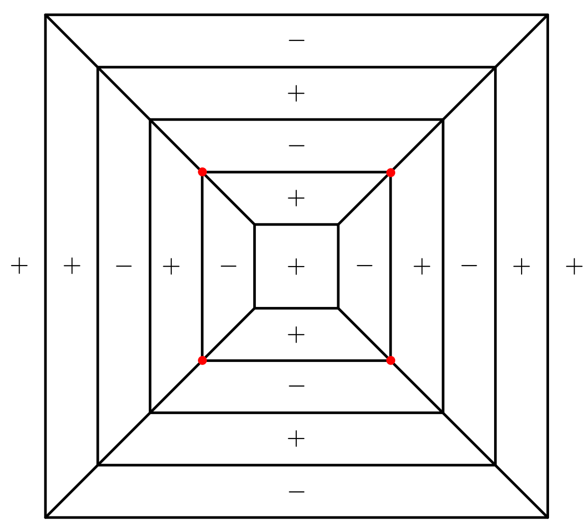

Figure 3(b)

Performing the construction above on each L_j, results in a 4-valent planar graph P_j (see Figure 3(a)) and polyhedra Π_+^j and Π_-^j. As above, the graphs P_j each contains exactly four bigons, and on collapsing these bigons leads to the polyhedra shown in Figure 3(b). As is visible from the diagram, each of Π_+^j and Π_-^j is a union of j cubes, whose faces are identified as described above. To establish that for each $j \geq 2$ the manifold $S^3 \setminus L_j$ is cubical, and therefore arithmetic by Lemma 2.2, we need to ensure that the combinatorial decomposition described here can be realized geometrically.

Referring to Figure 3(b) we now view the polyhedra Π_+^j and Π_-^j as being built from copies of the regular ideal cube, so that edges of Π_+^j and Π_-^j have dihedral angle $\pi/3$ or $2\pi/3$, the latter occurring at edges where two cubes meet; e.g. the edges between those red vertices of Figure 3(b), and then the edges of all concentric squares except the "innermost" and "outermost" ones. From above, the polyhedra Π_+ and Π_- are identified by sending f_i to the corresponding face of Π_- using a rotation of $\sigma_i\pi/2$ (with + denoting clockwise). Using this we see that edges with dihedral angle $2\pi/3$ are identified via the $\pi/2$ rotation to an edge with dihedral angle $\pi/3$. Each such edge with dihedral angle $2\pi/3$ lies in two faces of adjacent cubes and so once the identifications are completed the angle sum is 2π. Edges of the innermost and outermost squares have dihedral angles $\pi/3$. They are identified via $\pi/2$ rotations to edges also with dihedral angles $\pi/3$. Six of these edges are identified to get angle sum 2π. This proves that each $S^3 \setminus L_j$ is cubical, and hence arithmetic.

Moreover, since any arithmetic link complement commensurable with Q_3 necessarily covers Q_3 (see for example [MR, Theorem 9.2.2] and note that $M(2, \mathbb{Q}(\sqrt{-3}))$ has type number one), the final part of Theorem 1.2 follows since, from above, the volume of $S^3 \setminus L_j$ is $10jv_0$, and the volume of Q_3 is $v_0/6$.

The case of $\mathcal{L}_j$ is handled in a completely similar manner using polyhedra arising as in Figure 4. We omit the details. □

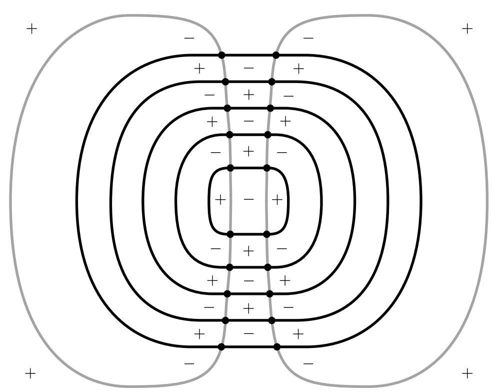

Figure 4

As was pointed out in [Tet] (see Remark 3.7), it is not always the case that a cubical manifold decomposes into regular ideal tetrahedra. However, this does hold for the manifolds $S^3 \setminus L_j$ and $S^3 \setminus \mathcal{L}_j$. The important point to note is that insertion of the diagonals on faces to create the five tetrahedra can be done so consistently (as was implict in [Ha]). In particular, each of $S^3 \setminus L_j$ and $S^3 \setminus \mathcal{L}_j$ is decomposed into $10j$ regular ideal tetrahedra, and so using this decomposition and [HaMu], a corollary of Theorem 1.2 is:

Corollary 2.5. *$S^3 \setminus L_j$ and $S^3 \setminus \mathcal{L}_j$ are manifolds of maximal volume amongst all hyperbolic manifolds admitting a decomposition into $10j$ tetrahedra.*

3. Closed embedded essential surfaces

We first show that for $j \geq 2$, $S^3 \setminus L_j$ contains a closed embedded essential surface. Deleting the component K of L_j results in the $j + 1$ component unlink. The result now follows from [CL, Theorem 4.1] since the $\mathrm{SL}(2,\mathbb{C})$ character variety of F_{j+1} has dimension $3(j+1)-3=3j$ and this is greater than $j+2$ for $j \geq 2$.

The case of $S^3 \setminus \mathcal{L}_j$ is handled in a similar manner. In this case, deleting the components K_1 and K_2 from $\mathcal{L}_j$ results in the $j+1$ component unlink and we now argue as above applying [CL, Theorem 4.1] on noting that $3(j+1)-3=3j$ is greater than $j+3$ for $j \geq 2$.

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IRMAR,
Université de Rennes 1,
35042 Rennes Cedex,
France.
E-mail address: mark.baker@univ-rennes1.fr

Department of Mathematics,
Rice University,
Houston, TX 77005, USA
E-mail address: alan.reid@rice.edu