Université de Rennes 1
Département de Mathématiques Année 2014-2015

Master1 de Mathématiques
“Analyse hilbertienne et applications"
Tutoriel problems n^{0}4

1 Orthogonality

1. Let E be a Hilbert space, F_1 and F_2 subspaces of E.
   (a) Show that $(F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp$
   (b) If moreover F_1 et F_2 are closed, show that $(F_1 \cap F_2)^\perp = \overline{F_1^\perp + F_2^\perp}$
2. Let $E = \mathscr{M}_n(\mathbb{R})$ be the $n \times n$ matrices endowed with the inner product $\langle M, N\rangle = \mathrm{tr}({}^tMN)$.
   (a) Compute the norme of the identity I and the matrix $M = (i + j)_{1\leq i,j\leq n}$
   (b) We denote by $\mathscr{S}_n(\mathbb{R})$ the space of symmetric matrices and by $\mathscr{A}_n(\mathbb{R})$ des antisymmetric matrices. Show that $\mathscr{S}_n(\mathbb{R})^\perp = \mathscr{A}_n(\mathbb{R})$ and $E = \mathscr{S}_n(\mathbb{R}) \oplus \mathscr{A}_n(\mathbb{R})$
   (c) Compute $\inf \left\{ \sum_{1\leq i,j\leq n} (a_{ij} - i - j)^2 \,|\, \text{où } A = (a_{ij}) \in \mathscr{A}_n(\mathbb{R}) \right\}$.

2 Dual and adjoint

1) Let H be a Hilbert space.
   (a) Let $a \in H \setminus \{0\}$. Show that $\forall u \in H$, $d(u, \{a\}^\perp) = \frac{|<u,a>|}{\|a\|}$.
   (b) Let $f \in H' \setminus \{0\}$. Show that $\forall x \in H$, $d(x, \ker f) = \frac{|f(x)|}{\|f\|}$.
   (c) Let F subspace of $L^2([0, 1])$ defined by $F = \{f \in L^2([0, 1]), \int_0^1 f(x)\,dx = 0\}$. Show that F is a closed subspace of $L^2([0, 1])$ and compute the distance from $f : x \mapsto e^x$ to F.
2) Let H be a Hilbert space, x_n and $x \in H$. Show that the following conditions are equivalent:
   (a) $x_n \to x$
   (b) $x_n \rightharpoonup x$ et $\|x_n\| \to \|x\|$.
   (c) $\lim_{n\to+\infty}\langle x_n, y\rangle = \langle x, y\rangle$ uniformly for all $y \in H$ and $\|y\| = 1$.
3) Let $H = L^2([0, 1])$. For every $f \in H$, we set
$$Tf(x) = \int_0^x f(t)dt.$$
   (a) Show that T is a continuous operator of H.
   (b) Compute the adjoint map of T.

3 Isometry, inner product

1) Let H_1 et H_2 two Hilbert spaces and $\Phi : H_1 \to H_2$ a linear application. Show that the following conditions are equivalent:
   a) $\|\Phi(u)\| = \|u\|$, pour tout $u \in H_1$.
   b) $< \Phi(u), \Phi(v) >=< u, v >$ pour tout $u, v \in H_1$.

   Let Φ a linear isometry. Show that:
   i) $\Phi(H_1)$ is a Hilbert space.
   ii) If $(e_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis of H_1 then $\Phi((e_\lambda)_{\lambda\in\Lambda})$ is an orthonormal basis $\Phi(H_1)$.

2) Let E be a $\mathbb{C}$-vector space endowed with a norm which satisfies: the parallelogram identity: for all $u, v \in E$: $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$. We define $f : E \times E \to \mathbb{C}$ by $f(u,v) = \frac{1}{4}\sum_{k=0}^{3} i^k \|u + i^k v\|^2$. We propose to show that f is an inner product associated to $\|.\|$ (Von Neumann's theorem).
Show that for $u, v, w \in E$ and $\lambda \in \mathbb{C}$ we have

(a) $f(u,u) = \|u\|^2$
(b) $f(u,v) = \overline{f(v,u)}$
(c) $f(u+v,w) = 2f(u,\frac{w}{2}) + 2f(v,\frac{w}{2})$
(d) $f(u,v) = 2f(u,\frac{v}{2})$
(e) $f(u+v,w) = f(u,w) + f(v,w)$
(f) $f(\lambda u, v) = \lambda f(u,v)$.

conclude

3) Let $h^1(\mathbb{N}) = \{a \in \ell^2(\mathbb{N}), \sum_{n=0}^{\infty} n^2|a_n|^2 < +\infty\}$. For any $a = (a_n)$ and $b = (b_n)$ elements of $h^1(\mathbb{N})$ we let

$$< a, b >= \sum_{n=0}^{+\infty}(1+n^2)a_n\overline{b_n}.$$

(a) Show that $< .,. >$ is an inner product.
Show that $h^1(\mathbb{N})$ is a Hilbert space.
(b) Show that $h^1(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$.
(c) Show that the closed unit ball in $h^1(\mathbb{N})$ is compact in $\ell^2(\mathbb{N})$.

4 Legendre's polynomials

On $\mathbb{R}[X]$ we consider the bilinear form: $\langle P, Q\rangle = \int_{-1}^{1} P(t)Q(t)\,dt$.

1. Check that $\mathbb{R}[X]$ endowed with this bilinear form is an inner space.
2. Is it a Hibert space? if not what is it completion?
3. Applie to $(X^n)_{n\in\mathbb{N}}$ the Gram-Schmidt process, to show that there is a unique orthonormal family $(P_n)_{n\in\mathbb{N}}$ such that P_n is a polynomial of degree n and $\langle P_n, X^n\rangle > 0$.
4. Deduce that $(P_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2([-1,1])$.
5. We define Q_n by
$$Q_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$
Show that Q_n is of degree n and has n simple roots in $]-1,1[$.
Show that Q_n is orthogonal to any polynomial of degree $< n$ and deduce that $Q_n = \lambda_n P_n$.
Compute $\|Q_n\|^2$ and λ_n. Detremine $Q_n(1)$ and $Q_n(-1)$.
6. Compute $\min_{a,b,c\in\mathbb{R}} \int_{-1}^{1} |x^3 - ax^2 - bx - c|^2 dx$.

5 Polynomials

Let $w(x) = x^{-\ln x}$. We consider the Hilbert space $L^2_w([0,+\infty[) = \{f; f\sqrt{w} \in L^2([0,+\infty[)\}$, endowed with the inner product $\langle u, v\rangle = \int_0^{+\infty} u(x)\overline{v(x)}x^{-\ln x}dx$.

1. Show that $\mathbb{R}[X] \subset L^2_w([0,+\infty[)$.
2. Let $f(x) = \sin(2\pi \ln x)$, show that $f \in L^2_w([0,+\infty[)$.
3. Let u_n the polynomial function $x \mapsto x^n$. Show that for all $n \in \mathbb{N}$, $\langle u_n, f\rangle = 0$
Deduce that $\mathbb{R}[X]$ is not dense in $L^2_w([0,+\infty[)$.