Université de Rennes
Département de Mathématiques Année 2014-2015

Master1 de Mathématiques
“Analyse hilbertienne et applications"
Tutoriel problems n^{0}3

1 Norm of a linear map

1. We equip $\mathbb{R}[X]$ with the norm $\|P\|_\infty := \max_{x\in[0,1]} |P(x)|$. For $a \in \mathbb{R}$, we define a linear form $\Phi_a : \mathbb{R}[X] \to \mathbb{R}$ by $\Phi_a(P) = P(a)$. Determine for each $a \in \mathbb{R}$, Φ_a is continuous, and compute in this case $\|\Phi_a\|$.
2. Let $E = C([0,1];\mathbb{R})$, equipped with $\|.\|_\infty$. Let $(a_n)_{n\in\mathbb{N}}$ be a dense sequence of $[0,1]$. Show that the linear form ϕ defined by $\phi(f) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} f(a_n)$ is continuous on E, its norm is $\|\phi\| = 1$, but does not attains it.

2 Series in Banach

Let $(a_n)_{n\geq 0}$ be a sequence in $\mathbb{K}$ such that the power serie $\sum_{n\in\mathbb{N}} a_n z^n$ has $R > 0$ as convergence radius and let $(E, \|.\|)$ be a Banach space.

1. Let $L \in \mathscr{L}(E)$ such that $\|L\| < R$. Show that $\sum_{n\in\mathbb{N}} a_n L^n \in \mathscr{L}(E)$.
2. Let $L \in \mathscr{L}(E)$. Show that $e^L := \sum_{n\in\mathbb{N}} \frac{L^n}{n!}$ defines an element of $\mathscr{L}(E)$.
3. Let $L, L' \in \mathscr{L}(E)$ such that $L \circ L' = L' \circ L$. Show that
$e^L \circ e^{L'} = e^{L'} \circ e^L$.
4. Deduce from that, if $L \in \mathscr{L}(E)$ then e^L is invertible and its inverse is $e^{-L} \in \mathscr{L}(E)$.
5. Let $L \in \mathscr{L}(E)$ such that $\|L\| < 1$. Show that there exists $G \in \mathscr{L}(E)$ such that $G^2 = I_E - L$.

3 Banach-Steinhaus

1. Let $E = \mathbb{R}[X]$, equipped with the norm $\|P\|_\infty = \max\{|a_i| \mid \text{ if } P = \sum_{i=0}^n a_i X^i\}$, and for $n \in \mathbb{N}$ let $T_n : P \mapsto P^{(n)}(0)$. Show that Banach-Steinhaus theorem does not apply to this situation.
2. Let $a = (a_n)$ a real sequence ; we suppose that for every $b = (b_n)$ in c_0, the power serie $\sum_{n=0}^\infty a_n b_n$ converges. Deduce that $a \in \ell^1$.
3. Let E be a normed vector space and $B \subset E$ such that for every $\phi \in E'$, the set $\phi(B) = \{\phi(x), x \in B\}$ is a bounded subset of $\mathbb{R}$. Show that B is bounded.

4 Bilinear application

Let E_1 a Banach space, E_2 and F be two normed spaces. Let $B : E_1 \times E_2 \to F$ a bilinear application such that its partial maps are continuous, i.e. for every $x \in E_1$, the map from E_2 in F that maps y to $B(x,y)$ is continuous and for every $y \in E_2$, the map from E_1 in F that maps x to $B(x,y)$ is continuous. Show that B is continuous.

5 Closed Graph

1) Let E and F be Banach spaces and $T : E \to F$ a linear map. We suppose for each sequence (x_n) in E which converges to 0 and for all continuous linear form $f \in F'$, we have $\lim_{n\to\infty} f(T(x_n)) = 0$. Show that T is continuous.
2) Let F be a vector subspace of $C^1([0,1],\mathbb{R})$, such that F is closed in $C^0([0,1],\mathbb{R})$ equipped with $\|.\|_\infty$. Show that F is of finite dimension.

6 Sequence spaces

For $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$, we let $\|x\| = \sum_{n=0}^\infty |a_n x_n|$.

1. Give a necessary and sufficient condition on (a_n) for $\|.\|$ to became a norm on ℓ^∞.
2. Prove that $\|.\|$ is never equivalent to $\|.\|_\infty$ in ℓ^∞.
3. Prove that ℓ^∞ equipped with $\|.\|$ is not a Banach space .

7 Hahn-Banach

1. Let E a normed $\mathbb{K}$-vector space , $\phi, \phi_1, \ldots, \phi_n$, $n+1$ linear forms , such that $\bigcap_{i=1}^{n} \ker \phi_i \subset \ker \phi$.
   Show that, there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $\phi = \sum_{i=1}^{n} \lambda_i \phi_i$.

2. Let E a normed $\mathbb{K}$-vector space , F a closed vector subspace of E and $x_0 \in E \setminus F$.
   Show that there exists $\phi \in E'$ such that $\phi|_F = 0$, $\|\phi\| \leq 1$ and $\phi(x_0) = d(x_0, F)$.

3. For every $\alpha \in \mathbb{R}$, let $E_\alpha := \{f \in C([-1,1], \mathbb{R}) \ : f(0) = \alpha\}$. Show that E_α is dense convex subset of $L^2([-1,1], \mathbb{R})$. Show that $\alpha \neq \beta$: $E_\alpha \cap E_\beta = \emptyset$, but there is no continuous linear form $\ell \in \left(L^2([-1,1], \mathbb{R})\right)'$ which separates them.

8 Weak topology

a) Let E a normed vector space.

   1) Let $a \in E$. Show that the translation map by the vector a is a homeomorphism of $(E, \ \sigma(E, \ E'))$ in it self.
   2) Show that the addition map from $E \times E$ to E and the multiplication by a scalar from $\mathbb{K} \times E$ in E, are weakly continuous.

b) Let E and F be two normed vector spaces and T a linear map from E to F. Show that the following conditions are equivalent:

   1) T is continuous from $(E, \ \|.\|)$ in $(F, \ \|.\|)$.
   2) T is continuous from $(E, \ \sigma(E, \ E'))$ in $(F, \ \sigma(F, \ F'))$ (i.e. T is weakly continuous).
   3) T is continuous from $(E, \ \|.\|)$ in $(F, \ \sigma(F, \ F'))$.

9 Weak convergence

Let E a normed vector space and $\{x_n\}$ a sequence in E. Show that:

(i) If $x_n \rightharpoonup x$ then $\|x\| \leq \liminf_n \|x_n\|$ and $x \in \overline{\text{conv}\{x_n\}}$.

(ii) The strong convergence (i.e. norm convergence) implies the weak convergence, but the converse is in general not true:
These notion are not equivalent in ℓ^p for any $1 < p < +\infty$.

(iii) $x_n \rightharpoonup x$ in ℓ^p for $1 < p < +\infty$, if and only if $\{x_n\}$ is bounded and for every $k \in \mathbb{N}$, $x_n(k) \to x(k)$, where $x(k)$ is the kth coordinate of x.

(iv) Show that $x_n = (\underbrace{1, \cdots, 1}_{n}, 0, 0, \cdots)$ does not weakly converge in ℓ^∞.

10 Duality

(i) let A a dense subset of a normed vector space E. Show that $A' = E'$.

(ii) Let $1 < p_1 < p_2 < \infty$, q_1 and q_2 there respective conjugate exponents.

   (a) Show that $\ell^{p_1} \subset \ell^{p_2}$.
   (b) Show that ℓ^{p_1} is dense in ℓ^{p_2}.
   What can we say about $(\ell^{p_1})'$ and $(\ell^{p_2})'$?

(iii) Show that $(C_0)' = (C_{00})' = \ell^1$.

11 Reflexiveness

1. Prove that for any continuous linear form on a reflexive space E, attains its norm; in other words $\exists x_0 \in E$, $\|x_0\| = 1$ such that $\|f\| = |f(x_0)|$ (thus, in the definition of the norm $\|f\|$, the sup is a max).
   Use this to show that ℓ^1 is not reflexive.

2. Show that the Banach space $C([0,1], \mathbb{R})$ is not reflexive. (We may consider the linear form on E defined by $L(f) = \int_0^{\frac{1}{2}} f(t)\, dt - \int_{\frac{1}{2}}^{1} f(t)\, dt$.)

3. Show that $C^1([0,1], \mathbb{R})$, equipped with its natural norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, is not reflexive.
   (Hint : we may begin by showing that $E = \{f \in C^1([0,1], \mathbb{R}) : \ f(0) = 0\}$ equipped with the norm $\|f\|' = \|f'\|_\infty$ is not reflexive).