Université de Rennes 1
Département de Mathématiques

Master1 de Mathématiques
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Analyse hilbertienne et applications
Tutoriel problems n^{0}2

1 Sequence spaces

1) Let c_{00} be the space of sequences in $\mathbb{K}$ with only a finite number of non-zero terms. Show that c_{00} is dense in $\ell^p(\mathbb{N},\mathbb{K})$, for $p \in [1,+\infty[$. Deduce that $\ell^p(\mathbb{N},\mathbb{K})$, pour $p \in [1,+\infty[$ is separable.

2) What about ℓ^∞, c (the space of convergent sequences) and c_0 (the space of sequences which converge to 0)?

2 Image of a filter

Let X et Y be two topological spaces, $f : X \to Y$ a map and $\mathscr{F}$ a filter on X.

1) Show that $\mathscr{F}$ is an ultrafilter if and only if for any subset A of X we have either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.

2) Show that the direct image by f of an ultrafilter is an ultrafilter i.e. $\{B \subset Y \mid f^{-1}(B) \in \mathscr{F}\}$ is an un ultrafilter.

3) Show that if f is continuous and $\mathscr{F}$ converges to x then $f(\mathscr{F})$ converges to $f(x)$.

3 Examples of compact sets

a) Show that the n-sphere $\mathbb{S}^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1},\ x_0^2 + ... + x_n^2 = 1\}$ is compacte.

b) Consider the equivalence relation on $\mathbb{S}^n$ defined by $x \sim y \Leftrightarrow x = y$ or $x + y = 0$. The quotient space is the *real projective space of dimension* n and denoted by $\mathbb{RP}^n$. Let p be the canonical projection. Show that

$$\mathscr{O} = \{V \subset \mathbb{RP}^n,\ p^{-1}(V) \text{ is open in } \mathbb{S}^n\}$$

define a compact topology on $\mathbb{RP}^n$.

c) Show that $\|A\|^2 = \mathrm{Tr}\,({}^tAA)$ is a norm on $M_n(\mathbb{R})$; deduce that $d(A,B) = \|B - A\|$ is a distance on $M_n(\mathbb{R})$; and that

$$O_n = \{A \in M_n(\mathbb{R}),\ {}^tAA = I\}$$

is compact.

4 Cantor set

We denote by $w_0(x) = \dfrac{1}{3} \cdot x$ et $w_2(x) = \dfrac{1}{3} \cdot x + \dfrac{2}{3}$ the homotheties on $\mathbb{R}$ of ratio $\dfrac{1}{3}$ and certers 0 et 1 respectively. Let $C_0 = [0,1]$ and define by induction, for all $n \in \mathbb{N}$, $C_{n+1} = w_0(C_n) \cup w_2(C_n)$. The set $C = \bigcap_{n \geq 0} C_n$ is the (triadic) Cantor set.

Let $A = \{0,2\}^{\mathbb{N}}$. An element of A is a sequence $(a_n)_{n\in\mathbb{N}}$, with $a_n = 0$ or $a_n = 2$.

Show that

$$\varphi : A \to [0,1] \text{ définit par } \varphi((a_n)) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

is a homeomorphism from A into C.

Deduce from this, that C is a compact non countable space.

5 Polynomials spaces

We endowed $\mathbb{R}[X]$ with a norm $\|.\|$.

1) Show that $\mathbb{R}_n[X]$ the subspace of polynomials of degree $\leq n$ is a closed set with empty interior.

2) Show that $\mathbb{R}[X]$ is not a Banach space.

6 Baire's theorem

1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ continuous such that: for all $x \in \mathbb{R}_+$ we have $\lim_{n\to+\infty} f(nx) = 0$.
   Show that $\lim_{x\to+\infty} f(x) = 0$.

2. Let $f : \mathbb{R}_+ \to \mathbb{R}$ continuous and (a_n) a (strictly) increasing sequence of (strictly) positive numbers such that :

   (a) $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 1$ et $\lim_{n\to+\infty} a_n = +\infty$

   (b) pour tout $x > 0$ on a $\lim_{n\to+\infty} f(a_n x) = 0$.

   Show that $\lim_{x\to+\infty} f(x) = 0$.

3. Let $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$ be a real power serie with infinite radius of convergence, such that $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $f^{(n)}(x) = 0$.
   Show that f is a polynomial.

7 Weierstrass's function

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the Weierstrass function defined by :
   pour tout $x \in \mathbb{Q} - \{0\}$, $f(x) = \frac{1}{q}$, si $x = \frac{p}{q}$ where$p \in \mathbb{Z}$ et $q \in \mathbb{N} - \{0\}$ are mutually primes and, for all $x \in \mathbb{R} - \mathbb{Q}$, $f(x) = 0$, and $f(0) = 0$.

   Show that f is continuous at x if and only if $x \in \mathbb{R} - \mathbb{Q} \cup \{0\}$.

2. Show that there exist no continuous function on $\mathbb{Q}$ and discontinuous on $\mathbb{R} - \mathbb{Q}$.

8 Separability

Let (X, d) be a compact metric space.

1. Show that X is separable.

2. Let $Q = \{a_n \,|n \in \mathbb{N}\}$ a countable and dense subset of X.

   For all $n \in \mathbb{N}$ we define $f_n : X \to \mathbb{R}$ by $f_n(x) = d(x, a_n)$.

   Show that if $x \neq y$ then there exists n such that $f_n(x) \neq f_n(y)$.

3. Show that if F is a countable subset of $C(X, \mathbb{R})$, the subalgebra generated by F is separable.

4. Show that $C(X, \mathbb{R})$ is separable.

9 Stone-Weierstrass's theorem

1. Let $f \in \mathscr{C}([a, b], \mathbb{R})$ such that $\forall n \in \mathbb{N} \quad \int_a^b f(t)t^n \, dt = 0$. Show that f vanishes identiqually .

2. Show that a non constant element of $\mathscr{C}(\mathbb{R}, \mathbb{R})$ that admits a finite limit at $+\infty$ is not a uniform limit of polynomials in $\mathbb{R}[x]$.

3. Let E a compact space. Let f_i, $i = 1, \ldots, n$ be n elements of $\mathscr{C}(E, \mathbb{R})$ which separates points in E. Show that E ist homeomorphic to a subset of $\mathbb{R}^n$.

10 Arzela-Ascoli's theorem

1. Let E, F be normed spaces and (f_n) a sequence of maps from E to F which is equicontinuous at $a \in E$.
   Show that, if $(f_n(a))$ converges to b, then $(f_n(x_n))$ converges to b, where (x_n) is a sequence of E such that $\lim_{n\to\infty} x_n = a$.
   Is the sequence of real functions $f_n(x) = (1 + x)^n$ equicontinuous?

2. Let (E, d) a metric and $\mathscr{H}$ a family of equicontinuous maps from E to $\mathbb{R}$. Show that:

   (a) the set A of $x \in E$ such that $\mathscr{H}(x)$ is a bounded is an open and closed set.

   (b) let E be compact and connected and $x_0 \in E$ such that $\mathscr{H}(x_0)$ is bounded. Show that $\mathscr{H}$ est relatively compact in $\mathscr{C}(E, \mathbb{R})$.

3. Let consider the sequence $f_n(t) = \cos(\sqrt{t + 4(n\pi)^2})$, $t \in [0, \infty[$.

   (a) Show that it's equicontinuous and converges pointwise to 0.

   (b) Is $\{f_n\}_{n\in\mathbb{N}}$ relatively compact in $(\mathscr{C}([0, \infty[), \|.\|_\infty)$? What can we conclude from Arzela-Ascoli's theorem?