Université de Rennes 1
Département de Mathématiques

Master1 de Mathématiques
Année 2014-2015

Analyse hilbertienne et applications
Tutoriel problems n^{0}1

1 Sequences

1. Show that a sequence of non negative real numbers which does not tend to $+\infty$ admits a convergent subsequence.
2. Show that $\mathbb{N}$ is a closed set in $\mathbb{R}$.
3. Show that a Cauchy sequence converges if and only if it admits a convergent subsequence. Deduce that a compact metric space is complete.
4. Show that if a sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ of $\mathbb{Q}$ converges to an irrational $x \in \mathbb{R}\setminus\mathbb{Q}$, then $|p_n| \to +\infty$.

2 Density

Let E be a metric space, $A \subset E$ be an open set and $B \subset E$ any subset.

(a) Show that $A \cap \overline{B} \subset \overline{A \cap B}$.

(b) $A \cap B = \emptyset \Rightarrow A \cap \overline{B} = \emptyset$.

(c) If B is dense in E, then $\overline{A \cap B} = \overline{A}$.

(d) If A and B are dense in E, $A \cap B$ is dense in E.

(e) Give an example, A not open, dense whose intersection is not dense.

3 Cardinality

1. Let X be a set and F a countable subset of X synch that $X \setminus F$ is infinite. Show that X is in bijection with $X \setminus F$
2. Let X and Y be two sets. Show that there is an injection from X in Y, or an injection from Y in X i.e. cardinality of X and Y can be compared. (consider all $Z \in \mathscr{P}(X \times Y)$ whose restrictions projections of X and Y are injective and apply Zorn's lemma.)

4 Somme

Let E be a normed vector space and $A, B \subset E$ we denote by $A + B$ the set $\{x + y|\ x \in A,\ y \in B\}$.

1. We assume A is open; show that $A + B$ is open.
2. Assume A closed and B compact. Show $A + B$ is closed.
3. Is the result in 2) still hold if we assume only A and B closed?

5 Ultrametric spaces

A metric d on a set E is said to be *ultrametric* if it satisfied $d(x,\ z) \leq \max(d(x,\ y),\ d(y,\ z))$ forall $x,\ y$ and z.

1. Let (E, d) an ultrametric space, show that:
   (a) Every triangle is isosceles;
   (b) Every point inside a ball is its center;
   (c) Intersecting balls are contained in each other;
   (d) All balls are both open and closed sets;
   (e) A sequence (x_n) is a Cauchy sequence if and only if $d(x_n,\ x_{n+1})$ tend to 0.
2. Let X be a set. For $x\ y \in X$, we let $d(x,\ y) = 0$ if $x = y$ and $d(x,\ y) = 1$ if $x \neq y$. Show that d is an ultrametric distance on X.
3. ***p*-adique distance**. Let p be a prime. Let consider the p-adique valuation $v_p : \mathbb{Q} \to [0,\ +\infty]$ defined as follows : for $a \in \mathbb{Z}^*$, $v_p(a)$ is the power of p occurring in the prime factorization of a, then $v_p(a/b) := v_p(a) - v_p(b)$; finally $v(0) := +\infty$. For $x,\ y \in \mathbb{Q}$, we define $d_p(x,\ y) = p^{-v_p(x-y)}$.
   Show that d_p is an ultrametric distance on $\mathbb{Q}$. Compare this distance with the standard distance. Show that any p-adique neighborhood of 0 is dense in $\mathbb{Q}$ for the standard distance.

4. **Formal series** Let $E = \mathbb{K}[[X]]$ be the ring of formal series with coefficients in a field $\mathbb{K}$ (the elements of E are series of type $\sum_{n\geq 0} a_n X^n$ avec $a_0,\ a_1,\ \dots \in \mathbb{K}$). We consider the valuation $v:\ E \to [0,\ +\infty]$ defined by

$$v(0) := +\infty \text{ and if } S = \sum_{n\geq 0} a_n X^n \neq 0,\ v(S) := \min\{n \in \mathbb{N}/a_n \neq 0\}.$$

For $S,\ T \in E$, we define $d(S,\ T) = 2^{-v(S-T)}$. Show that d is an ultrametric distance on E.

6 Hyperplanes and linear forms

Let $(E, ||.||)$ be a normed vector space.

1. Show that two non-zero linear forms define the same hyperplane if and only if they are proportional.
2. Prove that a linear form is continuous if and only if it's kernel is closed.
3. Show that E is infinite-dimensional if and only if it has a non-continuous linear form.

Let $L : E \to \mathbb{R}$ be a non-zero linear form and $H := \ker L$ it associated hyperplane.

1. Show that $E - H$ is dense in E and H is connected.
2. Show that if L is continuous then $E - H$ has exactly two connected components.
3. Now suppose L not continuous (so $dimE = \infty$)
   (a) Prove that H is dense.
   (b) Deduce that $\{x \in E|\ L(x) = 1\}$ is dense.
   (c) Prove that $E - H$ is connected.

7 Norms on function space

Let $E = \mathscr{C}^0([0,1], \mathbb{R})$ endowed with $N_\infty(f) = \sup_{x\in[0,1]} |f(x)|$.

1. Consider $N_p(f) := (\int_0^1 |f(x)|^p dx)^{1/p}$ pour $p = 1, 2$. Show that N_1, N_2, N_∞ are norms on E and are not equivalents.
2. What is the closure (for each norm) of the subspace $\mathscr{P}$ of polynomial functions ?
3. Show that $P \in (\mathscr{P}, N_\infty) \mapsto P' \in (\mathscr{P}, N_\infty)$ is not continuous.

8 Dini's theorem

Let X be a compact space and $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence of functions in $C(X, \mathbb{R})$. If $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some $f \in C(X, \mathbb{R})$., then the convergence is uniform, i.e., $\|f_n - f\|_\infty \to 0$.

9 The circle

Let $\mathbb{T}$ be the quotient group of $\mathbb{R}$ by $\mathbb{Z}$ and $p : \mathbb{R} \to \mathbb{T}$ the quotient map. Let $\mathbb{S}^1 = \{z \in \mathbb{C},\ |z| = 1\}$ and $j : \mathbb{S}^1 \to \mathbb{C}$ the canonical inclusion.

a) Show $\mathbb{T}$ and $\mathbb{S}^1$ are homeomorphic.

b) Does the standard distance on $\mathbb{R}$ induce a distance on $\mathbb{T}$?

c) Show that a continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ induces a continuous $\phi : \mathbb{T} \to \mathbb{R}$.

10 Distance on the sphere

We denote by $\|.\|$ the euclidean norm on $\mathbb{R}^3$,and $\mathbb{S}^2$ the unit sphere. If p and q are points on $\mathbb{S}^2$, we let $C(p,\ q)$ be the set of piecewise C^1 path $\gamma:\ [0,1] \to \mathbb{S}^2$ such that $\gamma(0) = p$ and $\gamma(0) = q$. Set $L(\gamma) = \displaystyle\int_0^1 \|\gamma'(s)\| ds$ the length of such path, and define $d(p,\ q)\ := \inf\{L(\gamma)|\gamma \in C(p,\ q)\}$. We recall that a *great circle* of $\mathbb{S}^2$ is the intersection of $\mathbb{S}^2$ and a plane which passes through the origine of $\mathbb{R}^3$. If p and q two points of $\mathbb{S}^2$ which are not antipodal, we denote by $\gamma_{p,q}$ the shorter of the two arcs of the great circle between this two points. If p and q are antipodal, we denote by $\gamma_{p,q}$ one of the great circle joining p and q.

1. Show that d define a distance on $\mathbb{S}^2$ equivalent to the distance induced by $\mathbb{R}^3$.
2. Show that, for any p and q of $\mathbb{S}^2$, the distance $d(p,\ q)$ is equal to the arc length of $\gamma_{p,q}$.