The subelliptic heat kernel and the associated Brownian motion on the K-contact model spaces

Jing Wang
Joint work with Fabrice Baudoin

Purdue University

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Riemannian manifold

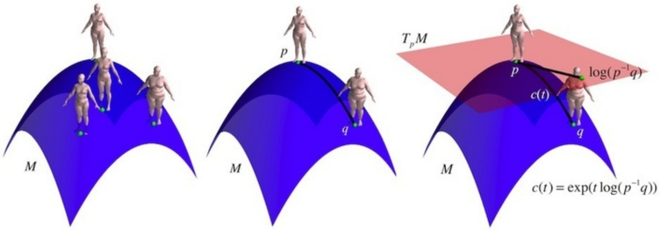

Figure: Riemannian manifolds

Riemannian manifold

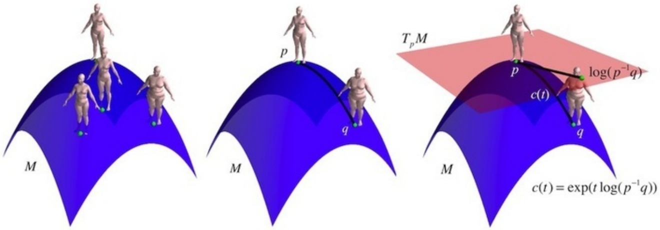

Figure: Riemannian manifolds

We call $(M, T(M), g(\cdot,\cdot)_{T(M)})$ a **Riemannian manifold** where

- M is a differentiable manifold of dimension n,
- $T(M)$ is a tangent bundle of M and $dim T(M) = n$,
- $g(\cdot,\cdot)$ is be a positive definite metric on $T(M)$.

Model spaces of Riemannian manifolds

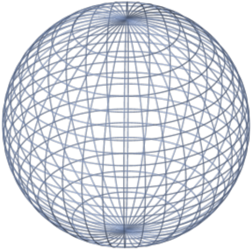

Figure: Sphere

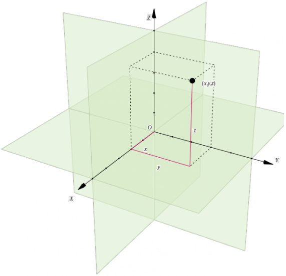

Figure: Euclidean space

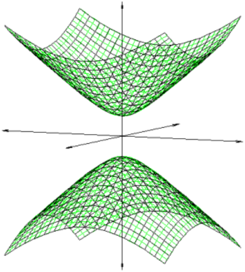

Figure: Hyperbolic space

Subriemannian manifolds: Riemannian manifolds with constraint on admissible direction of movement.

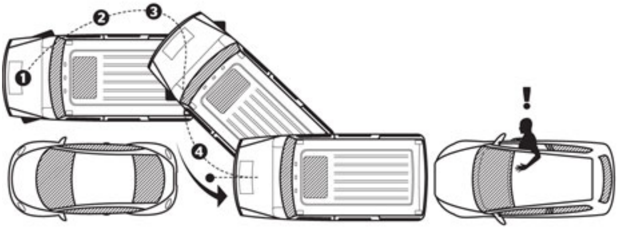

Figure: Parallel parking

Subriemannian manifold

$(M, H(M), g(\cdot,\cdot)_{H(M)})$ is a **Subriemannian manifold** where

- M is a differentiable manifold of dimension n,
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$H(M)$ is referred to as the horizontal space of $T(M)$.

Contact Riemannian manifolds

- A contact Riemannian manifold (M, θ, g) is a Subriemannian manifold of co-dimension 1, endowed with a contact form θ and a metric g.

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- The vertical direction is given by the Reeb vector field.
- A K-contact Riemannian manifold is a contact Riemannian manifold for which the Reeb vector field is a Killing vector field.

K-contact model spaces

- The Heisenberg group $\mathbf{H}^{2n+1} = \{(z_1, \cdots, z_n, t) \in \mathbb{C}^n \times \mathbb{R}\}$ with group law

$$(z, t)(z', t') = (z + z', t + t' + 2\mathbf{Im} z \cdot \overline{z}')$$

is a flat K-contact manifold.

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K-contact model spaces

- The K-contact sphere

$$\mathbb{S}^{2n+1} = \{z = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}, \|z\| = 1\}$$

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$$\mathbb{H}^{2n+1} = \{z \in \mathbb{C}^{n+1}, \|z\|_H^2 = |z_{n+1}|^2 - \sum_{k=1}^{n} |z_k|^2 = 1\}$$

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- $$\mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{CH}^n.$$

Content

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Subelliptic heat kernel on Heisenberg group $\mathbf{H}^{2n+1}$

Theorem (Gaveau, 1976)

The subelliptic heat kernel of the sub-Laplacian $\frac{1}{2}L_{\mathbf{H}^{2n+1}}$ *is given by*

$$p_s(z,t) = \frac{1}{(2\pi s)^{n+1}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau} \right)^n \exp\left(\frac{i\tau t}{s} - \left(\frac{\|z\|^2}{2s} \right) \frac{2\tau}{\tanh 2\tau} \right) d\tau$$

The small time behavior of the subelliptic kernel on $\mathbf{H}^{2n+1}$

[Beals, Gaveau, Greiner, 2000] The small time asymptotics of the kernel are:

- On the degenerated cut-locus, i.e. $(0, 0)$,

$$p_s(0,0) \sim \frac{C_n}{s^{n+1}}$$

- On the cut-locus, i.e. $(0, t)$, $\theta \neq 0$,

$$p_s(0,\theta) \sim \frac{C}{s^{2n}}$$

- Outside of the cut-locus, i.e. (z, t), $z \neq 0$, $t \neq 0$,

$$p_s(z,t) \sim \frac{C'}{s^{n+\frac{1}{2}}}$$

Hopf fibration

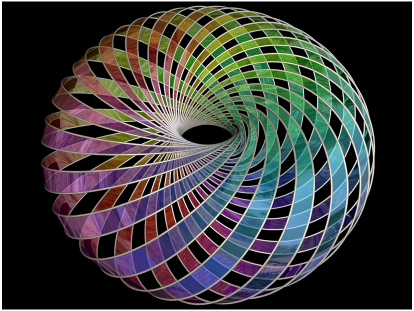

Figure: Hopf fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n, \quad \mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{CH}^n.$$

Subelliptic heat kernel on $SU(2)$

A basis of $\mathfrak{su}(2)$ is formed by the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

for which the following relationships hold

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X.$$

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for which the following relationships hold

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X.$$

The sub-Laplacian is then given by

$$\mathcal{L} = X^2 + Y^2.$$

Subelliptic heat kernel on $SU(2)$

[Baudoin, Bonnefont, 2008] To study $\mathcal{L}$, we will use the cylindric coordinates:

$$\begin{aligned}(r,\theta,z) &\to \exp\left(r\cos\theta X + r\sin\theta Y\right)\exp(zZ)\\ &= \begin{pmatrix} \cos(r)e^{iz} & \sin(r)e^{i(\theta-z)} \\ -\sin(r)e^{-i(\theta-z)} & \cos(r)e^{-iz} \end{pmatrix},\end{aligned}$$

with

$$0 \le r < \frac{\pi}{2},\ \theta \in [0,2\pi],\ z \in [-\pi,\pi].$$

We therefore obtain the radial part of sub-Laplacian

$$\tilde{\mathcal{L}} = \frac{\partial^2}{\partial r^2} + 2\ \text{cotan}\ 2r\frac{\partial}{\partial r} + \tan^2 r\frac{\partial^2}{\partial z^2}$$

Subelliptic heat kernel on $SU(2)$

We have the integral representation of the subelliptic heat kernel:

Proposition (Baudoin, Bonnefont, 2008)

For $t > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$,

$$p_t(r, z) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+iz)^2}{4t}} q_t(\cos r \cosh y) dy,$$

where q_t is the Riemannian heat kernel on S^3

Geometry of $\mathbb{S}^{2n+1}$

Taking into account the Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$

$$(z_1, \cdots, z_n) \to (e^{i\theta} z_1, \cdots, e^{i\theta} z_n).$$

Introduce a new set of coordinates

$$(w_1, \cdots, w_n, \theta) \longrightarrow \left(\frac{w_1 e^{i\theta}}{\sqrt{1+\rho^2}}, \cdots, \frac{w_n e^{i\theta}}{\sqrt{1+\rho^2}}, \frac{e^{i\theta}}{\sqrt{1+\rho^2}} \right),$$

where $\rho = \sqrt{\sum_{j=1}^n |w_j|^2}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $(w_1, \cdots, w_n) \in \mathbb{CP}^n$.

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where $\rho = \sqrt{\sum_{j=1}^n |w_j|^2}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $(w_1, \cdots, w_n) \in \mathbb{CP}^n$. Then

$$T = \frac{\partial}{\partial \theta}.$$

- The associated sub-Laplacian L on $\mathbb{S}^{2n+1}$ is the lift of the Laplacian on $\mathbb{CP}^n$.
- By the symmetries, it's enough to compute the radial part of L with respect to the cylindrical variables (ρ, θ).

$$\tilde{L} = \left(1+\rho^2\right)^2 \frac{\partial^2}{\partial \rho^2} + \left(\frac{(2n-1)(1+\rho^2)}{\rho} + (1+\rho^2)\rho \right) \frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \theta^2}.$$

sub-Laplacian on K-contact manifolds

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Let $\rho = \tan r$, then

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + ((2n-1)\cot r - \tan r)\frac{\partial}{\partial r} + \tan^2 r \frac{\partial^2}{\partial \theta^2}.$$

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- The Riemannian distance δ form the north pole satisfies

$$\cos \delta = \cos r \cos \theta.$$

Spectral decomposition of the subelliptic heat kernel

- K-contact structure $\Leftrightarrow$ T and L commute.
- The idea is to expand $p_t(r,\theta)$ as a Fourier series in θ, by letting

$$p_t(r,\theta) = \sum_{k=-\infty}^{+\infty} e^{ik\theta}\phi_k(t,r),$$

Spectral decomposition of the subelliptic heat kernel

Proposition

For $t > 0$, $r \in [0, \frac{\pi}{2})$, $\theta \in [-\pi, \pi]$, the subelliptic kernel has the following spectral decomposition:

$$p_t(r,\theta) = \frac{\Gamma(n)}{2\pi^{n+1}} \sum_{k=-\infty}^{+\infty} \sum_{m=0}^{+\infty} (2m+|k|+n) \binom{m+|k|+n-1}{n-1} \cdot e^{-\lambda_{m,k} t + ik\theta} (\cos r)^{|k|} P_m^{n-1,|k|}(\cos 2r),$$

where $\lambda_{m,k} = 4m(m+|k|+n) + 2|k|n$ and $P_m^{n-1,|k|}(x)$ is a Jacobi polynomial.

Sub-Laplacian in Cylindric coordinates

- L can also be observed as

$$Lu = \mathbf{div}(\nabla^H u), \quad \forall u \in C^2(M),$$

  where ∇^H is the horizontal gradient.
- Let Δ denote the Laplace-Beltrami operator of the standard Riemannian structure on M,

$$L = \Delta - T^2$$

Since

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and

$$LT = TL \quad \Leftrightarrow \quad \text{K-contact structure}$$

We formally have

$$e^{tL} = e^{-t\frac{\partial^2}{\partial\theta^2}} e^{t\Delta}$$

The subelliptic heat kernel on $\mathbb{S}^{2n+1}$

Let q_t be the Riemannian heat kernel, such that

$$\frac{\partial}{\partial t}(q_t(\cos r \cos \theta)) = \Delta(q_t(\cos r \cos \theta)),$$

we have the integral representation of the subelliptic heat kernel:

Proposition

For $t > 0$, $r \in [0, \pi/2)$, $\theta \in [-\pi, \pi]$,

$$p_t(r, \theta) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+i\theta)^2}{4t}} q_t(\cos r \cosh y) dy.$$

Green function of the conformal sub-Laplacian on $\mathbb{S}^{2n+1}$

We recover the Green function of the conformal sub-Laplacian $-L + n^2$:

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Proposition

The Green function of the conformal sub-Laplacian $-L+n^2$ on $\mathbb{S}^{2n+1}$ is given by

$$G(r,\theta)=\frac{\Gamma\left(\frac{n}{2}\right)^2}{8\pi^{n+1}(1-2\cos r\cos\theta+\cos^2 r)^{n/2}}$$

This coincides with the result by Geller (1980).

Asymptotics of the subelliptic heat kernel in small times

- On the degenerated cut-locus, i.e. $(0, 0)$,

$$p_t(0,0) \sim \frac{C_n}{t^{n+1}}$$

- On the cut-locus, i.e. $(0, \theta)$, $\theta \neq 0$,

$$p_t(0,\theta) \sim \frac{C_n}{t^{2n}}$$

- Outside of the cut-locus, i.e. (r, θ), $r \neq 0$, $\theta \neq 0$,

$$p_t(r,\theta) \sim \frac{C_n}{t^{n+\frac{1}{2}}}$$

The Subriemannian distance

By Léandre's result, the Subriemannian distance $d(r,\theta)$ on $\mathbb{S}^{2n+1}$ is

$$\lim_{t\to 0} t \ln p_t(r,\theta) = -\frac{d^2(r,\theta)}{4}$$

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- For $\theta \in [-\pi, \pi]$,

$$d^2(0, \theta) = 2\pi|\theta| - \theta^2$$

- For $\theta \in [-\pi, \pi]$, $r \in \left(0, \frac{\pi}{2}\right)$,

$$d^2(r, \theta) = \frac{(\varphi(r, \theta) + \theta)^2 \tan^2 r}{\sin^2(\varphi(r, \theta))}$$

where $\varphi(r, \theta)$ is the unique solution in $[-\pi, \pi]$ to the equation

$$\varphi(r, \theta) + \theta = \cos r \sin \varphi(r, \theta) \frac{\arccos(\cos \varphi(r, \theta) \cos r)}{\sqrt{1 - \cos^2 r \cos^2 \varphi(r, \theta)}}.$$

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- For $\theta \in [-\pi,\pi]$, $r \in \left(0,\frac{\pi}{2}\right)$,

$$d^2(r,\theta) = \frac{(\varphi(r,\theta)+\theta)^2 \tan^2 r}{\sin^2(\varphi(r,\theta))}$$

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$$\varphi(r,\theta)+\theta = \cos r \sin\varphi(r,\theta)\frac{\arccos(\cos\varphi(r,\theta)\cos r)}{\sqrt{1-\cos^2 r\cos^2\varphi(r,\theta)}}.$$

In particular, the sub-Riemannian diameter of $\mathbb{S}^{2n+1}$ is π.

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When $n = 1$, $\mathbb{H}^3$ is isomorphic to the Lie group $\mathbf{SL}(2, \mathbb{R})$,

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$$SO(2) \to \mathbf{SL}(2, \mathbb{R}) \to H^2$$

where H^2 is the 2-dimensional hyperbolic space.
A basis of its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is formed by the matrices:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for which the following relations hold

$$[X, Y] = 2Z, \quad [X, Z] = 2Y, \quad [Y, Z] = -2X.$$

The sub-Laplacian hence writes:

$$L = X^2 + Y^2.$$

Subelliptic heat kernel on $\mathbb{H}^3$

[Bonnefont, 2012] By introducing the cylindrical coordinates:

$$(r, \theta, z) \to \exp\left(r \cos\theta X + r \sin\theta Y\right) \exp(zZ)$$

with

$$r > 0,\ \ \theta \in [0, 2\pi],\ \ z \in [-\pi, \pi].$$

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Subelliptic heat kernel on $\mathbb{H}^3$

Proposition (Bonnefont, 2012)

The heat kernel on $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ *is given for* $t>0, r>0, z\in\mathbb{R}$ *by*

$$\tilde{p}_t(r,z) = \frac{1}{4\pi}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r \cosh y)dy$$

$$= \frac{e^{-t}}{(4\pi t)^2}\int_{-\infty}^{+\infty} e^{-\frac{\mathrm{arch}^2(\cosh r\cosh y)-(y-iz)^2}{4t}} \frac{\mathrm{arch}(\cosh r \cosh y)}{\sqrt{\cosh^2 r\cosh^2 y-1}}dy$$

where s_t *is the heat kernel associated with the Lalpacian.*

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where s_t *is the heat kernel associated with the Lalpacian.*

- The subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$ is then just obtained by wrapping the one of $\widetilde{\mathbf{SL}(2,\mathbb{R})}$.

Hopf fibration of $\mathbb{H}^{2n+1}$

By taking into account the symmetries of the fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{CH}^n,$$

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$$(w_1, \cdots, w_n, \theta) \longrightarrow \left(\frac{w_1 e^{i\theta}}{\sqrt{1-\rho^2}}, \cdots, \frac{w_n e^{i\theta}}{\sqrt{1-\rho^2}}, \frac{e^{i\theta}}{\sqrt{1-\rho^2}} \right),$$

where $\rho = \sqrt{\sum_{j=1}^n |w_j|^2}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $w \in \mathbb{CH}^n$.

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where $\rho = \sqrt{\sum_{j=1}^n |w_j|^2}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $w \in \mathbb{CH}^n$. The radial part of the sub-Laplacian on $\mathbb{H}^{2n+1}$ is

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + ((2n-1)\coth r + \tanh r)\frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial \theta^2},$$

where $\rho = \tanh r$.

Sub-Laplacian on $\mathbb{H}^{2n+1}$

On the universal covering of $\widetilde{\mathbb{H}^{2n+1}}$, the Hopf fibration is

$$\mathbb{R}^1 \longrightarrow \widetilde{\mathbb{H}^{2n+1}} \longrightarrow \mathbb{CH}^n.$$

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.Then we can obtain the subelliptic kernel on $\widetilde{\mathbb{H}^{2n+1}}$:

$$p_t^{\widetilde{\mathbb{H}^{2n+1}}}(r,\theta) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-i\theta)^2}{4t}} q_t(\cosh r \cosh y) dy,$$

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where $q_t(\cosh\delta) = \frac{\Gamma(n+1)e^{-n^2 t}}{(2\pi)^{n+1}\sqrt{\pi t}} \int_0^{+\infty} \frac{e^{\frac{\pi^2-u^2}{4t}} \sinh u \sin\frac{\pi u}{2t}}{(\cosh u+\cosh\delta)^{n+1}} du$ is the Riemannian heat kernel associated to the Laplacian $\tilde{\Delta}$ issued from the north pole (Gruet, 1996).

The subelliptic kernel on $\mathbb{H}^{2n+1}$ and its universal covering

Proposition

For $t > 0$, $r \in [0, +\infty)$, $\theta \in (-\infty, +\infty)$, the subelliptic kernel on $\widetilde{\mathbb{H}^{2n+1}}$ is then

$$p_t^{\widetilde{\mathbb{H}^{2n+1}}}(r,\theta) = \frac{\Gamma(n+1)e^{-n^2 t+\frac{\pi^2}{4t}}}{(2\pi)^{n+2}t} \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{e^{\frac{(y-i\theta)^2-u^2}{4t}} \sinh u \sin\left(\frac{\pi u}{2t}\right)}{(\cosh u + \cosh r \cosh y)^{n+1}} du dy.$$

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We can then easily deduce the subelliptic heat kernel on $\mathbb{H}^{2n+1}$:

Proposition

For $t > 0$, $r \in [0, +\infty)$, $\theta \in [-\pi, \pi]$, the subelliptic heat kernel on $\mathbb{H}^{2n+1}$ is given by

$$p_t^{\mathbb{H}^{2n+1}}(r,\theta) = \frac{\Gamma(n+1)e^{-n^2t+\frac{\pi^2}{4t}}}{(2\pi)^{n+2}t} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{e^{\frac{(y-i\theta-2k\pi i)^2-u^2}{4t}} \sinh u \sin\left(\frac{\pi u}{2t}\right)}{(\cosh u + \cosh r \cosh y)^{n+1}} dudy$$

Small time asymptotics of the subelliptic kernel on $\widetilde{\mathbb{H}^{2n+1}}$

- On the degenerated cut-locus, i.e. $(0,0)$,

$$p_t(0,0) \sim \frac{C_n}{t^{n+1}}$$

- On the cut-locus, i.e. $(0,\theta)$, $\theta \neq 0$,

$$p_t(0,\theta) \sim \frac{C_n}{t^{2n}}$$

- Outside of the cut-locus points, i.e. (r,θ), $r \neq 0$, $\theta \neq 0$,

$$p_t(r,\theta) \sim \frac{C_n}{t^{n+\frac{1}{2}}}$$

The Subriemannian distance

By symmetry, the Subriemannian distance from the north pole to any point on $\widetilde{\mathbb{H}^{2n+1}}$ only depends on r and θ. Then

- For $\theta \in \mathbb{R}$,

$$d^2(0,\theta) = 2\pi|\theta| + \theta^2$$

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- For $\theta \in \mathbb{R}$, $r \in (0,+\infty)$,

$$d^2(r,\theta) = \frac{(\varphi(r,\theta)-\theta)^2 \tanh^2 r}{\sin^2(\varphi(r,\theta))}$$

  where $\varphi(r,\theta)$ is the unique solution in $\left(-\arccos\left(\frac{1}{\cosh r}\right), \arccos\left(\frac{1}{\cosh r}\right)\right)$ to the equation

$$\varphi(r,\theta) - \theta = \cosh r \sin\varphi(r,\theta) \frac{\cosh^{-1}(\cosh r \cos\varphi(r,\theta))}{\sqrt{\cosh^2 r \cos^2\varphi(r,\theta) - 1}}.$$

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The above formulas also work for $\mathbb{H}^{2n+1}$ if we restrict θ to $[-\pi,\pi]$.

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Cayley transform of Brownian Motion on $\mathbf{H}^{2n+1}$ to $\mathbb{S}^{2n+1}$

Work in progress

Brownian Motion on the K-contact model spaces

- The Brownian Motion on the Heisenberg group $\mathbf{H}^{2n+1}$ writes: (B_t, β_t, Z_t) where $(B_t, \beta_t) = (B_t^1, \cdots, B_t^n, \beta_t^1, \cdots, \beta_t^n)$ is a Brownian Motions in $\mathbb{R}^{2n}$ and Z_t is given by

$$Z_t = \sum_{i=1}^{n} \int_0^t (B_s^i d\beta_s^i - \beta_s^i dB_s^i)$$

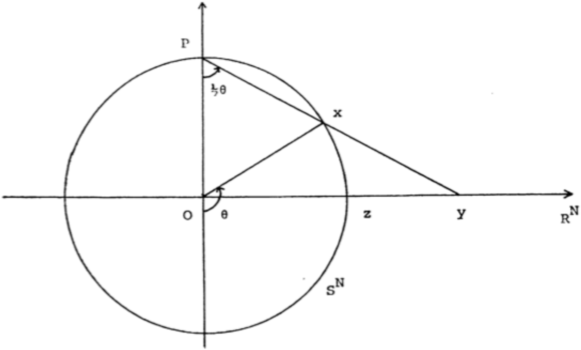
- What about on the other K-contact model spaces $\mathbb{S}^{2n+1}$ and $\mathbb{H}^{2n+1}$?
- One way to understand it is via Cayley transfom.

The Riemannian case: Stereographic projection

Consider the unit sphere S^N and hyperplane

$$R^N = \{y = (y_1, \cdots, y_{N+1}) : y_{N+1} = 0\},$$

The **Stereographic Projection** from the north pole $P = (0, \cdots, 0, 1)$ of S^N maps $y \in R^N$ to $x \in S^N \setminus \{P\}$



Stereographic projection of Brownian Motion on R^n to S^n

- Let $r = \|y\|$ for $y \in \mathbb{R}^N$, and $\tan \frac{1}{2}\theta = r$.
-

$$\frac{1}{2}\Delta_{R^N} = (1+\cos\theta)^2 \left(\frac{1}{2}\Delta_{S^N} + \frac{1}{2}(N-2)\tan\frac{1}{2}\theta \frac{\partial}{\partial\theta} \right)$$

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- The Brownian Motion on Riemannian sphere $BM(S^N)$ conditioned to be at North pole P at time T is generated by

$$\frac{1}{2}\Delta_{S^N} + \frac{1}{2}(N-2)\tan\frac{1}{2}\theta\frac{\partial}{\partial\theta}$$

where T is a random time independent of $BM(S^N)$.

Stereographic projection of Brownian Motion on R^n to S^N

T is a negative exponential distributed random variable with parameter $N(N-2)/8$ and independent of $BM(S^N)$.

Proposition (T.K.Carne, 1985)

$BM(R^N)$ is mapped by stereographic projection to a time-changed version of $BM(S^N)$ conditioned to be at north pole at time T.

Cayley transformation

- Cayley transforms are conformal mappings as analogue of stereographic projection.

Cayley transformation

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-
$$\mathcal{C}_1 : \mathbf{H}^{2n+1} \longrightarrow \mathbb{S}^{2n+1} \setminus \{-e_{n+1}\}$$
  where $-e_{n+1}$ is the south pole of $\mathbb{S}^{2n+1}$.
- $\mathcal{C}_1$ maps the origin on $\mathbf{H}^{2n+1}$ to the north pole on $\mathbb{S}^{2n+1}$. i.e.,
$$\mathcal{C}_1 : 0 \longmapsto e_{n+1}$$

Cayley transformation

- Recall that on $\mathbb{S}^{2n+1}$ we introduced the cylindric coordinates:

$$(\zeta_1, \cdots \zeta_{n+1}) = \left(\frac{w_1 e^{i\theta}}{\sqrt{1+\rho^2}}, \cdots, \frac{w_n e^{i\theta}}{\sqrt{1+\rho^2}}, \frac{e^{i\theta}}{\sqrt{1+\rho^2}} \right),$$

where $w \in \mathbb{CP}^n$, $\rho = \sqrt{\sum_{j=1}^n |w_j|^2} = \tan r_S$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

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  where $w \in \mathbb{CP}^n$, $\rho = \sqrt{\sum_{j=1}^n |w_j|^2} = \tan r_S$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.
- By symmetry we only consider the radial coordinates (r_S, θ) on $\mathbb{S}^{2n+1}$.
- For $(z_1, \cdots, z_n, t) \in \mathbf{H}^{2n+1}$ where $z \in \mathbb{C}^n$, let $r_H = \sqrt{\sum_{j=1}^n |z_j|^2}$, we consider the radial coordinates (r_H, t).

Cayley transformation

Cayley transform gives

$$\mathcal{C}_1(r_H, t) = \left(\frac{\sin r_S}{\sqrt{1 + \cos^2 r_S + 2\cos r_S \cos\theta}}, \frac{\cos r_S \sin\theta}{\sqrt{1 + \cos^2 r_S + 2\cos r_S \cos\theta}} \right)$$

and

$$\begin{aligned} &\mathcal{C}_1^{-1}(\sin r_S, \sin\theta) \\ = &\left(\frac{2r_H}{\sqrt{(1 + r_H^2)^2 + 4t^2}}, \frac{4t}{\sqrt{(1 + r_H^2)^2 + 4t^2}\sqrt{(1 - r_H^2)^2 + 4t^2}} \right) \end{aligned}$$

Cayley transformation

Theorem

For any function $F \in C^\infty(\mathbf{H}^{2n+1})$, *the relation between* $-L_{\mathbf{H}^{2n+1}}$ *and* $-L_{\mathbb{S}^{2n+1}} + n^2$ *via Cayley transform writes:*

$$(-L_{\mathbb{S}^{2n+1}} + n^2)\left(h_{\mathcal{C}}^{-\frac{n}{2}}(F \circ \mathcal{C}_1^{-1})\right) = h_{\mathcal{C}}^{-(\frac{n}{2}+1)}(-L_{\mathbf{H}^{2n+1}}F) \circ \mathcal{C}_1^{-1}$$

where $h_{\mathcal{C}} = 1 + 2\cos r_S \cos\theta + \cos^2 r_S$.

Cayley transformation

Corollary

For any function $f \in C^\infty(\mathbb{S}^{2n+1})$*, we have that*

$$L_{\mathbf{H}^{2n+1}}(f \circ \mathcal{C}_1) = (h_{\mathcal{C}} \circ \mathcal{C}_1)\left(L_{\mathbb{S}^{2n+1}} f + \frac{2\langle \nabla_H h, \nabla_H f\rangle}{h}\right) \circ \mathcal{C}_1$$

where $h_{\mathcal{C}} = 1 + \cos^2 r_S + 2\cos r_S \cos\theta$ *and* $h = h_{\mathcal{C}}^{-\frac{n}{2}}$.

Time changed Doob's transform

Theorem (Time changed Doob's transform)

Let Y_t be the Brownian motion on $\mathbf{H}^{2n+1}$ generated by $\frac{1}{2}L_{\mathbf{H}^{2n+1}}$, and X_t be the Brownian motion on $\mathbb{S}^{2n+1}$ generated by $\frac{1}{2}L_{\mathbb{S}^{2n+1}}$. Then Cayley transformation maps Y_t to a time changed process $X^h_{\mathcal{A}_t}$ with $\mathcal{A}_t = \int_0^t H(Y_s)^{-1}ds$, $H(r_H, t) = \frac{4}{(1+r_H^2)+4t^2}$, i.e.,

$$\mathcal{C}_1(Y_t) = X^h_{\mathcal{A}_t}$$

where X^h_t is X_t conditioned to be at the south pole $-e_{n+1}$ at time T

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$$\mathcal{C}_1(Y_t) = X^h_{\mathcal{A}_t}$$

where X^h_t is X_t conditioned to be at the south pole $-e_{n+1}$ at time T and T is an independent random variable with distribution

$$\mathbb{P}_x\left[T > t\right] = \frac{\int_t^{+\infty} e^{-n^2 s} p_s(0,x)ds}{\int_0^{+\infty} e^{-n^2 t} p_t(0,x)dt}.$$

Outline of the proof

Recall

$$L_{\mathbf{H}^{2n+1}}(f \circ \mathcal{C}_1) = (h_{\mathcal{C}} \circ \mathcal{C}_1)\left(L_{\mathbb{S}^{2n+1}} f + \frac{2\Gamma_{\mathbb{S}^{2n+1}}(h, f)}{h} \right) \circ \mathcal{C}_1$$

- Let

$$L^h f = L_{\mathbb{S}^{2n+1}} f + \frac{2\Gamma_{\mathbb{S}^{2n+1}}(h, f)}{h} = \frac{L_{\mathbb{S}^{2n+1}}(hf)}{h} - n^2 f,$$

and X_t^h and X_t be Markov processes generated by $\frac{1}{2} L^h$ and $\frac{1}{2} L_{\mathbb{S}^{2n+1}}$.

Lemma (Doob's transform)

X_t^h is X_t conditioned to be at the south pole $-e_{n+1}$ at time T, where T is a random time with distribution

$$\mathbb{P}_x[T > t] = \frac{\int_t^{+\infty} e^{-n^2 s} p_s(0, x) ds}{\int_0^{+\infty} e^{-n^2 t} p_t(0, x) dt}.$$

Outline of the proof

- On the other hand

$$L_{\mathbf{H}^{2n+1}}(f \circ \mathcal{C}_1) = (h_{\mathcal{C}} L^h f) \circ \mathcal{C}_1.$$

Let Y_t be the Markov process generated by $\frac{1}{2} L_{\mathbf{H}^{2n+1}}$, then

Lemma

Y_t is mapped by Cayley transform to a time-changed version of X^h-process:

$$X^h_{\mathcal{A}_t} = \mathcal{C}_1(Y_t)$$

with $\mathcal{A}_t = \int_0^t H(Y_s)^{-1} ds$, $H = \frac{4}{(1+r_H^2)+4t^2}$.

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