Large time behavior of solutions to semilinear equations and long term portfolio choice

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joint work with
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Setup

Given $E \subset \mathbb{R}^d$, define

$$\mathfrak{F}[\phi](x) := \frac{1}{2}\sum_{i,j=1}^{d} A_{ij} D_{ij}\phi + \frac{1}{2}\sum_{i,j=1}^{d} \overline{A}_{ij} D_i\phi D_j\phi + \sum_{i=1}^{d} B_i D_i\phi + V,$$

- A is locally elliptic,
- V is unbounded,
- $\underline{\kappa} A \leq \overline{A} \leq \overline{\kappa} A$, for some $0 < \underline{\kappa} \leq \overline{\kappa}$.

Consider the Cauchy problem:

$$\begin{aligned} \partial_t v &= \mathfrak{F}[v], \quad (t,x) \in (0,\infty) \times E, \\ v(0,x) &= v_0(x). \end{aligned} \tag{1}$$

Problems

Q1: Large time behavior of $v(t, \cdot)$ as $t \to \infty$.

Q2: Large time behavior of $v(T - t, X_t)$ as $T \to \infty$.

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Consider the ergodic analogue of (1):

$$\lambda = \mathfrak{F}[v], \quad x \in E. \tag{2}$$

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Consider the ergodic analogue of (1):

$$\lambda = \mathfrak{F}[v], \quad x \in E. \tag{2}$$

A: $\exists(\hat{\lambda}, \hat{v})$ solving (2), set $\hat{w}(t,x) = \hat{\lambda}t + \hat{v}(x)$, Then as $t \to \infty$,

- $(v - \hat{w})(t,\cdot) \to C$;
- $\nabla(v - \hat{w})(t,\cdot) \to 0$, pointwise, $\mathbb{L}^2$.

Easy-to-check conditions are presented for $E = \mathbb{R}^d$ or $\mathbb{S}_d^{++}$.

Merton's problem vs risk-sensitive control

Financial market: risk-free S^0, risky $S = (S^1, \dots, S^n)$:

$$\frac{dS_t^0}{S_t^0} = r(X_t)dt, \quad \frac{dS_t^i}{S_t^i} = r(X_t)dt + dR_t^i.$$

μ and σ of R depend on X, (R, X) is Markov.

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Given $U(w) = w^p/p$, $0 \neq p < 1$,

- Merton's problem:

$$\mathbb{E}\left[(\mathcal{W}_T^\pi)^p/p\right] \to \max;$$

- Risk-sensitive control:

$$\limsup_{T\to\infty} \frac{1}{pT} \log \mathbb{E}\left[(\mathcal{W}_T^\pi)^p\right] \to \max.$$

[Fleming-McEneaney 95], [Bensoussan-Frese-Nagai 98], [Bielecki-Pliska 00], [Fleming-Sheu 02], [Nagai 96, 03], [El-Karoui-Hamadène 03], [Kaise-Sheu 06], [Ichihara 11], [Ichihara-Sheu 13].

[Nagai-Peng 02], [Gerhold-Guasoni-Muhle-Karbe-Schachermayer 12]

Intuitive large time analysis

No dynamic programming principle for risk-sensitive control.

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Define v via

$$u^T(t,w,x) = \frac{w^p}{p} e^{v(T-t,x)}.$$

v is expected to satisfy (1).

Suppose for large T

$$v(T-t,x) \sim \hat{\lambda}(T-t) + \hat{v}(x).$$

Then

$$\frac{1}{pT} \log \mathbb{E}[(\mathcal{W}_T^{\pi^*})^p] \sim \frac{1}{pT}\Big(p \log w + \hat{\lambda}(T-t) + \hat{v}(x)\Big) \to \frac{\hat{\lambda}}{p}.$$

Optimal portfolio

Finite horizon:

$$\pi_t^T = \frac{1}{1-p}\Sigma^{-1}(\mu + \Upsilon \nabla v)(T-t, X_t).$$

Long run:

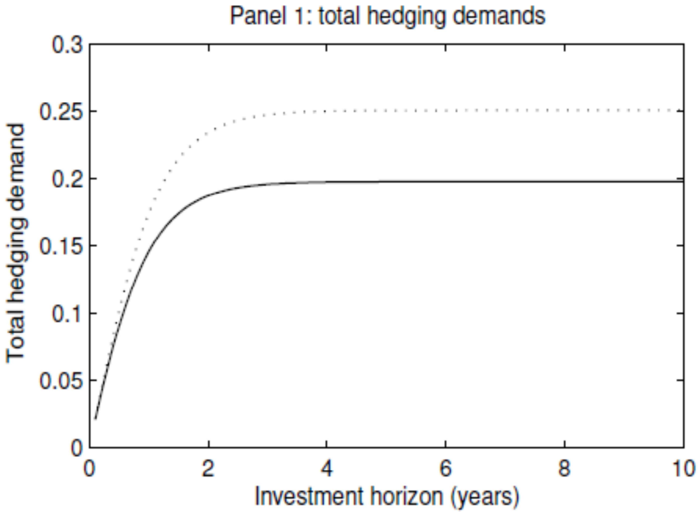
$$\hat{\pi}_t = \frac{1}{1-p}\Sigma^{-1}(\mu + \Upsilon \nabla \hat{v})(X_t).$$

If $\nabla v(T, \cdot) \to \nabla \hat{v}$ as $T \to \infty$, then $\pi^T \to \hat{\pi}$.

Empirical studies

Figure from [Buraschi-Porchia-Trojani 10]

2 risky assets, $\mathbb{S}_2^{++}$ Wishart model for factors, relative risk aversion 8.



Portfolio turnpike

Consider a generic utility U s.t.

$$\frac{U'(w)}{w^{p-1}} \to 1, \quad \text{as } w \uparrow \infty.$$

Market growth over time: $\lim_{T\to\infty} S^0_T = \infty$.

Portfolio turnpike asserts

$$\pi^{1\,T}_s - \pi^{0\,T}_s \to 0, \quad \text{for } s \in [0,t] \text{ as } T \to \infty.$$

[Mossin 68], [Leland 72], [Hakansson 74], [Huberman-Ross 83], [Cox-Huang 92], [Huang-Zariphopoulou 99], [Dybvig-Rogers-Back 99], [Guasoni-Kardaras-Robertson-X. 12].

Portfolio turnpike in factor models

[Guasoni-Kardaras-Robertson-X. 12] proved under mild condition,

$$\lim_{T\to\infty} \mathbb{P}^T \left(\int_0^t (\pi_s^{1\,T} - \pi_s^{0\,T})' \Sigma_s (\pi_s^{1\,T} - \pi_s^{0\,T}) \, ds \geq \epsilon \right) = 0.$$

In order to replace $\mathbb{P}^T$ with $\mathbb{P}$, need

$$\mathbb{P}^T \to \hat{\mathbb{P}} \sim \mathbb{P}, \qquad \text{on } \mathcal{F}_t.$$

Need to show

$$\mathbb{E}^{\hat{\mathbb{P}}} \left[\int_0^t \nabla(v - \hat{v})' \bar{A} \nabla(v - \hat{v})(T - s, X_s) \, ds \right] \to 0.$$

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For multivariate factor models in $\mathbb{R}^d$ and $\mathbb{S}_d^{++}$, we show, for any x, t,

$$\mathbb{P}^x - \lim_{T\to\infty} \int_0^t (\pi_s^{1\,T} - \pi_s^{0\,T})' \Sigma(X_s) (\pi_s^{1\,T} - \pi_s^{0\,T})\, ds = 0;$$

$$\mathbb{P}^x - \lim_{T\to\infty} \int_0^t (\pi_s^{1\,T} - \hat{\pi}_s)' \Sigma(X_s) (\pi_s^{1\,T} - \hat{\pi}_s)\, ds = 0;$$

Large time asymptotics of quadratic BSDE

Set $(Y^T, Z^T) = (v(T - \cdot, X_\cdot), a'\nabla v(T - \cdot, X_\cdot))$.

$$Y_t = v_0(X_T) + \int_t^T \left[Z_s' M(X_s) Z_s + V(X_s) \right] ds - \int_t^T Z_s' dW_s,$$

where $\underline{\kappa} I_d \le M(X_s) \le \overline{\kappa} I_d$.

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Let $(\hat{\lambda}, \hat{v})$ solves the ergodic equation.

Set $(\hat{Y}, \hat{Z}) = (\hat{v}(X_\cdot), a'\nabla \hat{v}(X_\cdot))$. $(\hat{Y}, \hat{Z})$ solves ergodic BSDE:

$$\hat{Y}_t = \hat{Y}_T + \int_t^T \left[\hat{Z}_s' M(X_s) \hat{Z}_s + V(X_s) - \hat{\lambda}\right] ds - \int_t^T \hat{Z}_s' dW_s, \quad \text{for any } t \leq T.$$

[Fuhrman-Hu-Tessitore 09], [Richou 09], [Debussche-Hu-Tessitore 11], [Cohen-Hu 12].

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Set $\mathcal{Y}^T = Y^T - \hat{Y} - \hat{\lambda}(T - \cdot)$ and $\mathcal{Z}^T = Z^T - \hat{Z}$. Our results imply

$$\lim_{T \to \infty} \mathbb{E}^{\hat{\mathbb{P}}^x}\left[\int_0^t \|\mathcal{Z}_s^T\|^2 ds\right] = 0 \quad \text{and} \quad \lim_{T \to \infty} \mathbb{E}^{\hat{\mathbb{P}}^x}\left[\sup_{0 \leq u \leq t} |\mathcal{Y}_u^T - \mathcal{Y}_0^T|\right] = 0.$$

Literature

PDE: [Fathi 98], [Barles-Souganidis 00, 01], [Fujita-Ishii-Loreti 06], [Fujita-Loreti-09]

PDE + stochastic analysis: [Nagai 96, 03], [Ichihara-Sheu 13]

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[Ichihara-Sheu 13] studied the pointwise convergence on $\mathbb{R}^d$ when $A = I_d$.

Our contribution:

- General space E + local ellipticity
  $\Longrightarrow$ treat equations with spatial variables in $\mathbb{S}_d^{++}$;
- $\mathbb{L}^2$ convergence
  $\Longrightarrow$ portfolio turnpikes in multivariate models.

Semilinear ergodic problem

Consider the ergodic problem

$$\lambda = \mathfrak{F}[v].$$

Suppose that

- $\xi' A(x)\xi \geq c(x)|\xi|^2$ for some $c(x) > 0$;
- $\underline{\kappa} A \leq \overline{A} \leq \overline{\kappa} A$ for some $0 < \underline{\kappa} \leq \overline{\kappa}$;
- $\exists \phi_0 \in C^3(E)$ s.t. $\lim_{x \to \partial E} \mathfrak{F}[\phi_0](x) = -\infty$.

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$$\text{Define} \quad \mathcal{L}^\phi = \frac{1}{2} \sum A_{ij} D_{ij} + \sum (B_i + \sum \overline{A}_{ij} D_j \phi) D_i.$$

Let $\mathbb{P}^{\phi,x}$ be the solution to the generalized martingale problem.

[Kaise-Sheu 06], [Ichihara 11] show that

i) $\exists \hat{\lambda}$, ergodic problem admits a classical soln. $\iff \lambda \geq \hat{\lambda}$.

ii) When $\lambda = \hat{\lambda}$, uniqueness holds up to additive constant;

iii) $\mathbb{P}^{\hat{v},x}$ is ergodic with invariant measure $\hat{m}$,

Convergence

Under slightly stronger conditions,

- Cauchy problem admits at least one solution v;
- Comparison result holds.

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Define

$$h(t,x) = v(t,x) - \hat{\lambda} t - \hat{v}(x), \quad (t,x) \in (0,\infty) \times E.$$

We want to show:

- $h(t,\cdot) \to C$?
- $\nabla h(t,\cdot) \to 0$, pointwise and $\mathbb{L}^2$?

An abstract result (intermediate step)

On E, suppose that

- Local ellipticity on A;
- $\underline{\kappa} A(x) \leq \overline{A}(x) \leq \overline{\kappa} A(x)$ for all $x \in E$;
- $\phi_0 \in C^3(E)$, nonnegative, and $\phi_0 \in \mathbb{L}^1(E, \hat{m})$ s.t. $\lim_{x \to \partial E} \mathfrak{F}[\phi_0] = -\infty$;
- Uniform upper bound of h: $\exists T_0$ and $\mathcal{J} \in C(E) \cap \mathbb{L}^1(E, \hat{m})$, s.t.

$$\sup_{T \geq T_0} h(T, x) \leq \mathcal{J}(x), \quad x \in E.$$

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Theorem (Abstract convergence)

Let $\hat{\mathbb{P}}^x$ be the solution to the martingale problem for $\mathcal{L}^{\hat{v}}$.

i) $\lim_{T \to \infty} h(T, \cdot) = C$;

ii) $\lim_{T \to \infty} \nabla h(T, \cdot) = 0$;

iii) $\lim_{T \to \infty} \mathbb{E}^{\hat{\mathbb{P}}^x} \left[\int_0^t (\nabla h)' \overline{A} \nabla h(T - s, X_s)\, ds \right] = 0$ *for any* x, t;

iv) $\lim_{T \to \infty} \mathbb{E}^{\hat{\mathbb{P}}^x} \left[\sup_{0 \le u \le t} |h(T, x) - h(T - u, X_u)| \right] = 0$ *for any* x, t.

All previous convergence are local uniformly in E.

$\mathbb{R}^d$ case

$E = \mathbb{R}^d$. The following growth assumptions are satisfied:

i) A bounded, B has at most linear growth;

ii) $\exists \beta_1 \in \mathbb{R}$, $C_1 > 0$,

$$B(x) \cdot x \le -\beta_1 |x|^2 + C_1;$$

iii) $\exists \gamma_1, \gamma_2 \in \mathbb{R}$, $C_2 > 0$,

$$-\gamma_2 |x|^2 - C_2 \le V(x) \le -\gamma_1 |x|^2 + C_2;$$

iv) $\max\{\beta_1, \gamma_1\} > 0$. Additionally

   a) When $\beta_1 \le 0$ and $\gamma_1 > 0$, $\exists \alpha_2, C_3 > 0$, s.t. $x' A(x) x \ge \alpha_2 |x|^2 - C_3$;

   b) When $\gamma_1 < 0, \beta_1 > 0$, $\beta_1^2 + 2\overline{\kappa}\alpha_1\gamma_1 > 0$;

   c) When $\beta_1 > 0$ and $\gamma_1 \ge 0$, no additional assumptions.

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c) When $\beta_1 > 0$ and $\gamma_1 \geq 0$, no additional assumptions.

The growth assumptions help to identify:

$$\text{Lyapunov function}: \quad \phi_0(x) = (c/2)|x|^2 \quad \text{for some } c > 0.$$

$$\text{Uniform bound for } h: \quad \mathcal{J}(x) = J(1 + |x|^2) \quad \text{for some } J > 0.$$

Financial model

W: $d-$dim BM, B: $n-$dim BM.

$$dR_t = \mu(X_t)dt + \sigma(X_t)\rho(X_t)dW_t + \sigma(X_t)\overline{\rho}(X_t)dB_t,$$
$$dX_t = b(X_t)dt + a(X_t)dW_t,$$

where ρ is the correlation matrix.

$$A = aa', \quad \overline{A} = a(I_d - q\rho'\rho)a', \quad B = b - q\Upsilon'\Sigma^{-1}\mu, \quad V = pr - \frac{q}{2}\mu'\Sigma^{-1}\mu.$$

Observe

- $1 - q \leq \underline{\kappa} \leq \overline{\kappa} \leq 1$ when $p < 0$;
- $1 \leq \underline{\kappa} \leq \overline{\kappa} \leq 1 - q$ when $0 < p < 1$;
- $V > 0$ when $0 < p < 1$; $V < 0$ when $p < 0$.

Long term investment

Theorem

Suppose

- *local ellipticity of A;*
- *growth restrictions;*

Then the following statements hold:

i) $\lim_{T\to\infty} \pi_t^T(\cdot) = \hat{\pi}_t(\cdot)$;

ii) $\mathbb{P}^x - \lim_{T\to\infty} \int_0^t (\pi_s^{1,T} - \pi_s^T)'\Sigma(\pi_s^{1,T} - \pi_u^T)'\, du = 0$;

iii) $\mathbb{P}^x - \lim_{T\to\infty} \int_0^t (\pi_s^{1,T} - \hat{\pi}_s)'\Sigma(\pi_s^{1,T} - \hat{\pi}_u)'\, du = 0$.

Linear Gaussian model

$$\begin{aligned} dR_t &= (\mu_0 + \mu_1 X_t)dt + \sigma\rho\, dW_t + \sigma\bar{\rho}\, dB_t, \\ dX_t &= bX_t dt + a\, dW_t, \\ r(X_t) &= r. \end{aligned}$$

Here $a, \sigma\sigma'$ are positive definite.

Growth assumptions:

i) When $p < 0$, no additional assumption;

ii) When $0 < p < 1$, $b - q\Upsilon'\Sigma^{-1}\mu_1$ is stable and $\beta_1^2 + 2\overline{\kappa}\alpha_1\gamma_1 > 0$:
- β_1: mean reverting speed of $b - q\Upsilon'\Sigma^{-1}\mu_1$;
- $\overline{\kappa}$ depends on $\rho'\rho$ and p: $p\uparrow, \rho'\rho\uparrow \implies \overline{\kappa}\uparrow$;
- γ_1 depends on p and $\mu_1'\Sigma^{-1}\mu_1$: $p\uparrow$, $\mu_1'\Sigma^{-1}\mu_1\uparrow \implies \gamma_1\uparrow$.

Convergence hold: mean reverting speed large, risk aversion large, market not too complete, Sharpe ratio not too large.

No large leverage in optimal portfolio!

$\mathbb{S}_d^{++}$ case

Given $F, G : \mathbb{S}_d^{++} \to \mathbb{M}_d$ and $B : \mathbb{S}_d^{++} \to \mathbb{S}_d$. Consider

$$dX_t = B(X_t)dt + F(X_t)dW_t G(X_t) + G(X_t)' dW_t' F(X_t)'.$$

Example: Wishart process

$$dX_t = (LL' + KX_t + X_t K')dt + \sqrt{X_t} dW_t \Lambda' + \Lambda dW_t' \sqrt{X_t}.$$

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The previous SDE can be rewritten as

$$dX_t^{ij} = B_{ij}(X_t)dt + \sum_{k,l=1}^{d} a_{kl}^{ij}\, dW_t^{kl}, \quad \text{where } a_{kl}^{ij} = F^{ik} G^{lj} + F^{jk} G^{li}.$$

Define

$$\mathfrak{F}[\phi](x) = \frac{1}{2} \sum_{i,j,k,l}^{d} \mathrm{Tr}\left(a^{ij} (a^{kl})' \right) D^2 \phi_{(ij),(kl)} + \frac{1}{2} \sum_{i,j,k,l}^{d} D\phi_{(ij)} \overline{A}_{(ij),(kl)} D\phi_{(kl)} + \sum_{i,j}^{d} b_{ij} D\phi_{(ij)} + V$$

where

$$D\phi_{(ij)} = \frac{\partial \phi}{\partial x_{ij}} \quad \text{and} \quad D^2 \phi_{(ij),(kl)} = \frac{\partial^2 \phi}{\partial x_{ij} \partial x_{kl}}.$$

Convergence problem

Consider Cauchy problem and its ergodic analogue

$$\partial_t v = \mathfrak{F}[v], \quad (t,x) \in (0,\infty) \times \mathbb{S}_d^{++}, \quad v(0,x) = v_0(x),$$

$$\lambda = \mathfrak{F}[v], \quad x \in \mathbb{S}_d^{++}.$$

Still define

$$h(t,x) = v(t,x) - \hat{\lambda} t - \hat{v}(x), \quad (t,x) \in [0,\infty) \times \mathbb{S}_d^{++}.$$

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Two types of boundaries for $\mathbb{S}_d^{++}$:

- $\{x \in \mathbb{S}^d \, : \, \|x\| = \infty\}$;
- $\{x \in \mathbb{S}^d \, : \, \det(x) = 0\}$.

Growth assumptions

Set $f = FF'$ and $g = G'G$. When $\|x\|$ is large,

i) B and $\mathrm{Tr}(f)\mathrm{Tr}(g)$ have at most linear growth;

ii) There exist $\beta_1 \in \mathbb{R}$ and $C_1 > 0$ such that

$$\mathrm{Tr}(B'x) \leq -\beta_1 \|x\|^2 + C_1, ;$$

iii) There exist constants $\gamma_1, \gamma_2 \in \mathbb{R}$ and $C_2 > 0$ such that

$$-\gamma_2 \|x\| - C_2 \leq V \leq -\gamma_1 \|x\| + C_2;$$

iv) $\max\{\beta_1, \gamma_1\} > 0$. Additionally

a) When $\beta_1 \leq 0$ and $\gamma_1 > 0$, these exists $\alpha_2, C_3 > 0$ such that

$$\mathrm{Tr}(fxgx) \geq \alpha_2 \|x\|^3 - C_3;$$

b) When $\beta_1 > 0$ and $\gamma_1 < 0$, $\beta_1^2 + 16\overline{\kappa}\alpha_1\gamma_1 > 0$;

c) When $\beta_1 > 0$ and $\gamma_1 \geq 0$, no additional condition.

Another growth assumption

When $\det(x)$ is small:

Define, for $\delta > 0$ and $x \in \mathbb{S}_d^{++}$,

$$H_\delta(x) = \mathrm{Tr}(bx^{-1}) - (1+\delta)\mathrm{Tr}(fx^{-1}gx^{-1}) - \mathrm{Tr}(fx^{-1})\mathrm{Tr}(gx^{-1}).$$

Assumption:
$\exists \epsilon > 0$ s.t. $H_\epsilon(x)$ is uniformly bounded from below on $\mathbb{S}_d^{++}$ and

$$\lim_{\det(x)\downarrow 0} H_\epsilon(x) = \infty.$$

[Mayerhofer-Pfaffel-Stelzer 11]:

H_0 bounded from below $\implies$ existence of $(X_t)_{t\in\mathbb{R}_+}$ to SDE.

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$$H_\delta(x) = \mathrm{Tr}(bx^{-1}) - (1+\delta)\mathrm{Tr}(fx^{-1}gx^{-1}) - \mathrm{Tr}(fx^{-1})\mathrm{Tr}(gx^{-1}).$$

Assumption:
$\exists \epsilon > 0$ s.t. $H_\epsilon(x)$ is uniformly bounded from below on $\mathbb{S}_d^{++}$ and

$$\lim_{\det(x)\downarrow 0} H_\epsilon(x) = \infty.$$

[Mayerhofer-Pfaffel-Stelzer 11]:

H_0 bounded from below $\implies$ existence of $(X_t)_{t\in\mathbb{R}_+}$ to SDE.

Up to constants, Lyapunov function

$$\phi_0(x) \sim -\log(\det(x)) \qquad \text{when } \det(x) \text{ is small;}$$
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A slightly stronger assumption $\implies$ the uniform upper bound of h:

Wishart factor model

W: $d \times d$ matrix BM, B: $d-$dim BM.

$$dX_t = (LL' + KX_t + X_t K')dt + \sqrt{X_t} dW_t \Lambda' + \Lambda dW_t' \sqrt{X_t},$$
$$dS_t = \text{diag}(S_t)(r(X_t)1 + \mu X_t dt + \sigma \sqrt{X_t} dZ_t),$$

where $Z_t = W_t \rho(X_t) + \sqrt{1 - \rho' \rho(X_t)} B_t$

[Buraschi-Porchia-Trojani 10], [Richter 13], [Hata-Sekine 13].

σ is of full rank, $\exists \nu \in \mathbb{R}^d$ s.t. $\mu = \sigma \sigma' \nu$.

i) $\Lambda\Lambda' > 0$;

ii) $LL' > (d+1)\Lambda\Lambda'$; $(LL' \geq (d+1)\Lambda\Lambda')$

iii) When $p < 0$, no additional assumption;

iv) When $0 < p < 1$, $K - q\Lambda\rho\nu'(x)\sigma$ is stable,

$\beta_1^2 + 16\overline{\kappa} d^{3/2} \|\Lambda\Lambda'\| \gamma_1 > 0$.

Intuition is the same as in $\mathbb{R}^d$ case!

A special case

$E = (\alpha, \beta) \subset \mathbb{R}$.

$$\partial_t v = \frac{1}{2} D^2 v + \frac{1}{2} (Dv)^2 + BDv + V, \quad v(0, x) = 0,$$

$$\lambda = \frac{1}{2} D^2 \hat{v} + \frac{1}{2} (D\hat{v})^2 + BD\hat{v} + V.$$

Then $u = e^v$ solves linear equation

$$\partial_t u = \frac{1}{2} D^2 u + BDu + Vu.$$

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Moreover,

$$H(t,x) := \frac{u(t,x)}{e^{\lambda t} \hat{u}(x)} \quad \text{satisfies}$$

$$\partial_t H = \mathcal{L}^{\hat{v}} H, \quad H(0,x) = e^{-\hat{v}(x)}, \quad \text{where } \mathcal{L}^{\hat{v}} = \frac{1}{2} D^2 + (B + D\hat{v}) D.$$

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Feynman-Kac implies

$$H(T,x) = \mathbb{E}^{\hat{\mathbb{P}}^x}\left[e^{-\hat{v}(X_T)}\right].$$

A special case cont

Consider

$$\lambda = \mathcal{L}v.$$

Theorem (Pinsky, Chap. 4)

Under mild conditions, $\exists \hat{\lambda} \in \mathbb{R}$, *such that*

i) *Ergodic equation admits solution* (λ, v) *only when* $\lambda \geq \hat{\lambda}$;

ii) *When* $\lambda = \hat{\lambda}$, $\hat{v}$ *is unique up to a constant;*

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iii) *When* $\lambda = \hat{\lambda}$, $\mathbb{P}^{\hat{v},x}$ *is ergodic with invariant measure* $\hat{m}$.

Now, if $e^{-\hat{v}} \in \mathbb{L}^1(E, \hat{m})$, then

$$H(T, x) = \mathbb{E}^{\hat{\mathbb{P}}^x}\left[e^{-\hat{v}(X_T)}\right] \to \int_E e^{-\hat{v}(x)}\hat{m}(dx), \quad \text{as } T \to \infty.$$

cf. [Pinchover 92, 04].

General case:

$$h(t,x) = v(t,x) - \hat{\lambda} t - \hat{v}(x) \quad \text{satisfies}$$

$$\partial_t h = \mathcal{L}^{\hat{v}} h + \frac{1}{2} \sum_{i,j}^{d} \overline{A}_{ij} D_i h D_j h,$$

$$h(0,x) = -\hat{v}(x).$$

If there is enough integrability,

$$\frac{1}{\underline{\kappa}} \log \mathbb{E}^{\hat{\mathbb{P}}^x} \left[e^{-\underline{\kappa} \hat{v}(X_T)} \right] \leq h(T,x) \leq \frac{1}{\overline{\kappa}} \log \mathbb{E}^{\hat{\mathbb{P}}^x} \left[e^{-\overline{\kappa} \hat{v}(X_T)} \right].$$

General case:

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Difficulties:

- $\hat{v}$ is not bounded;
- $e^{-\underline{\kappa}\hat{v}} \in \mathbb{L}^1(E, \hat{m})$, but $e^{-\overline{\kappa}\hat{v}} \in \mathbb{L}^1(E, \hat{m})$???
- No control on $\hat{m}$ in multivariate diffusion;
- Only gives

$$\underline{C} \leq \liminf_{T\to\infty} h(T,x) \leq \limsup_{T\to\infty} h(T,x) \leq \overline{C}.$$

Proof

Let

$$f^{t,T}(x) := \frac{1}{2}\mathbb{E}^{\hat{\mathbb{P}}^x}\left[\int_0^t (\nabla h)'\overline{A}\nabla h(T-s, X_s)\, ds\right].$$

1. If $\sup_{T\geq T_0} h(T,x) \leq \mathcal{J}(x)$ for some $\mathcal{J} \in C(E) \cap \mathbb{L}^1(E, \hat{m})$. Then

$$\lim_{T\to\infty}\int_E f^{t,T}(x)\hat{m}(dx) = 0, \quad \text{for any } t.$$

2. Show $\{f^{t,T}(\cdot); T > t + T_0\}$ is uniformly bounded and equicontinuous on any compact subdomain of E.
3. The previous two combined imply

$$\lim_{T\to\infty} f^{t,T}(x) = 0 \quad \text{local uniformly.}$$

Uniform upper bound of h

Let v be the solution to the Cauchy problem. Define

$$\mathcal{L}^{v,T-t} := \frac{1}{2}\sum A_{ij}D_{ij} + \sum \left(B_i + \sum \overline{A}_{ij}D_j v(T-t,x)\right) D_i$$

and $\mathbb{P}^{T,x}$ be the solution to the martingale problem.

$$h(T,x) \leq -\mathbb{E}^{\mathbb{P}^{T,x}}[\hat{v}(X_T)], \qquad \text{for any } T \geq 0.$$

Lemma

If there exists $\delta > 1, \alpha > 0$ such that

$$M = \sup_{x\in E} \left(\mathfrak{F}[\delta\phi_0] + \alpha(\delta\phi_0 - \hat{v})\right) < \infty.$$

Then

$$-\mathbb{E}^{\mathbb{P}^{T,x}}[\hat{v}(X_T)] \leq K(1+\phi_0(x) - \hat{v}(x)), \qquad \textit{for any } T > 0.$$

Conclusion

1. We clarify assumptions which lead to convergence:
   - Local ellipticity + $\underline{\kappa}A \leq \overline{A} \leq \overline{\kappa}A$ for $0 < \underline{\kappa} \leq \overline{\kappa}$;
   - Growth restrictions $\implies$ Lyapunov function + uniform bounds on h.

2. Can be extended to other cases;

3. Give examples for ergodic theorem for nonlinear expectation.

   How about non-Markovian BSDE?

Thanks for your attention!