Recent advances in empirical regression schemes for BSDEs

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Main issue: for solving BSDE using empirical regressions, how to *optimally* tune

- ✓ the number of discretization dates,
- ✓ the approximation spaces,
- ✓ the number of simulations?

Agenda

- ✓ Different discrete-time Dynamic Programming (DP) Equations:
  - **ODP**: One step forward DP equation [BT04][GLW05]
  - **MDP**: Multi-step forward DPE ($\approx$ [BD07] without Picard iterations)
  - **Mal.MDP**: Malliavin MDP (alternative representation of Z)

  Pros and cons: error norms and stability, independent clouds of simulations, basis functions, managing constraints...
- ✓ Handling irregular/quadratic BSDE
- ✓ Generic variance reductions
- ✓ Conclusion, perspectives, works in progress, open questions

BSDE Setting

Generalized BSDE with *fixed terminal time* T:

$$\mathbf{Y_t} = \xi + \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{f}(\mathbf{s}, \mathbf{Y_s}, \mathbf{Z_s})\mathbf{ds} - \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{Z_s dW_s} - (\mathbf{L_T} - \mathbf{L_t}),$$

under various assumptions, for instance:

- ✓ driving noise = Brownian Motion W and Poisson measure,
- ✓ L martingale orthogonal to W,
- ✓ quadratic driver, ...

but under **Markovian assumptions**: $\mathbf{f}(\mathbf{s}, \omega, \mathbf{y}, \mathbf{z}) = \mathbf{f}(\mathbf{s}, \mathbf{X_s}, \mathbf{y}, \mathbf{z})$, $\xi = \mathbf{g}(\mathbf{X_T})$, X is a jump-diffusion ➠ $\mathbf{Y_t} = \mathbf{u}(\mathbf{t}, \mathbf{X_t}), \mathbf{Z_t} = \nabla\mathbf{u}(\mathbf{t}, \mathbf{X_t})\sigma(\mathbf{t}, \mathbf{X_t})$.

Multidimensional: $X \in \mathbb{R}^d$, $Y \in \mathbb{R}$, $Z \in \mathbb{R}^q$.

Simulating BSDE = 2 problems:

1. computing u and ∇u (hard)
2. simulate the path of X (easy)

Conditional expectations representations

$$Y_t = \mathbb{E}^{\mathcal{F}_t}\Big(\xi + \int_t^T f(s, X_s, Y_s, Z_s)ds\Big).$$

Solving the BSDE requires the computation of nested conditional expectations.

Advantages of the empirical approach:

✓ black box algorithm (no need to know the model):
input = model simulations ➠ output = BSDE solutions.

✓ uniform controls w.r.t. the model, models may be degenerate, machine learning techniques. But presumably too conservative estimates (worst-case).

Overview of global error decomposition:

$$\text{quadratic error} \le \underbrace{\text{discretization error}}_{\underset{N\to+\infty}{\longrightarrow} 0} + \underbrace{\text{approximation error}}_{\underset{K\to+\infty}{\longrightarrow} 0, \quad \underset{N\to+\infty}{\longrightarrow} +\infty}$$

$$+ \underbrace{\text{statistical error}}_{\underset{M\to+\infty}{\longrightarrow} 0, \quad \underset{N\to+\infty}{\longrightarrow} +\infty, \quad \underset{K\to+\infty}{\longrightarrow} +\infty} + \underbrace{\text{interdependency error}}_{\underset{M\to+\infty}{\longrightarrow} 0, \quad \underset{N\to+\infty}{\longrightarrow} +\infty, \quad \underset{K\to+\infty}{\longrightarrow} +\infty}.$$

TIME DISCRETIZATION OF $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - (L_T - L_t)$

Standard discretization along deterministic time grid $\pi := \{0 = t_0 < \ldots < t_N = T\}$:

- ✓ $(i+1)$-th time-step is $\Delta_i = t_{i+1} - t_i$;
- ✓ mesh size $|\pi| := \max_{0 \le i < N} \Delta_i$;
- ✓ related Brownian motion increments $\Delta W_i := W_{t_{i+1}} - W_{t_i}$.

Discrete time BSDE (Y, Z):

$$\begin{cases} Y_i & = \mathbb{E}_i \left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i \right), \quad 0 \le i < N, \\ \Delta_i Z_i & = \mathbb{E}_i \left(Y_{i+1} \Delta W_i^\top \right), \quad 0 \le i < N, \\ Y_N & = \xi, \end{cases}$$

where $\mathbb{E}_i(\cdot) := \mathbb{E}(\cdot | \mathcal{F}_{t_i})$.

- ✓ Because of $f_i(\mathbf{Y_{i+1}}, \ldots)$, **explicit scheme.**
- ✓ Differences with implicit scheme have not been really studied.

1) ODP scheme

One-step forward **D**ynamic **P**rogramming equation

$$\begin{cases} Y_i &= \mathbb{E}_i\left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i\right), \quad 0 \le i < N, \qquad Y_N = \xi. \\ \Delta_i Z_i &= \mathbb{E}_i\left(Y_{i+1}\Delta W_i^\top\right), \quad 0 \le i < N. \end{cases} \qquad \textbf{(ODP)}$$

X could be approximated by a path-wise approximation (the Euler scheme for SDE).

We do not discuss here the L_2-convergence of the discrete appproximation $(X_i, Y_i, Z_i)_{0\le i<N}$ towards the limit $(X_t, Y_t, Z_t)_{0\le t\le T}$, see [Zha04][GLW05][BT04].

✓ Usually, the L_2-rate is equal to $N^{\frac{1}{2}}$.

✓ The speed $N^{\frac{1}{2}}$ is achieved by taking appropriate choice of times grids according to fractional smoothness of ξ: see [GM10][GGG12]...

2) MDP scheme

From

$$Y_i = \mathbb{E}_i \left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i\right), \quad \Delta_i Z_i = \mathbb{E}_i \left(Y_{i+1}\Delta W_i^\top\right),$$

replugging Y_{i+1} and iterating over i until N gives the **M**ulti-Step forward **D**ynamic **P**rogramming equation:

$$\begin{cases} Y_i & = \mathbb{E}_i \left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k\right), \\ \Delta_i Z_i & = \mathbb{E}_i \left([\xi + \sum_{k=i+1}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k]\Delta W_i^\top\right). \end{cases} \qquad \textbf{(MDP)}$$

If no extra approximation is incorporated, then **ODP$\Longleftrightarrow$MDP.**

? differences regarding regression schemes?

3) Malliavin scheme: Mal.MDP

Based on Ma-Zhang representation theorem [MZ02] (Bismut type formula for the gradient): **under ellipticity conditions**, we have

$$\mathbf{Z_t} = \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{g}(\mathbf{X_T})\mathbf{I_{t,T}} + \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{f}(\mathbf{s}, \mathbf{X_s}, \mathbf{Y_s}, \mathbf{Z_s})\mathbf{I_{t,s}}\mathbf{ds}\right)$$

for some explicit stochastic integral $I_{t,s}$.

✓ In the case $X = BM$, $I_{t,s} = \frac{(W_s - W_t)^\top}{s-t}$.

✓ In general, $|I_{t,s}|_{L_2} \le c(s-t)^{-\frac{1}{2}}$ (singular weights but integrable).

✓ Leads to a discrete-time version

$$\begin{cases} Y_i &= \mathbb{E}_i\left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k\right), \\ Z_i &= \mathbb{E}_i\left(\xi I_{i,N} + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k) I_{i,k}\Delta_k\right). \end{cases} \qquad (\textbf{Mal.MDP})$$

✓ Discretization error analysis performed by Turkedjiev (2013): usual convergence rates.

To sum up, 3 DP equations

$$\begin{cases} Y_i &= \mathbb{E}_i\left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i\right), \quad 0 \le i < N, \qquad Y_N = \xi. \\ \Delta_i Z_i &= \mathbb{E}_i\left(Y_{i+1}\Delta W_i^\top\right), \quad 0 \le i < N. \end{cases} \qquad \textbf{(ODP)}$$

$$\begin{cases} Y_i &= \mathbb{E}_i\left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k\right), \\ \Delta_i Z_i &= \mathbb{E}_i\left([\xi + \sum_{k=i+1}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k]\Delta W_i^\top\right). \end{cases} \qquad \textbf{(MDP)}$$

$$\begin{cases} Y_i &= \mathbb{E}_i\left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k\right), \\ Z_i &= \mathbb{E}_i\left(\xi I_{i,N} + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)I_{i,k}\Delta_k\right). \end{cases} \qquad \textbf{(Mal.MDP)}$$

First clear and intuitive differences:

- ✓ ODP computationally simpler (iteration and memory)
- ✓ Mal.MDP requires more to simulate under ellipticity

? Other differences: L_2-stability, empirical regression versions,...

Standing assumptions. f is (locally in time) Lipschitz in (y, z):
$\mathbf{|f_k(y,z) - f_k(y',z')| \le \frac{L_f}{(T-t_k)^{(1-\theta)/2}}(|y-y'| + |z-z'|)}$ for some $\theta \in (0, 1]$.

1ST COMPARISON: STABILITY RESULTS

Computation of theoretical regression functions (so far, no empirical projections, $M = +\infty$)

We are given a **family of projection operator** $\mathcal{P}_i^Y, \mathcal{P}_i^Z$ from $\mathbb{L}_2(\mathcal{F}_T)$ into a linear vector space of $\mathbb{L}_2(\mathcal{F}_{t_i})$:

$$\mathcal{S}_\mathbf{i} = \mathrm{Span}(\mathbf{\Phi_{k,i}} : \mathbf{1} \le \mathbf{k} \le \mathbf{K}_\mathcal{F}),$$

where $\Phi_{k,i} \in \mathbb{L}_2(\mathcal{F}_{t_i})$. **Property (usual iterated projection).** Projecting the r.v. U on $\mathcal{S}_i$ or its conditional expectation $\mathbb{E}_i(U)$ is the same.

PROOF.

$$\begin{aligned}\mathcal{P}_\mathbf{i}(\mathbf{U}) := \arg\inf_{U_i \in \mathcal{S}_i} |U - U_i|^2_{\mathbb{L}_2(\mathbb{P})} &= \arg\inf_{U_i \in \mathcal{S}_i} (|U - \mathbb{E}_i(U)|^2_{\mathbb{L}_2(\mathbb{P})} + |\mathbb{E}_i(U) - U_i|^2_{\mathbb{L}_2(\mathbb{P})}) \\ &= \arg\inf_{U_i \in \mathcal{S}_i} |\mathbb{E}_i(U) - U_i|^2_{\mathbb{L}_2(\mathbb{P})} = \mathcal{P}_\mathbf{i}(\mathbb{E}_\mathbf{i}(\mathbf{U})).\end{aligned}$$

□

Propagation of approximation errors in DP equations

$$\begin{cases} \widehat{Y}_i &= \mathcal{P}_{\mathbf{i}}^{\mathbf{Y}}\big(\widehat{Y}_{i+1} + \Delta_i f_i(\widehat{Y}_{i+1}, \widehat{Z}_i)\big), \\ \Delta_i \widehat{Z}_{i+1} &= \mathcal{P}_{\mathbf{i}}^{\mathbf{Z}}\big(\widehat{Y}_{i+1}\Delta W_i^\top\big), \end{cases} \qquad (\textbf{ODP+regression } M = +\infty)$$

$$\begin{cases} \check{Y}_i &= \mathcal{P}_{\mathbf{i}}^{\mathbf{Y}}\big(\xi + \sum_{k=i}^{N-1} \Delta_k f_k(\check{Y}_{k+1}, \check{Z}_k)\big), \\ \Delta_i \check{Z}_i &= \mathcal{P}_{\mathbf{i}}^{\mathbf{Z}}\big([\xi + \sum_{k=i+1}^{N-1} \Delta_k f_k(\check{Y}_{k+1}, \check{Z}_k)]\Delta W_i^\top\big). \end{cases} \qquad (\textbf{MDP+regression } M = +\infty)$$

Proposition. Consider the $L_\infty(L_2(\Omega), \{0 : N-1\})$ and $L_1(L_2(\Omega), \{0 : N-1\})$ norms: $\mathcal{E}_\infty(U) = \sup_{0\le i\le N-1} \mathbb{E}|U_i|^2$ and $\mathcal{E}_1(U) = \sum_{i=0}^{N-1} \mathbb{E}|U_i|^2\Delta_i$.

$$\textbf{(ODP)}\ \mathcal{E}_\infty(\widehat{Y} - Y) + \mathcal{E}_1(\widehat{Z} - Z) \le_c \sum_{\mathbf{0\le i\le N-1}} \mathbb{E}|Y_i - \mathcal{P}_i^Y(Y_i)|^2 + \sum_{i=0}^{N-1} \mathbb{E}|Z_i - \mathcal{P}_i^Z(Z_i)|^2\Delta_i,$$

$$\textbf{(MDP)} \begin{cases} \mathcal{E}_1(\check{Y} - Y) + \mathcal{E}_1(\check{Z} - Z) \le_c \sum_{\mathbf{0\le i\le N-1}} \mathbb{E}|Y_i - \mathcal{P}_i^Y(Y_i)|^2\boldsymbol{\Delta_i} + \sum_{i=0}^{N-1} \mathbb{E}|Z_i - \mathcal{P}_i^Z(Z_i)|^2\Delta_i, \\ \mathbb{E}|\check{\mathbf{Y}}_\mathbf{i} - \mathbf{Y}_\mathbf{i}|^2 \le_\mathbf{c} \mathbb{E}|\mathbf{Y}_\mathbf{i} - \mathcal{P}_\mathbf{i}^\mathbf{Y}(\mathbf{Y}_\mathbf{i})|^2 + \mathcal{E}_\mathbf{1}(\check{\mathbf{Y}} - \mathbf{Y}) + \mathcal{E}_\mathbf{1}(\check{\mathbf{Z}} - \mathbf{Z}). \end{cases}$$

$$\textbf{(Mal. MDP)}\begin{cases}\widetilde{Y}_i &= \mathcal{P}^{\mathbf{Y}}_{\mathbf{i}}\big(\xi + \sum_{k=i}^{N-1}\Delta_k f_k(\widetilde{Y}_{k+1},\widetilde{Z}_k)\big),\\ \widetilde{Z}_i &= \mathcal{P}^{\mathbf{Z}}_{\mathbf{i}}\big(\xi I_{i,N} + \sum_{k=i+1}^{N-1}\Delta_k f_k(\widetilde{Y}_{k+1},\widetilde{Z}_k)I_{i,k}\big).\end{cases}$$

Proposition. We have

$$\mathcal{E}_1(\widetilde{Y}-Y)+\mathcal{E}_1(\widetilde{Z}-Z) \le_c \sum_{\mathbf{0\le i\le N-1}} \mathbb{E}|Y_i - \mathcal{P}^Y_i(Y_i)|^2 \boldsymbol{\Delta}_{\mathbf{i}} + \sum_{i=0}^{N-1}\mathbb{E}|Z_i - \mathcal{P}^Z_i(Z_i)|^2\Delta_i,$$

$$\mathbb{E}|\widetilde{Y}_i - Y_i|^2 \le_c \mathbb{E}|Y_i - \mathcal{P}^Y_i(Y_i)|^2 + \mathcal{E}_1(\widetilde{Y}-Y) + \mathcal{E}_1(\widetilde{Z}-Z),$$

$$\mathbb{E}|\widetilde{\mathbf{Z}}_{\mathbf{i}} - \mathbf{Z}_{\mathbf{i}}|^2 \le_{\mathbf{c}} \mathbb{E}|\mathbf{Z}_{\mathbf{i}} - \mathcal{P}^{\mathbf{Z}}_{\mathbf{i}}(\mathbf{Z}_{\mathbf{i}})|^2 + \sum_{\mathbf{j=i+1}}^{\mathbf{N-1}} \frac{\mathbb{E}|\mathbf{Y}_{\mathbf{j}} - \mathcal{P}^{\mathbf{Y}}_{\mathbf{j}}(\mathbf{Y}_{\mathbf{j}})|^2 + \mathbb{E}|\mathbf{Z}_{\mathbf{j}} - \mathcal{P}^{\mathbf{Z}}_{\mathbf{j}}(\mathbf{Z}_{\mathbf{j}})|^2}{\sqrt{\mathbf{t}_{\mathbf{j}} - \mathbf{t}_{\mathbf{i}}}}\boldsymbol{\Delta}_{\mathbf{j}}.$$

Proof. Used unusual Gronwall lemma with non-bounded weights:

Lemma. Let $\alpha \ge 0, \beta > 0$. Assume for two sequences $\{u_l\}_{l\ge k}$ and $\{w_l\}_{l\ge k}$

$$u_j \le w_j + C\sum_{l=j+1}^{N-1}\frac{u_l\Delta_l}{(T-t_l)^{\frac12-\beta}(t_l-t_j)^{\frac12-\alpha}}.$$

Then, $u_j \le C'w_j + C'\sum_{l=j+1}^{N-1}\frac{w_l\Delta_l}{(T-t_l)^{\frac12-\beta}(t_l-t_j)^{\frac12-\alpha}}$. □

To sum up

✓ Different DP equations measure the error on (Y, Z) in different norms:

- average over i
- uniformly in i

✓ Different DP equations average differently the projection approximation error:

ODP $<$ **MDP** $<$ **Mal. MDP**

✓ But **Mal.MDP** requires ellipticity and additional weights to simulate.

2nd comparison: ordinary least-squares (OLS) (regression method)

To simplify the problem, computation of $\mathbb{E}(H|X = x)$ with

- ✓ $X \in \mathbb{R}^d$ (random observation=design) and $H \in \mathbb{R}$ (random response),
- ✓ data sample $D_N := (H_i, X_i)_{1\le i\le N}$, **i.i.d. realizations** of (H, X),
- ✓ approximation vector space: $\mathcal{F} = \mathrm{Span}(\boldsymbol{\Phi}_\mathbf{k} : \mathbf{1} \le \mathbf{k} \le \mathbf{K}_\mathcal{F})$,

➥ Nonparametric estimation of $m(x) = \mathbb{E}(H|X = x)$: **[Gyorfi *et al.* 2002, Tsybakov 2009, . . .]**, machine learning (distribution-free estimates).

- ✓ Empirical Regression function: $\mathbf{m_N} = \arg \inf_{\mathbf{f}\in\mathcal{F}} \frac{\mathbf{1}}{\mathbf{N}} \sum_{\mathbf{i=1}}^{\mathbf{N}} |\mathbf{H_i} - \mathbf{f}(\mathbf{X_i})|^{\mathbf{2}}.$

Theorem ([GKKW02]). Assume $\Sigma^2 = \sup_{x\in\mathbb{R}^{d_x}} \mathbb{V}\mathrm{ar}(H|X = x) < +\infty$. Then

$$\mathbb{E}\left[\|m_N - m\|^2_{L_2(\mu_N)}\right] \le \underbrace{\Sigma^2 \frac{K_\mathcal{F}}{N}}_{\text{estimation error}} + \underbrace{\inf_{f\in\mathcal{F}} \mathbb{E}\left[(f(X) - m(X))^2\right]}_{\text{approximation error}}.$$

Consequences: what to expect for the OLS+DP equations?

▷ For instance, in the MDP scheme, the global error estimates
$\mathcal{E}_1(Y-\check{Y})+\mathcal{E}_1(Z-\check{Z}) \leq_c \sum_{0\leq i\leq N-1}\mathbb{E}|Y_i-\mathcal{P}_i^Y(Y_i)|^2\Delta_i+\sum_{i=0}^{N-1}\mathbb{E}|Z_i-\mathcal{P}_i^Z(Z_i)|^2\Delta_i$
become (*in the best case*) for Y

$$\sum_{\mathbf{i=0}}^{\mathbf{N-1}}\mathbb{E}\Big(\frac{\mathbf{1}}{\mathbf{M}}\sum_{\mathbf{m=1}}^{\mathbf{M}}(\mathbf{y_i^M}(\mathbf{X_i^{i,m}})-\mathbf{y_i}(\mathbf{X_i^{i,m}}))^{\mathbf{2}}\Big)\mathbf{\Delta_i} \leq_{\mathbf{c}} \sum_{\mathbf{i=0}}^{\mathbf{N-1}}\left(\frac{\mathbf{K_i^Y}}{\mathbf{M}}+\inf_{\mathbf{f}\in\mathcal{F}_\mathbf{i}^\mathbf{Y}}\mathbb{E}|\mathbf{y_i}(\mathbf{X_i})-\mathbf{f}(\mathbf{X_i})|^{\mathbf{2}}\right)\mathbf{\Delta_i}$$

and similarly for Z.

Parameters tuning:

✓ If $y_i \in C^k$: using local polynomials of deg. $k-1$ on hypercubes of size δ gives

- app. error: $\inf_{f\in\mathcal{F}_i^Y}\mathbb{E}|y_i(X_i)-f(X_i)|^2 \leq c\delta^{2k}$ + tail-truncation error.
- $K_i^Y \sim \delta^{-d}$.

✓ To get N^{-1} for global error, $\delta \sim \mathbf{N^{-1/(2k)}}$ and $\mathbf{M} \sim \mathbf{NK_i^Y} \sim \mathbf{N^{1+d/(2k)}}$

✓ **Complexity** $\sim \mathbf{NM} \sim \mathbf{N^{2+d/(2k)}}$: trade-off dimension/smoothness

▷ As a comparison, using ODP, the global error estimates read (for Y)

$$\sum_{i=0}^{N-1} \mathbb{E}\Big(\frac{1}{M}\sum_{m=1}^{M}(y_i^M(X_i^{i,m}) - y_i(X_i^{i,m}))^2\Big) \le_c \sum_{i=0}^{N-1}\Big(\frac{K_i^Y}{M} + \inf_{f\in\mathcal{F}_i^Y}\mathbb{E}|y_i(X_i) - f(X_i)|^2 + \dots\Big)$$

Parameters tuning

With local polynomials of deg. $k-1$ on hypercubes of size δ and C^k-solution y_i:

✓ app. error: $\inf_{f\in\mathcal{F}_i^Y}\mathbb{E}|y_i(X_i) - f(X_i)|^2 \le c\delta^{2k}$ + tail-truncation error.

✓ $K_i^Y \sim \delta^{-d}$

✓ To get N^{-1} for global error, $\delta \sim N^{-\mathbf{2}/(2k)}$ and $M \sim N^{\mathbf{2}} K_i^Y \sim N^{\mathbf{2}+\mathbf{2}d/(2k)}$

➡ **Similar to double the dimension!! Huge impact.**

✓ Complexity $\sim NM \sim N^{1+\mathbf{2}+\mathbf{2}d/(2k)}$.

✓ The trade-off accuracy/complexity is worse compared to MDP.

What does "best case" means?

- ✓ Simulations used for regression at i have to be independent of other ones (no interdependency error).
  - ▸ N independent cloud of simulations
  - ▸ worse complexity $\sim N^{\mathbf{2}}M \sim N^{3+d/(2k)}$ for MDP or $N^{4+\mathbf{2}d/(2k)}$ for ODP
- ✓ What is the *interdependency cost* of having a unique cloud of independent simulations for all regressions at once? Depends on DP equations.

Theorem (uniform large/small deviation estimate).
Let $T_B\mathcal{F} := \{-B \vee f(\cdot) \wedge B : f \in \text{vector space } \mathcal{F}\}$ the truncated $\mathcal{F}$.
Then, there is a universal constant $c > 0$ s.t. for any $\mathcal{X}_1, \dots, \mathcal{X}_M$ i.i.d.r.v.

$$\mathbb{E}\Big[\sup_{g\in T_B\mathcal{F}}\Big(\int_{\mathbb{R}^d} g^2(x)\mathbb{P}\circ\mathcal{X}_1^{-1}(dx) - \frac{\mathbf{2}}{M}\sum_{m=1}^{M} g^2(\mathcal{X}_m)\Big)_+\Big] \le cB^2\frac{(\dim(\mathcal{F})+1)\log(cM)}{M},$$

$$\mathbb{E}\Big[\sup_{g\in T_B\mathcal{F}}\Big(\int_{\mathbb{R}^d} g^2(x)\mathbb{P}\circ\mathcal{X}_1^{-1}(dx) - \frac{\mathbf{1}}{M}\sum_{m=1}^{M} g^2(\mathcal{X}_m)\Big)_+\Big] \le cB^2\sqrt{\frac{(\dim(\mathcal{F})+1)\log(cM)}{M}}.$$

Since $\dim(\mathcal{F})/M$ should be $\sim N^{-1}$, it may much deteriorates the global error.

The case of quadratic driver

Principle

- ✓ plugging of a priori PDE estimates to force the Lipschitzianity
- ✓ transfer of space irregularity to time singularity

Assumptions

- ✓ For a given constant $c \geq 0$,

$$|f(t,x,y,z)| \leq c\,(1+|y|+|z|^2),$$
$$|f(t,x,y,z)-f(t,x,y',z')| \leq c\,(1+|z|+|z'|)(|y-y'|+|z-z'|)$$

for any $(t,x,y,y',z,z') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$.

- ✓ The terminal function g is θ-Hölder continuous and bounded.

Theorem ([DG06]). The associated semi-linear PDE is s.t.
$$\mathbf{(T-t)^{(1-\theta)/2}|\nabla u(t,x)\sigma(t,x)| \leq C_u}, \quad \forall \mathbf{(t,x) \in [0,T)\times\mathbb{R}^d}.$$

Corollary. Define the new driver

$$\bar{f}(t,x,y,z) := f\big(t,x,y,\varphi_t(z_1)\ \dots,\varphi_t(z_d)\big)$$

where $\varphi_t : \zeta \in \mathbb{R} \mapsto \varphi_t(\zeta) = \text{sign}(\zeta)\min\big(|\zeta|, \frac{C_u}{(T-t)^{(1-\theta)/2}}\big)$.

Then $\bar{f}(t, X_t, Y_t, Z_t) = f(t, X_t, Y_t, Z_t)$.

➠ equivalent to solve the BSDE with driver f or $\bar{f}$. in practice $C_u =?$

BSDE with globally Lipschitz driver locally in time. The driver $\bar{f}$ is now globally Lipschitz in y, z with a time-dependent constant:

$$|\bar{f}(t,x,y,z) - \bar{f}(t,x,y',z')| \le \frac{L_f}{(T-t)^{(1-\theta)/2}}(|y-y'| + |z-z'|),$$

for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q$.

All previous discussions and error estimates apply to that setting.

In that analysis, the time grid can be of the form $t_k = T - T(1-k/N)^{1/\theta_\pi}$ with $\theta_\pi \in (0,1]$, like for fractional smoothness conditions.

Variance reductions

- ✓ Usually, the variance of OLS-ODP scheme is lower than OLS-MDP.
- ✓ Variance reduction techniques are complementary tools to speed-up any OLS schemes.

Two ways:

- ✓ taking a proxy (inspired by what is done in PDE)
- ✓ preliminary control variates (for automatic and data-driven improvement)

Numerical tests in progress (or ask Plamen!)

Shifting the driver around a proxy

Assumption (from user *expertise*). The BSDE solution (Y_t, Z_t) is close to $(v(t, X_t), \nabla v(t, X_t)\sigma(t, X_t))$, where v is explicit.

Then, v captures a significant part of the solution and it remains to solve numerically the BSDE residual $(Y_t^0, Z_t^0) := (Y_t - v(t, X_t), Z_t - \nabla v(t, X_t)\sigma(t, X_t))$ with data

- ✓ terminal function: $g(.) - v(T, .)$
- ✓ driver:

$$f^0(t, x, y, z) := f(t, x, y + v(t, x), z + \nabla v(t, x)\sigma(t, x)) - \partial_t v(t, x) - \mathcal{L}v(t, x).$$

Example. If $g(x) = (x - K)_+$ and the driver f comes from the two-interest rates BSDE [Ber95][EPQ97], take for v the Black-Scholes price with a given volatility and a given interest rate.

Example. v can be the solution with zero-driver $\partial_t v(t, x) + \mathcal{L}v(t, x) = 0$. It is known [GM10] that the time-regularity Z^0 behaves better than Z as $t \to T$.

Using Preliminary Control Variates

Generic method for speeding-up the regression computations.

We explain it in the context $\mathbb{E}(H|X)$ with $H = h(U)$.

Assumption. Some regression functions $\mathbb{E}(P_k(U)|X) = m_k(X)$ are known (called PCV): w.l.o.g. $\forall \mathbf{1} \leq \mathbf{k} \leq \mathbf{K}_{\text{pcv}} : \mathbb{E}[\mathbf{P_k}(\mathbf{U})|\mathbf{X}] = \mathbf{0}$.

▷ Heuristics about PCV

✓ No modification of the regression function: for any α

$$\mathbb{E}(H - \sum_{k=1}^{K_{\text{pcv}}} \alpha_k P_k(U)|X) = m(X).$$

✓ Variance reduction:

$$\begin{aligned}(\widehat{\alpha}_k)_{1\leq k\leq K_{\text{pcv}}} &= \arg\inf_{\alpha} \mathbb{E}\left[\left|H - \alpha \cdot P(U)\right|^2\right] \\ &= \arg\inf_{\alpha} \mathbb{V}\text{ar}\big(H - \alpha \cdot P(U)\big) = \arg\inf_{\alpha} \mathbb{E}\left[\mathbb{V}\text{ar}(\mathbf{H} - \alpha \cdot \mathbf{P}(\mathbf{U})|\mathbf{X})\right].\end{aligned}$$

Exemples of PCV inspired by stochastic processes

▷ **Localized functions $P_k(\cdot)$**

✓ **Dimension $d_z = d_x = 1$ and $\bigwedge$-functions.** Consider a Brownian motion W, and let $X := W_t$, $U := (W_t, W_T)$ for $0 < t < T$. Suppose we are interesting to compute $\mathbb{E}[h(W_T)|W_t]$ for some function h and for $t \leq T$. For $K_{\text{pcv}} = 2l + 1$, define

$$\mathbf{p^1_{x_k,\Delta}(x) = \left(1 - \left|\frac{x - x_k}{\Delta}\right|\right)_+,} \qquad \mathbf{x_k = (k - l - 1)\Delta,} \qquad \mathbf{\Delta = \frac{2\sqrt{T}}{l+1}}.$$

Define
$$\mathbf{P_k(W_t, W_T) := p^1_{x_k,\Delta}(W_T)} - \underbrace{\mathbb{E}\left[p^1_{x_k,\Delta}(W_T)|W_t\right]}_{\text{explicit formula}}.$$

✓ **Dimension $d_z = d_x > 1$ and $\bigwedge$-functions.** Immediate extension to BM $W = (W^1, \ldots, W^d)$:

$$\mathbf{P_{k_1,\ldots,k_d}(W_t, W_T) = \prod_{i=1}^{d} p^1_{x_{k_i},\Delta}(W^i_T) - \prod_{i=1}^{d} \mathbb{E}\left[p^1_{x_{k_i},\Delta}(W^i_T)|W^i_t\right]}.$$

▷ Non-localized functions $P_k(\cdot)$: polynomials and martingales

✓ Let W be a scalar **Brownian Motion** and let $(H_k)_k$ be the Hermite polynomials: set $X = W_t, U = (W_t, W_T)$ and

$$\mathbf{P_k(U) = T^{k/2} H_k\left(\frac{W_T}{\sqrt{T}}\right) - t^{k/2} H_k\left(\frac{W_t}{\sqrt{t}}\right).}$$

Straightforward multidimensional extension.

✓ Let N be a **Poisson process** and let $(C_k)_k$ be the Charlier polynomials: set $X = N_t, U = (N_t, N_T)$ and

$$\mathbf{P_k(U) = C_k\left(N_T, T\right) - \left(\frac{t}{T}\right)^k C_k\left(N_t, t\right).}$$

✓ For many other distributions and stochastic processes (**affine processes, processes with quadratic diffusion coefficients, Lévy-driven SDEs with affine vector fields...**), see **[Schoutens '01, Cucheiro *et al.* '12]**

✓ ...

The PCV empirical regression algorithm

Define

✓ the **PCV parameter** set $\mathcal{A}$ (non empty closed convex),

✓ **PCV functions**: $\mathcal{G}^{\mathcal{A}} = \left\{\sum_{k=1}^{K_{\rm pcv}} \alpha_k P_k : \alpha \in \mathcal{A}\right\}$ (non empty closed convex),

✓ the **PCV-modified response**: $H^{\alpha} = H - \sum_{k=1}^{K_{\rm pcv}} \alpha_k P_k$ for $\alpha \in \mathcal{A}$.

▷ **Two-steps algorithm:**

Step 1. Variance Reduction: $(\widetilde{\alpha}_k)_{1\leq k\leq K_{\rm pcv}} = \arg\inf_{\alpha\in\mathcal{A}} \frac{1}{N}\sum_{i=1}^{N}\left|H_i - \sum_{k=1}^{K_{\rm pcv}} \alpha_k P_k(U_i)\right|^2.$

Step 2. Least Squares Regression:

$$(\widetilde{\beta}_k)_{1\leq k\leq K_{\mathcal{F}}} = \arg\inf_{(\beta_k)_k} \frac{1}{N}\sum_{i=1}^{N}\left|H_i - \sum_{k=1}^{K_{\rm pcv}} \widetilde{\alpha}_k P_k(U_i) - \sum_{k=1}^{K_{\mathcal{F}}} \beta_k \Phi_k(X_i)\right|^2.$$

➠ set $\widetilde{m}_N = \sum_{k=1}^{K_{\mathcal{F}}} \widetilde{\beta}_k \Phi_k.$

Technical assumption

- ✓ $\|P_k\|_\infty \le 1$ and $\exists L \ge 1 : \|h\|_\infty \le L$.
- ✓ We choose $\mathcal{A} := \{\alpha \in \mathbb{R}^{K_{\mathrm{pcv}}} : \sum_{i=1}^{K_{\mathrm{pcv}}} |\alpha_i| \le L\}$.

Theorem. Denote by $\widehat{\alpha}$ the optimal PCV parameter and set $\Sigma^2(\widehat{\alpha}) = \sup_{x\in\mathbb{R}^{d_x}} \mathbb{V}\mathrm{ar}(H^{\widehat{\alpha}}|X = x) < +\infty$. Then, for any $\rho > 0$,

$$\mathbb{E}\left[\|\widetilde{\mathbf{m}}_{\mathbf{N}} - \mathbf{m}\|^{\mathbf{2}}_{\mu_{\mathbf{N}}}\right] \le (\mathbf{1}+\rho^{-\mathbf{1}})\mathbf{L}^{\mathbf{4}}\left\{\frac{\mathbf{c_1} + (\mathbf{c_2} + \mathbf{c_3}\log(\mathbf{N}))(\mathbf{K}_{\mathrm{pcv}} + \mathbf{1})}{\mathbf{N}}\right\}$$
$$+ (1+\rho)\frac{K_{\mathcal{F}}}{N}\boldsymbol{\Sigma}^{\mathbf{2}}(\widehat{\alpha}) + (1+\rho)\mathbb{E}\left\{\inf_{f\in\mathcal{F}_N}\|f - m\|^2_{\mu_N}\right\}$$

for some universal constants c_1, c_2 et c_3.

- ✓ We still achieve the best approximation error up to the factor $(1+\rho)$.
- ✓ Reduction of the estimation error : $\boldsymbol{\Sigma}^{\mathbf{2}}(\mathbf{0}) \Rightarrow \boldsymbol{\Sigma}^{\mathbf{2}}(\widehat{\alpha})$.
- ✓ Additional term (error estimation on $\widetilde{\alpha}$): $\frac{\mathbf{c_1}}{\mathbf{N}} + \frac{(\mathbf{c_2}+\mathbf{c_3}\log(\mathbf{N}))(\mathbf{K}_{\mathrm{pcv}}+\mathbf{1})}{\mathbf{N}}$.
- ✓ Better choice: $K_{\mathrm{pcv}} \ll K_{\mathcal{F}}$.

Optimal variance for piecewise constant functions

Theorem. If Φ_k are piecewise constants on statistically equivalent blocks (containing approximately the same number of data) or approximately equi-probabilistic blocks (defined by a constant $c_I \geq 1$), then

$$\begin{aligned}\mathbb{E}[\|\widetilde{m}_N - m\|^2_{\mu_N}] \leq &(1+\rho^{-1})L^4 \left\{\frac{c_1 + (c_2 + c_3 \log(N))(K_{\text{pcv}} + 1)}{N}\right\} \\ &+ (1+\rho)c_I \frac{K_{\mathcal{F}}}{N} \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\mathbb{V}\text{ar}(\mathbf{H}^\alpha | \mathbf{X})\right] \\ &+ (1+\rho)\mathbb{E}\left\{\inf_{\Phi \in \mathcal{F}_N} \|\Phi - m\|^2_{\mu_N}\right\}.\end{aligned}$$

Example in dimension $d_x = 2$

- **Goal:** estimate $m(x) = \mathbb{E}[h(W_2, B_2)|W_1 = x, B_1 = x]$ where

$$\mathbf{h(W_2, B_2) = e^{-\frac{W_2^2+B_2^2+\rho W_2 B_2}{2}}} \qquad \text{with } \rho = \mathbf{0.5}.$$

- **Model:** $U = (W_1, B_1, W_2, B_2)$ with (W, B) BM.

- **PCV:** choose $K_{\text{pcv}} = (2l + 1)(2l + 1)$ and set $\Psi(x) = (1 - |x|)_+$and define

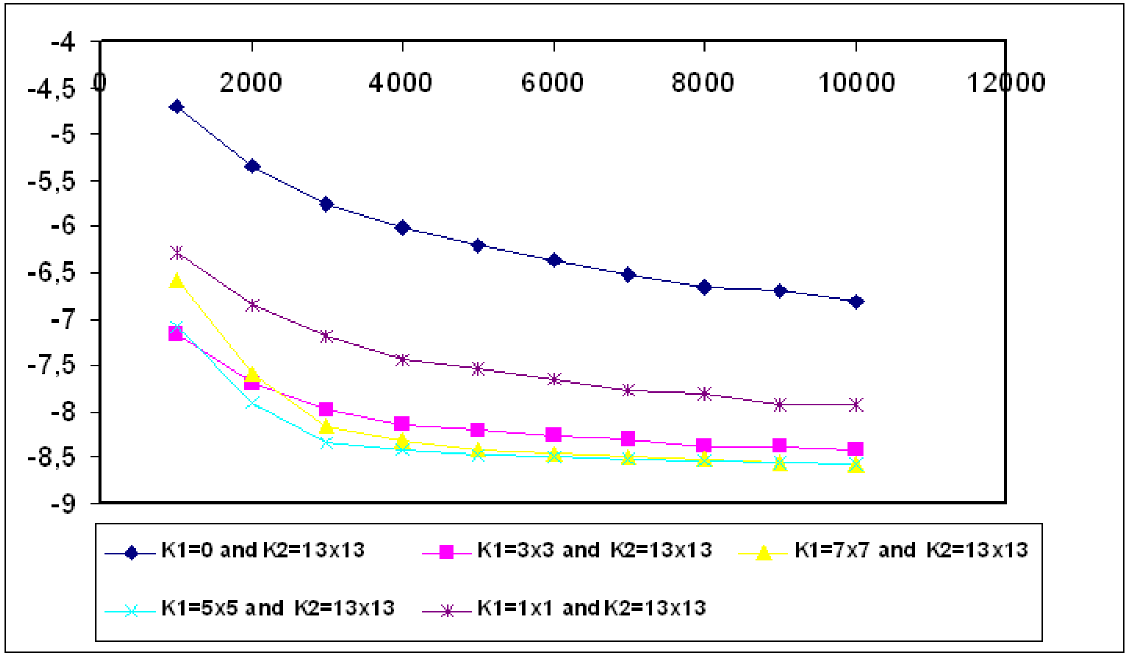
$$\mathbf{P_k(W_1, W_2) = \Psi(\frac{W_2 - (k - l - 1)\Delta}{\Delta}) - \mathbb{E}[\Psi(\frac{W_2 - (k - l)\Delta}{\Delta})|W_1]},$$

where $\Delta = \frac{2\sqrt{2}}{l+1}$. Define $\mathbf{Q_{i,j}(U) = P_i(W_1, W_2)P_j(B_1, B_2)}$, for $1 \leq i, j \leq 2l + 1$.

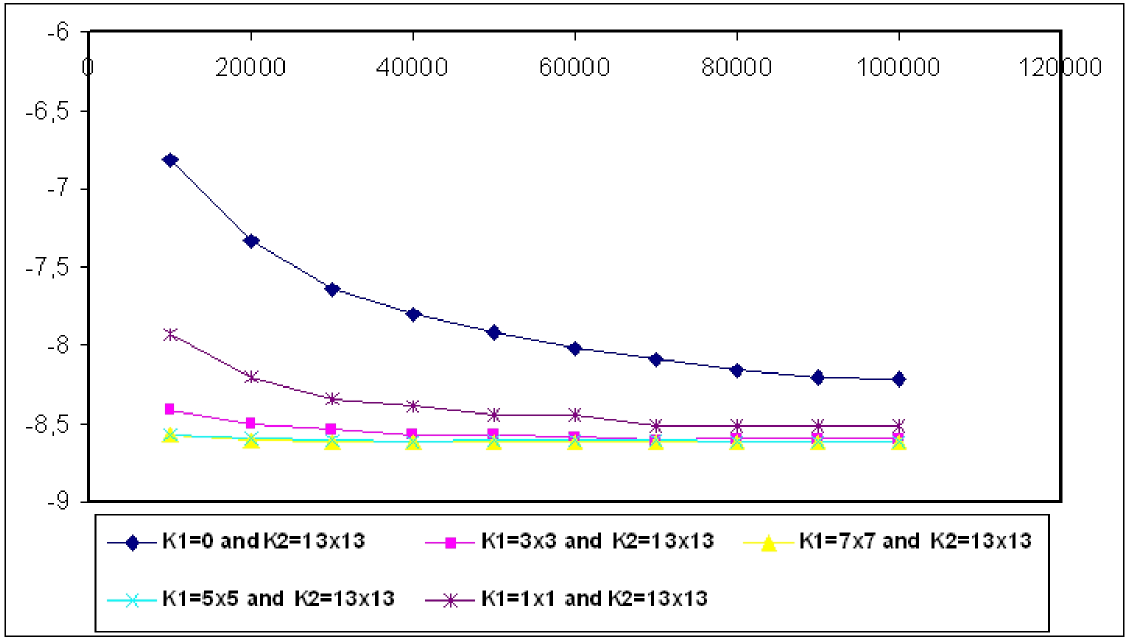
- **Regression basis functions:**

$$\mathbf{\Phi_{i,j}(W_1, B_1) = \Psi(\frac{W_1 - (i - r - 1)\Delta}{\Delta})\Psi(\frac{B_1 - (j - r - 1)\Delta}{\Delta})}$$

for $1 \leq i, j \leq 2r + 1$ with $K_{\mathcal{F}} = (2r + 1)(2r + 1)$.



Empirical error (in log-scale) as a function of $N \leq 12000$.



Empirical error (in log-scale) as a function of $N \geq 10000$.

$PCV\ (N = 1000) \approx Standard\ (N = 20000)$

➠ **Efficiency improvement $\approx$ 20**

Conclusion, perspectives, open problems

▷ **Mathematical aspects.**

✓ Schemes much sensitive to the dimension and the regularity of solution (estimated by a priori PDE estimates).

✓ Most efficient (theoretically) schemes are those based on MDP, but the effect of larger variance is not yet deeply analyzed.

▷ **Programming and algorithmic aspects.**

✓ Local polynomials can be implemented very efficiently by taking advantage of local basis. Crucial trick to make it fast.

✓ Storing in computer memory all coefficients for MDP may become a issue, more critical than for ODP.

✓ Good idea to improve schemes by incorporating theoretical information about the true solution (proxy, upper bound to stabilize the estimates, refined hypercubes near singularity)

✓ Data-driven basis (see experiments in [Lem05], [BW12]). Not yet fully covered by theoretical results.

✓ Parallel computations (Labart-Lelong '13)

▷ **Works in progress.**

✓ Mal. MDP with one single cloud of simulations.

✓ Large dimension and effective dimension of a BSDE regression problems.

✓ Non-linear least-squares regression and sparse representations.

✓ Jump components.

▷ **Open problems.**

✓ How to take advantage of the knowledge of fractional smoothness conditions?

✓ How to design optimal stochastic discretization grids for BSDEs? see [GL12] for optimal discretization stochastic integrals.

✓ BSDE with space constraints (RBSDE and switching, random terminal time).

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