Backward stochastic differential equations and point processes

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Spring Semester: Perspectives in Analysis and Probability

Workshop on BSDEs

Centre Henri Lebesgue - Rennes, May $22^{nd}, 2013$

Plan

1. Some motivation from the theory of BSDEs.

2. Marked point processes (random measures) and related topics.

3. BSDEs driven by random measures. Existence and uniqueness.

4. Stochastic optimal control: formulation and solution via BSDEs.

5. Stochastic Hamilton-Jacobi-Bellman equation.

6. The Markovian case and other possible developments.

Backward stochastic differential equations

$(\Omega, \mathcal{F}, \mathbb{P})$ basic probability space.

$(W_t)_{t\geq 0}$ Wiener process in $\mathbb{R}^d$, with natural completed filtration $(\mathcal{F}_t^W)$.

$$Y_t + \int_t^T Z_s\,dW_s = \xi + \int_t^T f_s(Y_s, Z_s)\,ds, \qquad t \in [0,T],$$

or

$$-dY_t = -Z_t\,dW_t + f_t(Y_t, Z_t)\,dt, \qquad Y_T = \xi.$$

Unknown $(\mathcal{F}_t^W)$-progressive processes:

$Y_t(\omega) : \Omega \times [0,T] \to \mathbb{R}$, $Z_t^j(\omega) : \Omega \times [0,T] \to \mathbb{R}$, $(j = 1, \ldots, d)$.

Given data: $\xi(\omega) : \Omega \to \mathbb{R}$ $\mathcal{F}_T^W$-measurable,
$f_t(\omega, y, z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ $(\mathcal{F}_t^W)$-progressive in (ω, t),

In the original paper by Pardoux-Peng (1990) two basic ingredients:

- representation theorem for $(\mathcal{F}_t^W)$ martingales;
- Lipschitz conditions on $(y, z) \mapsto f_t(\omega, y, z)$.

Some early extensions beyond the Brownian case:
El Karoui - Peng - Quenez. Math. Finance 7 (1997).
El Karoui - Huang. In: Pitman R.N.M. 364, 1997.

Aim: study BSDEs using the filtration of a marked point process.

Marked (multivariate) point processes

$(T_n, \xi_n)_{n \geq 1}$ random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

ξ_n take values in $(K, \mathcal{K})$, a Lusin space called state (mark) space.

$T_0 := 0$. T_n take values in $[0, \infty]$, are increasing and satisfy, for $n \geq 0$,

$$T_n < \infty \quad \Rightarrow \quad T_n < T_{n+1}.$$

Random measure $p(dt\, dy)$ on $(0, \infty) \times K$: for $C \in \mathcal{B}((0, \infty)) \otimes \mathcal{K}$,

$$p(\omega, C) = \sum_{n \geq 1} 1((T_n(\omega), \xi_n(\omega)) \in C).$$

Counting processes: for $t \geq 0$, $B \in \mathcal{K}$,

$$N_t(B) = p((0, t] \times B), \quad N_t = N_t(K) = \sum_{n \geq 1} 1(T_n \leq t),$$

and associated filtration $(\mathcal{F}_t)_{t \geq 0}$:

$$\mathcal{F}_t = \sigma(N_s(B) \ : \ s \in [0, t],\, B \in \mathcal{K}).$$

$\mathcal{P}$ = $(\mathcal{F}_t)$-predictable σ-algebra.

State (forward) process $(X_t)_{t \geq 0}$:

$$X_t = \sum_{n \geq 0} \xi_n\, 1(T_n \leq t < T_{n+1}),$$

where $\xi_0 \equiv x \in K$ (deterministic).

Dual predictable projections (compensators)

Given $(T_n, \xi_n)_{n\geq 1}$, $N_t = \sum_{n\geq 1} 1(T_n \leq t)$, $p(dt\,dy)$, the compensator of N is an increasing, right-continuous, predictable process A, with $A_0 = 0$ such that

$$\mathbb{E}\int_0^\infty H_t\,dN_t = \mathbb{E}\int_0^\infty H_t\,dA_t$$

for every predictable ($\mathcal{P}$-measurable) $H_t(\omega) \geq 0$.

Standing assumption: A has continuous trajectories. This implies

$$T_\infty :=\uparrow \lim_n T_n \equiv +\infty.$$

The compensator of $p(dt\,dy)$ is a predictable random measure $\tilde{p}(dt\,dy)$ such that

$$\mathbb{E}\int_0^\infty \int_K H_t(y)\,p(dt\,dy) = \mathbb{E}\int_0^\infty \int_K H_t(y)\,\tilde{p}(dt\,dy)$$

for every $\mathcal{P}\otimes\mathcal{K}$-measurable $H_t(\omega, y) \geq 0$.

$\tilde{p}(dt\,dy)$ exists and has the form

$$\tilde{p}(dt\,dy) = \phi_t(dy)\,dA_t$$

where $B \mapsto \phi_t(\omega, B)$ is a probability on $\mathcal{K}$, and $(\omega, t) \mapsto \phi_t(\omega, B)$ is predictable.

Example: the Poisson random measure on $\mathbb{R}^N$ has compensator

$$\tilde{p}(dt\, dy) = \lambda(dy)\, dt$$

for some (deterministic, fixed) intensity measure λ on $\mathbb{R}^N$.

$\phi_t(dy)\, dA_t$ may be thought of as a "generalized intensity".

Stochastic integrals and martingale representation

Suppose $H_t(\omega, y)$ is $\mathcal{P} \otimes \mathcal{K}$-measurable over $\Omega \times [0, T]$ and

$$\mathbb{E} \int_0^T \int_K |H_t(y)|\, \phi_t(dy)\, dA_t < \infty. \tag{1}$$

Then one defines the compensated stochastic integral: for $t \in [0, T]$

$$M_t := \int_0^t \int_K H_s(y)\, q(ds\, dy) := \int_0^t \int_K H_s(y)\, p(ds\, dy) - \int_0^t \int_K H_s(y)\, \phi_s(dy)\, dA_s. \tag{2}$$

Shortly: $q(ds\, dy) = p(ds\, dy) - \phi_s(dy)\, dA_s$.

Martingale representation: M defined in (2) is a càdlàg martingale; conversely, any càdlàg martingale has the form (2) for some process H satisfying (1).

BSDE driven by point processes

$$Y_t + \int_t^T \int_K Z_s(y)\, q(ds\, dy) = \xi + \int_t^T f_s(Y_s, Z_s(\cdot))\, dA_s, \qquad t \in [0, T],$$

or

$$-dY_t = -\int_K Z_t(y)\, q(dt\, dy) + f_t(Y_t, Z_t(\cdot))\, dA_t, \qquad Y_T = \xi.$$

A = compensator of N. $q(ds\, dy) = p(ds\, dy) - \phi_s(dy)\, dA_s$.

Given data: f, ξ. Unknown processes:

$Y_t(\omega) : \Omega \times [0, T] \to \mathbb{R}$, $(\mathcal{F}_t)$-adapted càdlàg;

$Z_t(\omega, y) : \Omega \times [0, T] \times K \to \mathbb{R}$, $\mathcal{P} \otimes \mathcal{K}$-measurable.

Earlier results

BSDEs driven by a noise "Wiener + Poisson":

$$Y_t + \int_t^T Z'_s \, dW_s + \int_t^T \int_K Z_s(y) \, q(ds \, dy) = \xi + \int_t^T f_s(Y_s, Z_s(\cdot), Z'_s) \, ds,$$

Here $p(dt \, dy)$ is a Poisson random measure on $K = \mathbb{R}^N \backslash \{0\}$,
hence $\tilde{p}(dt \, dy) = \lambda(dy) \, dt$ with

$$\int_{\mathbb{R}^N} (1 \wedge |y|^2) \, \lambda(dy) < \infty.$$

Many results:

- Tang, S. Li, X. SIAM J. Control Optim. (1994).
- M. Royer. Stoch. Proc. Appl. (2006).
- Barles-Buckdahn-Pardoux. Stochastics (1997).

More general BSDE in: Xia, J. Acta Math. Appl. Sinica (2000).

$$\begin{aligned} Y_t + \int_t^T Z'_s \, dM_s + \int_t^T \int_K Z_s(y) \, q(ds \, dy) \\ = \xi + \int_t^T f'_s(Y_s, Z'_s) \, dN_s + \int_t^T \int_K f_s(y, Y_s, Z_s(y)) \, \lambda(ds \, dy). \end{aligned}$$

Here $K = \mathbb{R}$, M is a martingale, N is increasing and λ is another random measure $(0, \infty) \times K$.

BSDEs related to Markov chains: Cohen-Elliott (2008, 2010); Cohen-Szpruch (2012).

Solution of the BSDE: L^2 theory

$$-dY_t = -\int_K Z_t(y)\, q(dt\, dy) + f_t(Y_t, Z_t(\cdot))\, dA_t, \qquad Y_T = \xi.$$

$Y_t(\omega)$ càdlàg adapted in $\mathcal{L}^{2,\beta}$, $Z_t(\omega, y)$ $\mathcal{P} \otimes \mathcal{K}$-measurable in $\mathcal{L}^{2,\beta}(p)$, i.e.

$$\|Y\|^2_{\mathcal{L}^{2,\beta}} := \mathbb{E} \int_0^T e^{\beta A_t} |Y_t|^2 dA_t < \infty,$$

$$\|Z\|^2_{\mathcal{L}^{2,\beta}(p)} := \mathbb{E} \int_0^T \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy)\, dA_t < \infty.$$

Assumptions:

- $\xi(\omega)$ is $\mathcal{F}_T$-measurable.
- $f_t(\omega, r, z(\cdot))$ is defined for $r \in \mathbb{R}$, $z(\cdot) \in \mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy))$, such that $f_t(\omega, r, Z_t(\omega))$ is progressive for $Z \in \mathcal{L}^{2,\beta}(p)$.
- $|f_t(\omega, r, z(\cdot)) - f_t(\omega, r', z'(\cdot))| \le L'|r - r'| + L \left(\int_K |z(y) - z'(y)|^2 \phi_t(\omega, dy) \right)^{\frac{1}{2}}$
- $\mathbb{E} \int_0^T e^{\beta A_t} |f_t(0,0)|^2 dA_t + \mathbb{E} e^{\beta A_T} |\xi|^2 < \infty.$

Theorem (Confortola, F.; SICON, to appear) Suppose $\beta > 2L' + L^2$. Then the BSDE has a unique solution $(Y, Z) \in \mathcal{L}^{2,\beta} \times \mathcal{L}^{2,\beta}(p)$.

$$-dY_t = -\int_K Z_t(y)\, q(dt\, dy) + f_t(Y_t, Z_t(\cdot))\, dA_t, \qquad Y_T = \xi.$$

Proof: representation theorem for $(\mathcal{F}_t)$-martingales, Ito's formula to compute $d(e^{\beta A_t}|Y_t|^2)$ and get

$$\mathbb{E}\, e^{\beta A_t}|Y_t|^2 + \mathbb{E}\int_t^T \beta e^{\beta A_s}|Y_s|^2\, dA_s + \mathbb{E}\int_t^T\int_K e^{\beta A_s}|Z_s(y)|^2\, \phi_s(dy)\, dA_s$$
$$= \mathbb{E} e^{\beta A_T}|\xi|^2 + 2\mathbb{E}\int_t^T e^{\beta A_s} Y_s\, f_s(y, Y_s, Z_s(y))\, dA_s.$$

Note: to solve even with $\xi =$ constant one needs

$$\mathbb{E} e^{\beta A_T} < \infty.$$

If p is Poisson with compensator $\lambda(dy)\, dt$ one needs $\lambda(\mathbb{R}^N \backslash \{0\}) < \infty$.

Solution of the BSDE: L^1 theory

$$-dY_t = -\int_K Z_t(y)\,q(dt\,dy) + f_t(Y_t, Z_t(\cdot))\,dA_t, \qquad Y_T = \xi.$$

$Y_t(\omega)$ càdlàg adapted, $Z_t(\omega, y)$ $\mathcal{P} \otimes \mathcal{K}$-measurable in $\mathcal{L}^1_{loc}(p)$, i.e.

$$\int_0^T \int_K |Z_t(y)|\phi_t(dy)\,dA_t < \infty, \qquad \mathbb{P} - a.s.$$

Assumptions:

- $\xi(\omega)$ is $\mathcal{F}_T$-measurable, $\mathbb{E}|\xi| < \infty$.
- $f_t(\omega, r, z(\cdot))$ is defined for $r \in \mathbb{R}$, $z(\cdot) \in \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$, such that $f_t(\omega, r, Z_t(\omega))$ is progressive for $Z \in \mathcal{L}^1_{loc}(p)$.
- $|f_t(\omega, r, z(\cdot)) - f_t(\omega, r', z'(\cdot))| \leq L'|r - r'| + L \int_K |z(y) - z'(y)|\phi_t(\omega, dy)$,
- $\int_0^T |f_t(0,0)|dA_t < \infty$, $\mathbb{P}$-a.s.

Theorem (Confortola, F., Jacod; in progress)
Suppose $0 < T_1 \leq T_2 = T_3 = \ldots = \infty$ (one jump case) with $\mathbb{P}(T_1 > T) > 0$.
Then the BSDE has a unique solution (Y, Z), $Z \in \mathcal{L}^1_{loc}(p)$.

Solution of the BSDE: L^1 theory and pathwise solutions

For $t < T_1(\omega)$ we have

$$f_t(\omega, r, z(\cdot)) = f_t(r, z(\cdot)), \quad \phi_t(\omega, dy) = \phi_t(dy).$$

Moreover there exist $u \in \mathbb{R}$ and deterministic functions $a(t), v(t, y)$ such that

$$dA_t(\omega) = da(t)\, 1_{t \le T_1(\omega)}, \quad \xi(\omega) = u\, 1_{T_1(\omega) > T} + v(T_1(\omega), \xi_1(\omega))\, 1_{T_1(\omega) \le T},$$

($a(t)$ continuous increasing) and the solution (Y, Z) has the form

$$Y_t(\omega) = y(t)\, 1_{t < T_1(\omega)} + v(T_1(\omega), \xi_1(\omega))\, 1_{t \ge T_1(\omega)}, \quad Z_t(\omega, y) = z(t, y)\, 1_{t \le T_1(\omega)},$$

where $y(t)$ solves the ODE on $[0, T]$:

$$y(t) = u + \int_t^T \left[f_s(y(s), v(s, \cdot) - y(s)) + \int_K v(s, y)\, \phi_s(dy) - y(s) \right] da(s)$$

and

$$z(t, y) = v(t, y) - y(t).$$

Similar results are in preparation for the general (multi-jump) case.

Application: stochastic optimal control

Given $(T_n, \xi_n)_{n\geq 1}$, $X_t = \sum_{n\geq 0} \xi_n \, 1(T_n \leq t < T_{n+1})$.

The controller acts on the (generalized) intensity, i.e. on the compensator.

The control problem is defined in a weak form, i.e. via a change of probability measure.

This approach is classical, see e.g. the book by P. Brémaud, 1981. We need:

- a space of control actions U and a space of control processes;
- a function r specifying the effect of the choice of a control process;
- two cost functions l, g defining the cost functional.

i) $(U, \mathcal{U} = \mathcal{B}(U))$ compact metric space: the space of control actions.

A control $u(\cdot)$ is a predictable processes $u : \Omega \times [0, T] \to U$. Then

$$u_t = \sum_{n\geq 0} u_t^{(n)} \, 1(T_n < t \leq T_{n+1}),$$

with $u^{(n)}$ $\mathcal{F}_{T_n} \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, $\mathcal{F}_{T_n} = \sigma(T_0, \xi_0, \ldots, T_n, \xi_n)$: at each T_n, the controller chooses his control actions for $t > T_n$ based on T_i, ξ_i $(0 \leq i \leq n)$ and updates his decisions only at time T_{n+1}.

ii) $r_t(\omega, y, u) : \Omega \times [0, T] \times K \times U \to [0, C_r]$, $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable, continuous in u.

Given $u(\cdot)$, let L be the solution of

$$L_t = 1 + \int_0^t \int_K L_{s-}(r_s(y, u_s) - 1)\, q(ds\, dy).$$

Let $\gamma > 1$, $\beta = \gamma + 1 + C_r^{\gamma^2}/(\gamma - 1)$. Then

$$\mathbb{E} \exp(\beta A_T) < \infty \quad \Rightarrow \quad \mathbb{E} L_T = 1, \ \sup_{t \in [0,T]} \mathbb{E}|L_t|^\gamma < \infty.$$

Define $\mathbb{P}_u(d\omega) = L_T(\omega)\, \mathbb{P}(d\omega)$. Then the compensator of p under $\mathbb{P}_u$ is

$$\tilde{p}^u(dt\, dy) = r_t(y, u_t)\, \tilde{p}(dt\, dy) = r_t(y, u_t)\, \phi_t(dy)\, dA_t.$$

"The choice of a control $u(\cdot)$ multiplies the intensity by $r_t(\cdot, u_t)$".

iii) $l_t(\omega, x, u) : \Omega \times [0, T] \times K \times U \to \mathbb{R}$, $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable, bounded, continuous in u; and
$g(\omega, x) : \Omega \times K \to \mathbb{R}$, $\mathcal{F}_T \otimes \mathcal{K}$-measurable, bounded (for simplicity).

The cost of a control $u(\cdot)$ is

$$J(u(\cdot)) = \mathbb{E}_u \int_0^T l_t(X_t, u_t)\, dA_t + \mathbb{E}_u\, g(X_T).$$

Optimal control problem via BSDEs

Hamiltonian function:

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, x, u) + \int_K z(y)\,(r_t(\omega, y, u) - 1)\,\phi_t(\omega, dy) \right\}.$$

Theorem (Confortola, F.; SICON, to appear) Assume $i)$-$ii)$-$iii)$ and $\mathbb{E}\exp(\beta A_T) < \infty$ for $\beta = 3 + C_r^4$. Then the BSDE

$$Y_t + \int_t^T \int_K Z_s(y)\,q(ds\,dy) = g(X_T) + \int_t^T f(s, X_s, Z_s(\cdot))\,dA_s,$$

has a unique solution $(Y, Z) \in \mathcal{L}^{2,\beta} \times \mathcal{L}^{2,\beta}(p)$. There exists a control $u^Z(\cdot)$ such that

$$f(t, X_{t-}, Z_t(\cdot)) = l_t(X_{t-}, u_t^Z) + \int_K Z_t(y)\,(r_t(y, u_t^Z) - 1)\,\phi_t(dy), \quad dA_t(\omega)\mathbb{P}(d\omega) - a.s.$$

Finally any such control is optimal and

$$Y_0 = J(u^Z(\cdot)) = \inf_{u(\cdot)} J(u(\cdot)).$$

An example with explicit solution

State space $K = \{a, b, c\}$. Single jump: $T_n = +\infty$ if $n \geq 2$.

- $X_0 = a$; at time T_1 the system jumps to ξ_1;
- $\mathbb{P}(\xi_1 = b) = \mathbb{P}(\xi_1 = c) = \frac{1}{2}$;
- $T_1(\omega) \in (0, \infty]$ has distribution function F;
- T_1 and ξ_1 are independent.

The compensator $\tilde{p}(dt\,dy) = \phi_t(dy)\,dA_t$ is

$$dA_t(\omega) = \frac{F(dt)}{1 - F(t)} 1_{\{t \leq T_1(\omega)\}}, \qquad \phi_t(a) = 0,\ \phi_t(b) = \phi_t(c) = \frac{1}{2}.$$

Assume $F(T) < 1 \Rightarrow A_T$ bounded. Take

$$r_t(\omega, b, u) = u, \quad r_t(\omega, c, u) = 2 - u, \qquad u \in U = [0, 2]$$

The compensator $\tilde{p}^u(dt\,dy)$ under $\mathbb{P}_u$ has

$$\phi_t(a) = 0, \quad \phi_t(b) = \frac{u_t}{2}, \quad \phi_t(c) = 1 - \frac{u_t}{2},$$

The control changes the probabilities of jumping to the state b or c.

Final cost g and running cost l:

$$g(a) = g(b) = 0,\ g(c) = 1, \qquad l_t(\omega, x, u) = \frac{\alpha u}{2}.$$

where $\alpha > 0$ is a parameter.

We will represent the optimal cost by the solution Y_0 of the BSDE

$$Y_t + \int_t^T \int_K Z_s(y) q(ds\, dy) = g(X_T) + \int_t^T \inf_{u \in [0,2]} \left[\frac{\alpha u}{2} + \int_K Z_s(y)(r_s(y,u) - 1)\phi_s(dy) \right] dA_s,$$

that can be written

$$Y_t + Z_{T_1}(\xi_1)\, 1_{\{t < T_1 \le T\}} = 1_{\{T_1 \le T\}}\, 1_{\{\xi_1 = c\}} + \int_t^{T \wedge T_1} [Z_s(c) \wedge (\alpha + Z_s(b))] \frac{F(dt)}{1 - F(t)}.$$

The solution is

$$Y_t = (1 \wedge \alpha) \left(1 - \exp\left(-\int_t^T \frac{F(ds)}{1 - F(s)} \right) \right) 1_{\{t < T_1\}} + 1_{\{T_1 \le t\}}\, 1_{\{\xi_1 = c\}},$$

$$Z_t(b) = (1 \wedge \alpha) \left(\exp\left(-\int_t^T \frac{F(ds)}{1 - F(s)} \right) - 1 \right) 1_{\{t \le T_1\}},$$

$$Z_t(a) = 0, \qquad Z_t(c) = (1 + Z_t(b)) 1_{\{t \le T_1\}}.$$

Optimal cost: $Y_0 = (1 \wedge \alpha) \left(1 - e^{-\int_0^T \frac{F(ds)}{1-F(s)}} \right)$.

Optimal control: $\begin{cases} u \equiv 0 & \text{if } \alpha \ge 1, \\ u \equiv 2 & \text{if } \alpha \le 1. \end{cases}$

Dynamic programming

We consider the point process $(X_s^{t,x})_{s\in[t,T]}$ starting at any time $t \in [0,T]$ from any $x \in K$. It is associated with the restriction of the random measure $p(dt\,dy)$ to $(t,T]\times K$. For any probability $\mathbb{P}_u$ associated to a control $u(\cdot)$ we introduce the random cost and value function

$$J_t(x,u(\cdot)) = \mathbb{E}_u\left[\int_t^T l_s(X_s^{t,x},u_s)\,dA_s + g(X_T^{t,x})\,\middle|\mathcal{F}_t\right], \quad v(t,x) = \operatorname*{ess\,inf}_{u(\cdot)} J_t(x,u(\cdot)).$$

Then we have similar results: there exists a unique solution to the BSDE

$$Y_s^{t,x} + \int_s^T\int_K Z_r^{t,x}(y)\,q(dr\,dy) = g(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Z_r^{t,x}(\cdot))\,dA_r, \quad s\in[t,T],$$

there exists an optimal control, and

$$Y_t^{t,x} = \operatorname*{ess\,inf}_{u(\cdot)} J_t(x,u(\cdot)), \qquad \mathbb{P}-a.s.$$

The stochastic Hamilton-Jacobi-Bellman equation (HJB)

Unknown processes:

$v(\omega, t, x) : \Omega \times [0, T] \times K \to \mathbb{R}$, $Prog \otimes \mathcal{K}$-measurable;

$V(\omega, t, x, y) : \Omega \times [0, T] \times K \times K \to \mathbb{R}$, $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{K}$-measurable.

$$
\begin{aligned}
v(t,x) &+ \int_t^T \int_K V(s,x,y)\, q(ds\, dy) \\
&= g(x) + \int_t^T \int_K \big(v(s,y) - v(s,x) + V(s,y,y) - V(s,x,y)\big)\, \phi_s(dy)\, dA_s \\
&+ \int_t^T f\big(s, X_s, v(s,\cdot) - v(s,x) + V(s,\cdot,\cdot)\big)\, dA_s.
\end{aligned}
$$

$\mathbb{P}$-a.s., this must hold for all $t \in [0, T]$, $x \in K$. We require

$$
\begin{aligned}
\sup_{x\in K} \mathbb{E} \int_0^T |v(t,x)|^2 e^{\beta A_t} dA_t &+ \mathbb{E} \int_0^T |v(t,X_t)|^2 e^{\beta A_t} dA_t \\
&+ \sup_{x\in K} \int_t^T \int_K |V(t,x,y)|^2\, \phi_t(dy)\, dA_t \\
&+ \mathbb{E} \int_0^T \int_K |v(t,y) + V(t,y,y)|^2\, \phi_t(dy) e^{\beta A_t}\, dA_t < \infty.
\end{aligned}
$$

Theorem (Confortola, F.; SICON, to appear) Assume $i)$-$ii)$-$iii)$ and K finite or countable. There exists $\beta_0 > 0$ (explicitly computable) such that if

$$\beta \geq \beta_0, \qquad \mathbb{E}[e^{\beta A_T}] < \infty,$$

then HJB has a unique solution (v, V).
We also have

$$Y_s^{t,x} = v(s, X_s^{t,x}), \quad Z_s^{t,x} = v(s-, y) - v(s-, X_{s-}^{t,x}) + V(s, y, y).$$

where $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ is the solution to the BSDE

$$Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q(dr\, dy) = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Z_r^{t,x}(\cdot))\, dA_r, \quad s \in [t, T],$$

In particular, $v(t, x) = Y_t^{t,x}$ coincides with the value function:

$$v(t, x) = \operatorname*{ess\,inf}_{u(\cdot)} J_t(x, u(\cdot)), \qquad \mathbb{P} - a.s.$$

Stochastic HJB introduced by Peng, SIAM J. Control Optim. (1992), in the diffusive case.

Proof. Uniqueness: Ito's formula for $dv(s, X_s^{t,x})$.
Existence: fixed point argument + estimates on the BSDE.

The Markovian case

We give an outline of some results in Confortola, F. - preprint arxiv 2013.

Let $(\Omega, X, \mathbb{P}^{t,x})$ a (non-homogeneous) Markov process on $(K, \mathcal{K})$.
We have $\mathbb{P}^{t,x}(X_t = x) = 1$ and we require:

- Pure jump process: each trajectory is piecewise constant, right-continuous.
- Non-explosive: jump times diverge to $+\infty$.

Given t, x, let (T_n) be the jumps times after t. The trajectories of X are determined by

$$M = (T_n, X_{T_n})_{n\geq 1}, \qquad\qquad (X_\infty := \Delta \notin K).$$

Under each $\mathbb{P}^{t,x}$, M is a time-homogeneous discrete Markov process, and it is our basic marked point process.

Let $\nu(t, x, dy)$ denote the rate transition measure of X (the rate matrix $\nu(x, \{y\})$ in the case of a stationary finite Markov chain). Then:

$$\tilde{p}(dt\, dy) = \nu(t, X_{t-}, dy)\, dt.$$

We assume $\sup_{t\geq 0, x\in K} \nu(t, x, K) < \infty$ and consider the BSDE

$$Y_s + \int_s^T \int_K Z_r(y)\, q(dr\, dy) = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r(\cdot))\, dr, \qquad s \in [t, T].$$

Under appropriate measurability and Lipschitz assumptions on the coefficients, the BSDE has a unique solution $(Y_s, Z_s)_{s\in[t,T]}$, such that:
$Y_s(\omega)$ càdlàg adapted, $Z_s(\omega, y)$ $\mathcal{P} \otimes \mathcal{K}$-measurable,

$$\mathbb{E}^{t,x} \int_t^T |Y_s|^2 ds + \mathbb{E}^{t,x} \int_t^T \int_K |Z_s(y)|^2 \nu(s, X_s, dy)\, ds < \infty.$$

Denote the solution $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$. The function

$$v(t,x) = Y_t^{t,x},$$

is the unique solution to the non-linear Kolmogorov equation:

$$v(t,x) = g(x) + \int_t^T \mathcal{L}_s v(s,x)\, ds + \int_t^T f(s,x,v(s,x),v(s,\cdot) - v(s,x))\, ds,$$

where $\mathcal{L}_t\phi(x) = \int_K \big(\phi(y) - \phi(x)\big)\, \nu(t,x,dy)$ is the generator of X, in the class of measurable functions $v : [0,T] \times K \to \mathbb{R}$ satisfying

$$\mathbb{E}^{t,x} \int_t^T |v(s,X_s)|^2 ds + \mathbb{E}^{t,x} \int_t^T \int_K |v(s,y) - v(s,X_s)|^2 \nu(s,X_s,dy)\, ds < \infty.$$

Moreover,

$$Y_s^{t,x} = v(s,X_s), \quad Z_s^{t,x}(y) = v(s,y) - v(s,X_{s-}), \qquad s \in [t,T],$$

Remark. The equation is easy to solve when f, g are bounded, but we only require $\mathbb{E}^{t,x} \int_t^T |f(s,X_s,0,0)|^2 ds + \mathbb{E}^{t,x} |g(X_T)|^2 < \infty$.

Optimal control problems in the Markov case can also be addressed. The non-linear Kolmogorov equation is the Hamilton-Jacobi-Bellman equation.

Further developments

- The semi-Markov case: non-linear Kolmogorov equations, optimal control problems (in preparation with F. Confortola and E. Bandini).
- Extensions to more general classes of processes.
- Infinite horizon, quadratic growth conditions.
- L^1 theory and pathwise solutions in more general cases (processes with explosion, discontinuous compensators etc.)

Thank you for your attention!

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