Continuous Time Random Walks and fractional HJB equations

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Outline

- HJB and fHJB
- Basics of fractional calculus
- Mittag-Leffler function
- Example of a fDE
- Payoff equation
- Existence and uniqueness
- Open problems

HJB and fHJB

General HJB:

$$\frac{\partial S}{\partial t} + LS + \sup_u [f(x,u)\frac{\partial S}{\partial x} + g(x,u)] = 0. \qquad (1)$$

Game theory: HJB Isaacs equation

$$\frac{\partial S}{\partial t} + LS + \sup_u \inf_v [f(x,u,v)\frac{\partial S}{\partial x} + g(x,u,v)] = 0. \qquad (2)$$

fHJB

$$\frac{\partial^\beta S}{\partial t^\beta} + LS + \sup_u [f(x,u)\frac{\partial S}{\partial x} + g(x,u)] = 0 \qquad (3)$$

We're particularly interested in the case when $L = a(x)D^{*\alpha}_{0,x}, \alpha \in (0,2]$.

CTRW and optimal payoff equation

A random variable γ in DOA (β-stable law) if as $n \to \infty$

$$\frac{\sum_{i=1}^{n} \gamma_i - a_n}{b_n} \to Z, \tag{4}$$

in distribution, for some a_n, b_n, where Z is stable. In other notation, as $n \to \infty$ if $\nu(dr)$ is the law for γ_i waiting times

$$\int_{|r|>n} \nu(dr) \sim \frac{1}{\Gamma[1-\beta]n^{\beta}} \tag{5}$$

for $\beta \in (0,1)$.

- Waiting times $\gamma_i \in \mathrm{DOA}(\beta\text{-stable law})$, $\beta \in (0,1)$.
- Denote $X(n) = \sum_{i=1}^{n} \gamma_i$ and $Z_X(t) = \inf_n\{n : X(n) > t\}$
- Jumps $\xi_i \in \mathrm{DOA}(\alpha\text{-stable law})$, $\alpha \in (0,2]$, $\xi_i \in \mathbb{R}^d$.
- The process $Y_{Z_X}(t) = \sum_{i=1}^{Z_X(t)} \xi_i$ is the CTRW.
- Control set U at every jump: $\xi_i(u_i)$, $i \in \mathbb{N}$.
- Set of all possible controls for all jump times:

$$\tilde{U} = \{\tilde{u} = (u_1, u_2, \ldots)\}, u_i \in U. \tag{6}$$

Basics

- Fractional integral defined via iterations for $f \in S$, Schwartz space: Let $If(x) = \int_{-\infty}^{x} f(y)dy$ for $f \in S$.

$$I^k f(x) = \frac{1}{(k-1)!} \int_{-\infty}^{x} (x-y)^{k-1} f(y)dy, \text{ for } k \in \mathbb{N}. \quad (7)$$

  Now replace integer k by fractional β:

$$I^\beta f(x) = \frac{1}{\Gamma[\beta]} \int_{-\infty}^{x} (x-y)^{\beta-1} f(y)dy, \text{ for } \beta > 0. \quad (8)$$

- Related fractional derivative for $\beta \in (0,1)$:

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{d}{dx} I^{1-\beta} f(x). \quad (9)$$

- The left-sided Caputo derivative for $\beta \in (0,1)$ is defined for $f \in S$ by

$$D_{0,x}^{*\beta} f(x) = \frac{1}{\Gamma[1-\beta]} \int_0^x \frac{df(y)}{dy} (x-y)^{-\beta} dy. \qquad (10)$$

- Fractional Laplacian operator $(-\nabla)^{\alpha}$:

$$-\frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{(f(x)-f(y))}{|x-y|^{1+\alpha}} dy. \qquad (11)$$

- The operators $-D_{0,y}^{*\beta}$ and $(-\nabla)^{\beta/2}$ are generators of stable Levy motions.
- A general β-stable Levy motion L_t can be described by the following log-characteristic function:

$$log\mathbb{E}[e^{i\theta L_t}] = -t\kappa^{\beta}|\theta|^{\beta}(1 - i\beta sign(\theta) tan(\beta\pi/2)) + itm\theta \quad (12)$$

for $\beta \neq 1$, β stability index.

Mittag-Leffler function: power series definition

A Mittag-Leffler function $E_{\beta,\alpha}(z)$ is defined as:

$$E_{\beta,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma[\beta k + \alpha]} = \frac{1}{2\pi i} \int_{\mathrm{Ha}} e^{\lambda} \frac{\lambda^{\beta-\alpha}}{\lambda^{\beta} - z} d\lambda, \tag{13}$$

for $\alpha, \beta > 0, z \in C$, and where Ha denotes a Hankel path, which is a contour starting and finishing at $-\infty$ and encircling the disc $|\lambda| \leq |z|^{1/\beta}$ counter clockwise.

Examples of Caputo derivatives of functions

- $D_{0,t}^{*\beta}(e^{\lambda t}) = t^{-\beta}E_{1,1-\beta}(\lambda t) - \frac{t^{-\beta}}{\Gamma[1-\beta]}$,
- $D_{0,t}^{*\beta}(\cosh(\sqrt{\lambda}t)) = t^{-\beta}E_{2,1-\beta}(\lambda t^2) - \frac{t^{-\beta}}{\Gamma[1-\beta]}$,
- $D_{0,t}^{*\beta}(H(t)) = 0$, where $H(t)$ is the unit Heaviside function.

Mittag-Leffler function: examples

For example,

$$E_{1,1}(z) = e^z \tag{14}$$

and

$$E_{1,2}(z) = \frac{e^z - 1}{z} \tag{15}$$

and

$$E_{2,2}(z^2) = \frac{\sinh(z)}{z}. \tag{16}$$

Simple fDE

For $\beta \in (0, 1], \lambda \in \mathbb{R}$,

$$D_{0,t}^{*\beta} y(t) = \lambda y(t), \tag{17}$$

with initial condition

$$y(0) = y_0. \tag{18}$$

Solution is given by

$$y(t) = y_0 E_{\beta,1}(\lambda t^{\beta}). \tag{19}$$

When $\beta = 1$, $y(t) = y_0 \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{\Gamma(n+1)} = y_0 e^{\lambda t}$, a standard result.

CTRW and optimal payoff equation

- Waiting times $\gamma_i \in \mathrm{DOA}(\beta\text{-stable law})$, $\beta \in (0,1)$.
- Denote $X(n) = \sum_{i=1}^{n} \gamma_i$
- Denote $Z_X(t) = \inf_n \{n : X(n) > t\}$
- Jumps $\xi_i \in \mathrm{DOA}(\alpha\text{-stable law})$, $\alpha \in (0,2]$, $\xi_i \in \mathbb{R}^d$.
- The process $Y_{Z_X}(t) = \sum_{i=1}^{Z_X(t)} \xi_i$ is the CTRW. Y considered at jump times only is a Markov process.
- Controlling jump sizes to optimise payoff:
- Control set U at every jump time
- Set of all possible controls for all jump times:

$$\tilde{U} = \{\tilde{u} = (u_1, u_2, \ldots)\}, u_i \in U. \tag{20}$$

- The optimal payoff function $S(t, y)$ is defined as follows:

$$S(t,y) = \sup_{\tilde{u}\in\tilde{U}} \mathbb{E} S_0(y + Y(t,\tilde{u})), \quad t \geq 0. \tag{21}$$

- Scaling jump sizes by $\tau^{1/\alpha}$ and waiting times by $\tau^{1/\beta}$, then $Y^\tau(u,t) = \sum_{i=1}^{Z_{X_n^\tau}(t)} \tau^{1/\alpha}\xi_i(u_i)$ and

$$S^\tau(t,y) = \sup_{\tilde{u}\in\tilde{U}} \mathbb{E} S_0^\tau(y + Y^\tau(t,\tilde{u})). \tag{22}$$

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$$S^\tau(t,y) = \sup_{u\in U}\Bigg[S_0^\tau(y)\int_t^\infty \nu(dr/\tau^{1/\beta}) + \int_0^t\int_{\mathbb{R}^d} S^\tau(t-r, y+\xi)\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})\Bigg]. \tag{23}$$

As $\tau \to 0$, assume $\forall t \geq 0$, $y \in \mathbb{R}^d$, $S^\tau(t,y) \to \tilde{S}(t,y)$, where $\tilde{S}$ belongs to domains of the stable generator L and $A_{0,t}^{*\beta}$.

Extensions and other versions

1. Motion during waiting times can be deterministic, e.g. in case $d = 1$, with a generator of the form $f(x)\frac{d}{dx}$. I.e. the process Y is piecewise deterministic.
2. This only slightly changes the equation for the optimal payoff, adding an extra term with the generator of the motion during waiting time intervals.
3. running costs- for waiting and for jumping, apart from arriving: represented by functions f and g.
4. Time and position dependence:

$$X_n^\tau = X_{n-1}^\tau + \gamma_n(X_{n-1}), \tag{24}$$

and

$$Y_n^\tau = Y_{n-1}^\tau + \xi_n(Y_{n-1}), \tag{25}$$

and the process we study is $Y_{Z_X(t)}$.

Construction of the jump process

For an arbitrary Feller process $\tilde{Y}$ with a generator L^u, it is always possible to construct a family of measures $\mu_{u,\tau}(y, d\xi)$, such that

$$Y_n^\tau = Y_{n-1}^\tau + \xi_n(Y_{n-1}^\tau), \tag{26}$$

where $\xi_n, n \in \mathbb{N}$ are r.v.'s with distributions given by $\mu_{u,\tau}$, and such that

$$\int_{\mathbb{R}^d} \frac{f(t, y+\xi) - f(t,y)}{\tau} \mu_{u,\tau}(y, d\xi) \to L^u f(t,y). \tag{27}$$

Then

$$Y_{[t/\tau]}^\tau \to_d \tilde{Y}. \tag{28}$$

Essentially, in such a case we obtain the same form of payoff equation.

fDE

Let $\tau \to 0$, obtain the limiting equation of the form:

$$D_{0,t}^{*\beta} S(t,y) = -LS(t,y) + H\left(u,t,y,D_y S\right). \tag{29}$$

First, we study

$$D_{0,t}^{*\beta} S(t,y) = -LS(t,y) + f(t,y), \tag{30}$$

We study the case when $L : D(L) \to C_\infty(\mathbb{R}^d)$, with the resolvent set $\rho(L)$ including the right half plane. More precisely this theory applies to $L = -|\nabla|^\alpha$, $\alpha \in (0,2]$ and to $L = \sum_{i,j=1}^d a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{d}{dx_i}$, where $a_{i,j}, b_i$ are constants.

Applying Laplace transform and re-arranging:

$$\hat{S}(\lambda) = \lambda^{\beta-1}(\lambda^\beta \mathbb{I} + L)^{-1} S_0(y) + (\lambda^\beta \mathbb{I} + L)^{-1} \hat{f}(\lambda). \tag{31}$$

Mittag-Leffler function

Other representation of the Mittag-Leffler function:

$$E_{\beta,\alpha}(-t^{\beta}L) = \frac{1}{2\pi i}\int_{Ha(\epsilon)} \frac{e^{stL^{1/\beta}} s^{\beta-\alpha}}{s^{\beta}+1} ds = \int_0^{\infty} e^{-rtL^{1/\beta}} K_{\beta,\alpha}(r) dr, \tag{32}$$

where

$$K_{\beta,\alpha}(r) = \frac{1}{\pi} \frac{r^{2\beta-\alpha}\sin(\alpha\pi) - r^{\beta-\alpha}\sin((\beta-\alpha)\pi)}{r^{2\beta}+2r^{\beta}cos(\beta\pi)+1}. \tag{33}$$

We will need

$$K_{\beta,\beta}(r) = \frac{1}{\pi} \frac{r^{\beta}\sin(\beta\pi)}{r^{2\beta}+2r^{\beta}cos(\beta\pi)+1}. \tag{34}$$

Here $e^{-rtL^{1/\beta}}$ is a bounded operator.

A Laplace transform and an unbounded operator L

Theorem

Let $L : D(L) \to C_\infty(\mathbb{R}^d)$, such as $L = (-\nabla)^\alpha, \alpha \in (1,2]$, and $\beta \in (0,1)$. Then the following holds:

$$\int_0^\infty e^{-\lambda t} t^{\alpha-1} E_{\beta,\alpha}(-t^\beta L) dt = \frac{\lambda^{\beta-\alpha}}{(\lambda^\beta \mathbb{I} + L)}. \tag{35}$$

Consider $L : D(L) \to L^2(\mathbb{R}^d)$ first. The power series representation of $E_{\beta,\alpha}(-t^\beta L)f$ holds for any vector f of L, satisfying

$$\sum_{n=1}^{\infty} \frac{\|L^n f\|_{L^2(\mathbb{R}^d)}}{n!} t^n < \infty \tag{36}$$

for any $t > 0$. This set of vectors is dense in $L^2(\mathbb{R}^d)$. $C_c(\mathbb{R}^d)$ dense in $L^2(\mathbb{R}^d)$. Approximate functions in $C_\infty(\mathbb{R}^d)$ by functions in $C_c(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$.

Apply the inverse Laplace transform to obtain the mild form.

Mild form of the equation

$$D^*_{0,t}S(t,y) = -LS(t,y) + f(t,y) \tag{37}$$

turns to

$$S(t,y) = E_{\beta,1}(-t^{\beta}L)S_0(y) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^{\beta}L)f(t,y)ds. \tag{38}$$

Application to the limiting payoff equation

Theorem

Let $\beta \in (0,1)$, $L = (-\nabla)^{\alpha}$, $\alpha \in (1,2]$ *or* $L = \sum_{i,j=1}^{d} a_{ij} \frac{d^2}{dx_i dx_j}$, *where* a_{ij} *are constants. The fDE*

$$D_{0,t}^{*\beta} S(t,y) = -LS(t,y) + H\left(t, y, u, D_y S(t,y)\right) \tag{39}$$

has the mild form

$$\begin{aligned} S(t,y) &= E_{\beta,1}(-t^{\beta} L) S_0(y) \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^{\beta} L) H\left(t, y, u, D_y S(t,y)\right) ds. \end{aligned} \tag{40}$$

Theorem 3

Assumptions:

- The domain of L is $D = C^2_\infty(\mathbb{R}^d)$, unbounded as an operator in $C_\infty(\mathbb{R}^d)$, e.g. $L = (-\nabla)^\alpha, \alpha \in (1, 2]$.
- $\|e^{-tL^{1/\beta}} f\|_{C^1(\mathbb{R}^d)} \leq t^{\epsilon(\beta)} \|f\|_{C^0(\mathbb{R}^d)}$.
- $H(u, t, y, p)$ is Lipschitz in p uniformly in y, i.e., with a Lipschitz constant κ independent of y, and

$$|H(u, y, 0)| \leq h \tag{41}$$

  for a constant h and all y.
- A restriction on stability parameters α, β:

$$\beta - \frac{\beta}{\alpha} + 1 > 0. \tag{42}$$

Under the above assumptions on H and L, for any $S_0 \in C^1_\infty(\mathbb{R}^d)$ there exists a unique $C^1_\infty(\mathbb{R}^d)$ solution $S(t, y)$ of the mild equation, for all $t \geq 0$.

The space $C([0,T], C^1_\infty(\mathbb{R}^d))$ has the norm

$$\|\phi(\cdot)\| = \sup_{t\in[0,T]} \|\phi\|_{C^1(\mathbb{R}^d)} \tag{43}$$

Denote by $B^T_{S_0}$ the space of continuous functions from the closed convex subset of $C([0,T], C^1_\infty(\mathbb{R}^d))$ with $\phi_0 = S_0$.

Sketch of the proof

Denote the RHS operator by $\Psi(\cdot)$. By triangle inequality:

$$\sup_{t\in[0,T]} \|\Psi(\phi)\|_{C^1(\mathbb{R}^d)} \leq \sup_{t\in[0,T]} \| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^{\beta} L) H\left(u, s, y, \frac{d\phi(s,y)}{dy}\right) ds \|_{C^1(\mathbb{R}^d)} + \sup_{t\in[0,T]} \|E_{\beta,1}(-t^{\beta} L) S(0)\|_{C^1(\mathbb{R}^d)}. \quad (44)$$

Let $\phi_1 \neq \phi_2 \in B_{S_0}^T$.

$$\sup_{t\in[0,T]} \|\Psi(\phi_t^1 - \phi_t^2)\|_{C^1(\mathbb{R}^d)} \leq$$
$$\sup_{t\in[0,T]} \left\| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta L) \right.$$
$$\left. \times \left(H(u,s,y,\frac{d\phi_s^1}{dy}) - H(u,s,y,\frac{d\phi_s^2}{dy}) \right) ds \right\|_{C^1(\mathbb{R}^d)}$$
$$\leq \omega(t) \left(\kappa \sup_{s\leq t} \|\phi_s^1 - \phi_s^2\|_{C^1(\mathbb{R}^d)} \right). \tag{45}$$

The RHS operator Ψ is a contraction in the closed convex subset of $C([0,T], C^1_\infty(\mathbb{R}^d))$ for small enough T.

As a consequence, it has a unique fixed point, i.e., there is a unique continuous solution $S(t, y)$ for the mild form of the fHJB,

which is approximated by the iterations of Ψ on $S_0(y)$ by Banach fixed point theorem.

Theorem 4

The assumptions of the previous theorem remain and we add a new one:

- $|H(u,t,y_1,p) - H(u,t,y_2,p)| \leq \tilde{\kappa}|y_1 - y_2|(1+|p|)$ with a certain constant $\tilde{\kappa} \geq 0$.

If $S_0(y) \in C^2_\infty(\mathbb{R}^d)$, the unique solution constructed previously belongs to $C^2_\infty(\mathbb{R}^d)$ and represents a classical solution to the original fDE.

Sketch of the proof

Denote by $B_{S_0}^{T,R,2}$ the space of functions which are twice continuously differentiable in y from the closed convex subset $B_{S_0}^T$ of $C([0,T], C_\infty^1(\mathbb{R}^d))$, have $\phi_0 = S_0$ and

$$\sup_{t\leq T} \|\phi_t\|_{C^2(\mathbb{R}^d)} \leq R \tag{46}$$

for some constant $R > 0$. Then

$$\sup_{t\leq T} \|\Psi(\phi_t)\|_{C^2(\mathbb{R}^d)} \leq \sup_{t\leq T} \left\|E_{\beta,1}(-t^\beta L)S(0)\right\|_{C^2(\mathbb{R}^d)} + \int_0^t \omega(s)R(\kappa+\tilde{\kappa})ds. \tag{47}$$

Want $\sup_{t\leq T}\|\Psi(\phi_t)\|_{C^2(\mathbb{R}^d)}\leq R$, so that Ψ maps $B^{T,R,2}_{S_0}$ to itself. Re-arrange to obtain an expression for R:

$$R \geq \sup_{t\leq T}\|E_{\beta,1}(-t^{\beta}L)S(0)\|_{C^2(\mathbb{R}^d)} \times \left(1-\int_0^T \omega(s)ds(\kappa+\tilde{\kappa})\right)^{-1}. \tag{48}$$

If necessary, reduce T to ηT for some $\eta\in(0,1)$, so that the denominator is positive.

If (48) is satisfied then Ψ maps $B^{R,T,2}_{S_0}$ into itself and consequently the limit of iterations of Ψ on S_0 is also of class $C^2(\mathbb{R}^d)$.

By the previous theorem we know that the solution to the fHJB is unique in $C^1(\mathbb{R}^d)$, hence, the solution $\lim_{k\to\infty}\Psi^k(S_0)$ is a unique solution of class $C^2(\mathbb{R}^d)$. So, by the principle of dynamic programming, there exists a unique classical solution for the simple fHJB on any interval $[0, T]$.

Reading

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- R. Gorenflo, F. Mainardi, Continuous time random walk, Mittag-leffler waiting time and fractional diffusion: mathematical aspects, Arxiv, 2008

Further research

- Still remains to prove $S^\tau \to S$.
- What if the restriction on α and β is not satisfied?
- If $L = \sup_{u\in U} a(y)g(u)D_y^{*\alpha}$, how to analyse the fHJB?
- SDE approach to the limiting process, links to fractional Fokker-Planck equation.
- Applications to insurance, for example considering a difference of two sums of jumps with waiting times dependent on related stability parameters β_1 and β_2 and introducing re-insurance as an additional running cost.
- CTRW with regenerations in limit order book theory
- Many-particle systems behaving as a similar CTRW, mean-field interaction of such systems in games
- CTRW approach to queueing theory

Thank You for listening.
Merci de votre attention.