Connectivity of the excursion sets of Gaussian fields with long-range correlations

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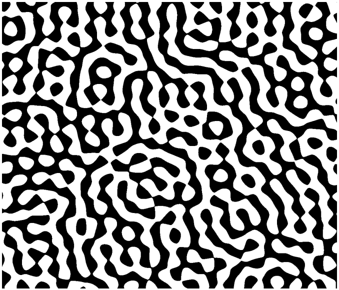

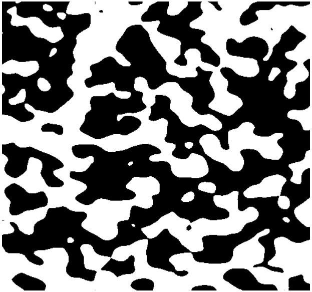

Level set percolation of smooth Gaussian fields

Let f be a smooth centred stationary ergodic Gaussian field on $\mathbb{R}^d$.

We consider the global connectivity of the excursion sets

$$\{f \leq \ell\} := \{x \in \mathbb{R}^d : f(x) \leq \ell\}$$

i.e. 'level set percolation of Gaussian fields'.

By monotonicity, there is a **critical level** $\ell_\mathsf{c} \in [-\infty, \infty]$ such that

$$\mathbb{P}(\{f \leq \ell\} \text{ has an unbounded component}) = \begin{cases} 0 & \text{if } \ell < \ell_\mathsf{c}, \\ 1 & \text{if } \ell > \ell_\mathsf{c}. \end{cases}$$

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(1) *If $\ell \le 0$, $\{f \le \ell\}$ has bounded components a.s.*

(2) *If $\ell > 0$, $\{f \le \ell\}$ has a unique unbounded component a.s.*

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3) Long-range dependent (oscillating): Correlations decay slowly and oscillate infinitely often
e.g. monochromatic random wave $K(x) = J_0(|x|)$
e.g. band-limited random wave $K(x) = J_1(|x|)/|x|$.

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e.g. monochromatic random wave: Lebesgue measure on $\mathbb{S}^1$
e.g. band-limited random wave: Lebesgue measure on $\mathbb{D}$

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$\ell > \ell_c$: unique unbounded component, diameter of finite components have exponential tails

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$\alpha < 1$: diameter of subcritical components have stretched exponential tails with exponent $\alpha < 1$

$\alpha = 0$: phase transition degenerates, 'scale-free' behaviour at all levels

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The belief is that the field is in the Bernoulli universality class if and only if

$$\int_{B(R)} \int_{B(R)} K(x-y)dxdy \ll R^{5/2}.$$

i.e. 'self-averaging' can cause this to be true even if correlations decay very slowly [Weinrib '84, Bogomolny & Schmidt '06]

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E.g. monochromatic random wave: correlations decay as $1/\sqrt{|x|}$ but nevertheless due to oscillations

$$\int_{B(R)} \int_{B(R)} K(x-y)dxdy \sim cR^{3/2}$$

so the monochromatic wave is believed to be in the Bernoulli universality class.

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The properties of **short-range dependency** and **positive correlations** were crucial in these works, since they allow for 'direct' comparison with Bernoulli percolation. However many important fields do not have these properties.

In recent work we established the full conjecture without assuming either of these properties:

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Theorem (M., Rivera & Vanneuville '20)

Let f be a smooth isotropic planar Gaussian field with correlations decaying as

$$|K(x)| \ll (\log\log|x|)^{-3}$$

and assume the support of the spectral measure contains an open set or a circle. Then $\ell_c = 0$.

In particular this is true for the monochromatic random wave, the band-limited random wave etc.

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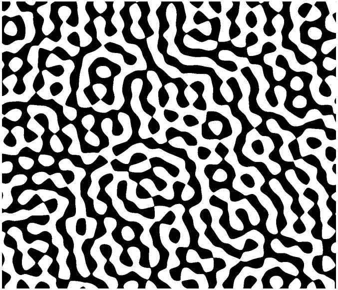
Proving it in general, e.g. for the monochromatic random wave, remains a fundamental open problem.



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As mentioned, physicists predict that in the short-range case components have an exponential tail, whereas long-range correlations should promote larger components.

We consider long-range correlated fields in the regularly varying case $K(x) = |x|^{-\alpha} L(x)$, $\alpha \in [0, d)$, and analyse how the diameter of the components depends on α.

For simplicity we restrict our attention to the one-parameter family of smooth isotropic fields on $\mathbb{R}^2$ with covariance kernels

$$K(x) = (q \star q)(x) \sim |x|^{-\alpha}, \quad q(x) \propto \frac{1}{(1+|x|^2)^{(2+\alpha)/4}}, \quad \alpha \in (0,2).$$

Theorem (M. & Severo '21+)

Let $\mathrm{Arm}_\ell(R)$ *be the event that* $f(0) \leq \ell$ *and the component of* $\{f \leq \ell\}$ *containing* 0 *intersects* $\partial B(R)$*. Then for* $\ell < \ell_c = 0$*:*

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3. *If* $\alpha \in (0,1)$, $-\log \mathbb{P}[\mathrm{Arm}_\ell(R)] \sim c_\alpha \ell^2 R^\alpha$, *where*

$$c_\alpha = \frac{1}{2\pi} B\Big(\frac{1+\alpha}{2}, \frac{1+\alpha}{2}\Big) \cos\Big(\frac{\pi\alpha}{2}\Big) \in (0, 1/2).$$

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3. *If $\alpha \in (0,1)$,*
$$\frac{D_{R,\ell}}{(\log R)^{1/\alpha}} \to \Big(\frac{2}{c_\alpha \ell^2}\Big)^{1/\alpha} \quad \textit{in probability.}$$

These results can be compared to recent work on the Gaussian free field [Snitzman '15, Popov & Rath '15, Goswami, Rodriguez & Severo '21] which has shown that, for $\ell < \ell_c(d) < 0$

$$-\log \mathbb{P}[\mathrm{Arm}_\ell(R)] = \begin{cases} \asymp R & d \geq 4, \\ \sim (\ell_c - \ell)^2 R/(4 \log R) & d = 3. \end{cases}$$

Recalling that $K(x) \sim |x|^{-(d-2)}$ for the GFF, our result shows that the subcritical behaviour of the GFF is 'generic' for Gaussian fields with regularly varying covariance with index $\alpha = d - 2$.

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For example, if $K(x) \sim (\log x)^{-1}$ then

$$\mathbb{P}[\mathrm{Arm}_\ell(R)] \sim R^{\frac{-\ell^2(1+o(1))}{2}},$$

i.e. the model has power-law decay in the subcritical phase.

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The best result in this direction [M., Rivera and Vanneuville '20]

$$-\log \mathbb{P}[\mathrm{Arm}_\ell(R)] \geq c\sqrt{\log R}$$

is very far from the (conjectural) truth.

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$$\mathrm{Cap}_K(D) = \Big(\min_{\mu \in \mathcal{P}(D)} \int_D \int_D K(x-y) d\mu(x) d\mu(y) \Big)^{-1}$$

or

$$\mathrm{Cap}_K(D) = \min \{ \|h\|_H^2 : h \in H, h \geq 1 \text{ on } D \}.$$

[Adler, Moldavskaya & Samorodnitsky, '14]

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2. A **local-global** decomposition of the field

$$f \overset{d}{=} g_L + h_L, \quad L \geq 1$$

into a **global field** g_L which carries the covariance of f on scales $R \gg L$, and a **local field** h_L which is L-range dependent. These are stationary GFs but **not** independent.

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Using regular variation, one can show that the global field carries the covariance on the scale $R \gg L$ in the sense that

$$\lim_{M\to\infty} \limsup_{L\to\infty} \sup_{|x|\geq ML} \left| \frac{\mathbb{E}[g_L(0)g_L(x)]}{K(x)} - 1 \right| = 0$$

The strategy of the proof is to show that $\mathbb{P}[\mathrm{Arm}_\ell(R)]$ is carried, in a large deviation sense, by the event $A_\ell(R)$ in which f has 'excess' mean of $\ell_c - \ell$ on the line-segment $[0, R] \subset \mathbb{R}^d$.

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Lower bound. If $A_\ell(R)$ occurs, the field 'looks supercritical' in a neighbourhood of $[0, R]$, so $\mathrm{Arm}_\ell(R)$ occurs with good probability.

The strategy of the proof is to show that $\mathbb{P}[\mathrm{Arm}_\ell(R)]$ is carried, in a large deviation sense, by the event $A_\ell(R)$ in which f has 'excess' mean of $\ell_c - \ell$ on the line-segment $[0, R] \subset \mathbb{R}^d$.

Lower bound. If $A_\ell(R)$ occurs, the field 'looks supercritical' in a neighbourhood of $[0, R]$, so $\mathrm{Arm}_\ell(R)$ occurs with good probability.

The Cameron-Martin theorem shows that $A_\ell(R)$ has probabilistic cost at most $e^{-\|h\|_H^2/2}$, where $h \in H$ is any function such that $h|_{[0,R]} \geq \ell_c - \ell$. This leads immediately to the lower bound

$$
\begin{aligned}
-\log \mathbb{P}[\mathrm{Arm}_\ell(R)] &\leq \frac{1}{2} \min \left\{ \|h\|_H^2 : h \geq \ell_c - \ell \text{ on } [0, R] \right\} + O(1) \\
&= \frac{1}{2}(\ell_c - \ell)^2 \mathrm{Cap}_K([0, R]) + O(1).
\end{aligned}
$$

Upper bound. For the upper bound we use a (one-step) renormalisation using the (topological) fact that $\mathrm{Arm}_\ell(R)$ implies the existence of a path in $\{f \leq \ell\}$ that crosses many well-separated annuli on some well-chosen mesoscopic scale $1 \ll L \ll R$.

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Fixing $\delta > 0$ small and recalling the local-global decomposition $f = g_L + h_L$, this implies that either:

- The local field excursion set $\{h_L \le \ell_c - \delta\}$ crosses a positive fraction of these annuli; or
- The global field has a high exceedence $\{g_L \ge \ell_c - \ell - \delta\}$ on a positive fraction of these annuli simultaneously.

Using some a priori control on $\mathbb{P}[\mathrm{Arm}_{\ell_c-\delta}(R)]$, since h_L is L-range dependent the first event is very unlikely by independence.

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On the other hand, one can show using regularly variation that the exceedence $\{g_L \geq \ell_c - \ell - \delta\}$ on many annuli has probability

$$\approx \exp\Big(- \frac{(\ell_c - \ell)^2}{2} \mathsf{Cap}_K(\text{'union of annuli'})\Big).$$

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Then by a ‘condensation' argument this capacity is $\sim \mathrm{Cap}_K([0, R])$.

Putting this together gives the matching upper bound

$$-\log \mathbb{P}[\mathrm{Arm}_\ell(R)] \leq \frac{1}{2}(\ell_c - \ell)^2 \mathrm{Cap}_K([0, R]) + O(R).$$

Completing the proof. To finish one can show using properties of regular variation that:

If $\alpha \geq 1$, the measure of minimal energy homogenises, so that

$$\mathrm{Cap}_K([0,R]) \sim \frac{R^2}{\int_0^R \int_0^R K(x-y)dxdy} \sim \frac{R}{2\int_0^R K(x)dx}.$$

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If $\alpha \in [0,1)$, after rescaling the measure of minimal energy of K approximates that for the Riesz kernel $K_\alpha(x) = |x|^{-\alpha}$, so that

$$\mathrm{Cap}_K([0,R]) \sim \frac{1/K(R)}{\min_{\mu \in \mathcal{P}([0,1])} \int_0^1 \int_0^1 |x-y|^{-\alpha} d\mu(x) d\mu(y)}$$

and we take $c_\alpha = (2\min_{\mu \in \mathcal{P}([0,1])} \int_0^1 \int_0^1 |x-y|^{-\alpha} d\mu(x) d\mu(y))^{-1}$.

Thank you!

S. Muirhead, A. Rivera and H. Vanneuville (with an appendix by L. Köhler-Schindler), The phase transition for planar Gaussian percolation models without FKG, preprint, 2020

S. Muirhead and F. Severo, Decay of subcritical connection probabilities for long-range correlated Gaussian fields, in preparation