ON THE CORRELATION BETWEEN CRITICAL POINTS AND CRITICAL VALUES FOR RANDOM SPHERICAL HARMONICS

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Random spherical harmonics

Spherical eigenfunctions are solutions of the Helmholtz equation

$$\Delta_{\mathbb{S}^2} f + \lambda_\ell f = 0,$$

$\Delta_{\mathbb{S}^2}$ is the spherical Laplacian, $\lambda_\ell = \ell(\ell+1)$ for $\ell \in \mathbb{N}$. For any eigenvalue $-\lambda_\ell$ choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(\cdot)\}_{m=-\ell,\dots,\ell}$ and consider random eigenfunctions

$$f_\ell(x) = \frac{\sqrt{4\pi}}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

the coefficients $\{a_{\ell m}\}$ are complex-valued Gaussian, for $m \neq 0$, $\mathrm{Re}(a_{\ell m})$, $\mathrm{Im}(a_{\ell m})$ are zero-mean, independent Gaussian with variance $1/2$, while $a_{\ell 0}$ is a standard Gaussian. The standardization is such that

$$\mathrm{Var}(f_\ell(x)) = 1.$$

The random fields $\{f_\ell(x) : x \in \mathbb{S}^2\}$ are isotropic centred Gaussian with covariance

$$\mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x,y)),$$

P_ℓ Legendre polynomial, $d(x,y) = \arccos\langle x, y\rangle$ geodesic distance on the sphere.

Critical points and critical values

The number of critical points of f_ℓ is denoted by

$$\mathcal{N}_\ell^c = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0\}.$$

Let $I \subseteq \mathbb{R}$ be any interval in the real line, the number of critical points of f_ℓ with value in I, number of critical values, is denoted by

$$\mathcal{N}_\ell^c(I) = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0, f_\ell(x) \in I\}.$$

We investigate how much the number of critical points and critical values characterizes the geometry of the random spherical eigenfunctions in the high frequency limit, i.e. the excursion sets

$$A_u(f_\ell) = \{x \in \mathbb{S}^2 : f_\ell(x) \geq u\},$$

for arbitrary levels $u \in \mathbb{R}$.

[Wigman, 2011], [Marinucci-Wigman, 2014], [Marinucci-Rossi, 2015], [Rossi, 2019], [C.-Marinucci, 2019 and 2020], **[C.-Todino, 2021]**

Asymptotic variance

[C.-Marinucci-Wigman, 2016] and [C.-Wigman, 2017] show that as $\ell \to \infty$

The expected number of critical values behaves like

$$\mathbb{E}[\mathcal{N}_\ell^c(I)] = \frac{2}{\sqrt{3}}\ell^2 \int_I \frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}\, dt + O(1),$$

the constant in the $O(\cdot)$ term is universal, i.e. the integral of the error term on any interval I is uniformly bounded by its value when $I = \mathbb{R}$.

The investigation of the **asymptotic variance** is more challenging

$$\text{Var}(\mathcal{N}_\ell^c(I)) = \ell^3[\nu^c(I)]^2 + O(\ell^{5/2}),$$

$$\nu^c(I) = \int_I \frac{1}{\sqrt{8\pi}}[2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4)]e^{-\frac{3}{2}t^2}\, dt, \qquad \nu^c(\mathbb{R}) = 0,$$

for $I = \mathbb{R}$ the leading term vanishes and

$$\text{Var}(\mathcal{N}_\ell^c) = \frac{1}{3^3\pi^2}\ell^2 \log \ell + O(\ell^2).$$

Similar results hold for extrema and saddles. Proof: via (approximate) Kac-Rice formula for moments.

Interpretation in terms of Wiener chaoses

These results on the asymptotic variance can be interpreted in terms of the $L^2(\Omega)$ expansion of critical points into Wiener chaoses

$$\mathcal{N}_\ell^c(I) = \sum_{q=0}^{\infty} \mathcal{N}_\ell^c(I)[q],$$

$\mathcal{N}_\ell^c(I)[q]$ denotes the projection of $\mathcal{N}_\ell^c(I)$ on the q-order chaos component that is the space generated by the L^2-completion of linear combinations of the form

$$H_{q_1}(\xi_1) \cdot H_{q_2}(\xi_2) \cdots H_{q_k}(\xi_k), \qquad k \geq 1,$$

H_{q_i} are Hermite polynomials, with $q_i \in \mathbb{N}$ such that $q_1 + \cdots + q_k = q$ and $(\xi_1, \ldots, \xi_k)$ standard real Gaussian vector.

It results that (after centring) a **single term dominates** the $L^2(\Omega)$ chaos expansion of $\mathcal{N}_\ell^c(I)$ and $\mathcal{N}_\ell^c$.

We define the random variables called sample polyspectra

$$h_{\ell,q} = \int_{\mathbb{S}^2} H_q(f_\ell(x))\,dx,$$

we have that [Marinucci-Wigman, 2014]

$$\mathrm{Var}(h_{\ell,2}) = (4\pi)^2 \frac{2}{2\ell+1}, \qquad \mathrm{Var}(h_{\ell,4}) = 576 \frac{\log \ell}{\ell^2} + O(\ell^{-2}),$$

and, for $q = 3$ and $q \geq 5$, the order of magnitude of the variances is smaller

$$\mathrm{Var}(h_{\ell,q}) = \frac{c_q}{\ell^2} + o(\ell^{-2}), \qquad c_q = \int_0^\infty \psi\, J_0(\psi)^q\, d\psi,$$

$J_0(\cdot)$ is the Bessel function of order zero.

These results suggest that the asymptotic behaviour of the total number of critical points is dominated by the projection into the fourth chaotic component, which can be expressed by the integral $h_{\ell,4}$.

The number of critical values in I is dominated by the projection into the second chaotic component, which can be expressed by $h_{\ell,2}$.

We introduce the random variables

$$\mathcal{S}_\ell(I) = \frac{\lambda_\ell}{2}\nu^c(I)\frac{1}{2\pi}\int_{\mathbb{S}^2} H_2(f_\ell(x))dx, \qquad \mathcal{F}_\ell = -\frac{\lambda_\ell}{2^3 3^2\sqrt{3}\pi}\int_{\mathbb{S}^2} H_4(f_\ell(x))dx.$$

[C.-Marinucci, 2020] shows that, as $\ell \to \infty$, for $I \subset \mathbb{R}$ such that $\nu^c(I) \neq 0$,

$$\mathcal{N}_\ell^c(I) - \mathbb{E}[\mathcal{N}_\ell^c(I)] = \mathcal{N}_\ell^c(I)[2] + R_\ell(I), \qquad \mathcal{N}_\ell^c(I)[2] = \mathcal{S}_\ell(I),$$

$\mathbb{E}[R_\ell^2(I)] = o(\ell^3)$ uniformly over I. $\mathcal{N}_\ell^c(I)$ **is fully correlated in the limit with** $\mathcal{S}_\ell(I)$

$$\text{Corr}(\mathcal{N}_\ell^c(I), \mathcal{S}_\ell(I)) = \frac{\text{Cov}(\mathcal{N}_\ell^c(I), \mathcal{S}_\ell(I))}{\sqrt{\text{Var}(\mathcal{N}_\ell^c(I))\text{Var}(\mathcal{S}_\ell(I))}} \to 1.$$

[C.-Marinucci, 2019] shows that, as $\ell \to \infty$

$$\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right] = \mathcal{N}_\ell^c[4] + o_{\mathbb{P}}(\sqrt{\ell^2 \log \ell}), \qquad \mathcal{N}_\ell^c[4] = \mathcal{F}_\ell,$$

i.e. $\mathcal{N}_\ell^c$ **is fully correlated in the limit with** $\mathcal{F}_\ell$

$$\text{Corr}(\mathcal{N}_\ell^c, \mathcal{F}_\ell) = \frac{\text{Cov}(\mathcal{N}_\ell^c, \mathcal{F}_\ell)}{\sqrt{\text{Var}(\mathcal{N}_\ell^c)\text{Var}(\mathcal{F}_\ell)}} \to 1.$$

Important consequences

- While the computation of $\mathcal{N}_\ell^c$ and $\mathcal{N}_\ell^c(I)$ via Kac-Rice formula requires the evaluation of **gradient and Hessian fields**, the dominant terms depend, in the high frequency limit, only on the second-order and fourth-order Hermite polynomials evaluated at the eigenfunctions f_ℓ, i.e. only on $h_{\ell,4}$ and $h_{\ell,2}$, respectively.

- $h_{\ell,2}$ is proportional to a sum of independent and identically distribute random variables with zero mean and finite variance

$$h_{\ell,2} = \int_{\mathbb{S}^2} f_\ell^2(x)\, dx - 4\pi = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2 - \mathbb{E}|a_{\ell m}|^2$$

as a simple corollary, a **quantitative Central Limit Theorem** for $\mathcal{N}_\ell^c(I)$. Similarly for $\mathcal{N}_\ell^c$, the limiting distribution of $h_{\ell;4}$ was studied in [Marinucci-Wigman, 2014], where it is shown a quantitative Central Limit Theorem for $h_{\ell;4}$.

Correlation structure between $\mathcal{N}_\ell^c(I)$ and $\mathcal{N}_\ell^c$

Partial correlation coefficient between two random variables X_i, $i = 1, 2$, with respect to a random variable Z

$$\text{Corr}_Z(X_1, X_2) = \frac{\text{Corr}(X_1, X_2) - \text{Corr}(X_1, Z)\text{Corr}(X_2, Z)}{\sqrt{1 - \text{Corr}^2(X_1, Z)}\sqrt{1 - \text{Corr}^2(X_2, Z)}} = \text{Corr}(X_1^*, X_2^*),$$

where the random variables X_i^* are defined by

$$X_i^* := (X_i - \mathbb{E}[X_i]) - \frac{\text{Cov}(X_i, Z)}{\text{Var}(Z)}(Z - \mathbb{E}[Z]).$$

In our context the random variables involved are

$$X_1 = \mathcal{N}_\ell^c, \qquad X_2 = \mathcal{N}_\ell^c(I), \qquad Z = ||f_\ell(x)||^2_{L^2(\mathbb{S}^2)},$$

the partial correlation coefficient **measures the liner dependence between $\mathcal{N}_\ell^c$ and $\mathcal{N}_\ell^c(I)$ after getting rid of the components depending on the random L^2-norm of the eigenfunctions f_ℓ.**

[C.-Todino, 2021] shows that the **correlation between $\mathcal{N}_\ell^c(I)$ and $\mathcal{N}_\ell^c$ is asymptotically zero when $I \neq \mathbb{R}$ and $\nu^c(I) \neq 0$, while the partial correlation, after controlling for the random L^2-norm on the sphere of the eigenfunctions, is asymptotically one.**

Assuming $I_1 \subseteq \mathbb{R}$ is such that $\nu^c(I_1) = 0$

$$\lim_{\ell\to\infty} \text{Corr}(\mathcal{N}_\ell^c(I_1), \mathcal{N}_\ell^c(I_2)) = \begin{cases} 0 & \text{if } \nu^c(I_2) \neq 0, \\ 1 & \text{if } \nu^c(I_2) = 0, \end{cases}$$

and for every $I_1, I_2 \subseteq \mathbb{R}$

$$\lim_{\ell\to\infty} \text{Corr}_{||f_\ell(x)||^2_{L^2(\mathbb{S}^2)}}(\mathcal{N}_\ell^c(I_1), \mathcal{N}_\ell^c(I_2)) = 1.$$

In particular for $I_1 = \mathbb{R}$.

Theorem

For subsets $I \subset \mathbb{R}$ such that $\nu^c(I) \neq 0$,

$$\lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{N}_\ell^c, \mathcal{N}_\ell^c(\mathrm{I})) = 0,$$

and for every $I \subseteq \mathbb{R}$

$$\lim_{\ell\to\infty} \mathrm{Corr}_{||f_\ell||^2_{L^2(\mathbb{S}^2)}}(\mathcal{N}_\ell^c, \mathcal{N}_\ell^c(I)) = 1.$$

Proof

Built the approximating sequence

$$\mathcal{N}^c_{\ell,\varepsilon}(I) = \int_{\mathbb{S}^2} |\text{det}\nabla^2 f_\ell(x)| \mathbb{I}_{\{f_\ell(x)\in I\}} \delta_\varepsilon(\nabla f_\ell(x)) dx, \quad \delta_\varepsilon(z) = \frac{1}{\varepsilon^2} \mathbb{I}_{\{z\in[-\varepsilon/2,\varepsilon/2]^2\}}$$

for every $\ell \in \mathbb{N}$, we have, both ω-a.s. and in $L^2(\Omega)$, that $\mathcal{N}^c_\ell(I) = \lim_{\varepsilon\to 0} \mathcal{N}^c_{\ell,\varepsilon}(I)$.

For $I_1, I_2 \subseteq \mathbb{R}$, compute the second and fourth order chaos

$$\mathcal{N}^c_\ell(I_i) - \mathbb{E}[\mathcal{N}^c_\ell(I_i)] = \mathcal{N}^c_\ell(I_i)[2] + \mathcal{N}^c_\ell(I_i)[4] + R_\ell(I_i),$$

$$\mathcal{N}^c_\ell(I_i)[2] = \frac{\lambda_\ell}{2} \nu^c(I) \frac{1}{2\pi} \int_{\mathbb{S}^2} H_2(f_\ell(x)) dx,$$

$$\mathcal{N}^c_\ell(I_i)[4] = \lambda_\ell \frac{51\mathcal{I}_0(I) - 2\cdot 11\mathcal{I}_2(I) + \mathcal{I}_4(I)}{2^3\pi} \int_{\mathbb{S}^2} H_4(f_\ell(x))\, dx,$$

Compute the variance with (approximate) Kac-Rice

$$\mathrm{Var}(\mathcal{N}^c_\ell(I)) = [\nu^c(I)]^2 \ell^3 + \frac{[51\mathcal{I}_0(I) - 2\cdot 11\mathcal{I}_2(I) + \mathcal{I}_4(I)]^2}{2^6\pi^2} \ell^2 \log \ell + O(\ell^2).$$

Use orthogonality of Wiener chaoses and $\mathcal{N}^c_\ell(I)^* = \sum_{q=3}^\infty \mathcal{N}^c_\ell(I)[q]$ to derive

$$\lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{N}^c_\ell(I_1), \mathcal{N}^c_\ell(I_2)), \quad \lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{N}^c_\ell(\mathbb{R}), \mathcal{N}^c_\ell(I_2)), \quad \lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{N}^c_\ell(I_1)^*, \mathcal{N}^c_\ell(I_2)^*).$$

Remark

$\mathcal{N}_\ell^c$ and $\mathcal{N}_\ell^c(I)$ are asymptotically independent, but, when the effect of random fluctuations of the norm of f_ℓ is properly subtracted, their joint distribution is completely degenerate and the behaviour of the fluctuations of $\mathcal{N}_\ell^c(I)$ is fully explained by $\mathcal{N}_\ell^c$, in the high energy limit.

More precisely, denoting with $\widehat{\mathcal{N}}_\ell^c$ and $\widehat{\mathcal{N}}_\ell^c(I)$ the standardised variables, as $\ell \to \infty$, for $I \subset \mathbb{R}$ such that $\nu^c(I) \neq 0$,

$$(\widehat{\mathcal{N}}_\ell^c, \widehat{\mathcal{N}}_\ell^c(I)) \overset{law}{\to} (Z_1, Z_2), \qquad (\widehat{\mathcal{N}}_\ell^{c*}, \widehat{\mathcal{N}}_\ell^{c*}(I)) \overset{law}{\to} (Z, Z),$$

(Z_1, Z_2) bivariate vector of standard independent Gaussian variables, Z standard Gaussian variable.

Excursion sets

Our result fits in the framework of the literature which has investigated the relationship between geometric functionals of excursion sets of f_ℓ at different levels u

$$A_u(f_\ell) = \{x \in \mathbb{S}^2 : f_\ell(x) \geq u\}.$$

The functionals which describe the geometry of such sets are the so called Lipschitz-Killing Curvatures, which correspond to

- area $\mathcal{L}_2(u, \ell)$
- (half of the) boundary length $\mathcal{L}_1(u, \ell)$
- Euler characteristic $\mathcal{L}_0(u, \ell)$

of the excursion sets.

Level curves and nodal lines

[Wigman 2010] showed that the behaviour of level curves $\mathcal{L}_1(u,\ell)$ is very different compared to the behaviour of nodal lines $\mathcal{L}_1(0,\ell)$; the variances are asymptotic to

$$\mathrm{Var}(\mathcal{L}_1(u,\ell)) \sim c_2 u^4 e^{-u^2}\ell, \qquad \mathrm{Var}(\mathcal{L}_1(0,\ell)) = \frac{1}{32}\log\ell + O(1).$$

[Wigman 2011] showed that the length of the level curves becomes asymptotically fully correlated for large ℓ; for $u_1, u_2 \neq 0$

$$\mathrm{Corr}(\mathcal{L}_1(u_1,\ell), \mathcal{L}_1(u_2,\ell)) = 1 + o_{\ell\to\infty}(1).$$

Proof: Kac-Rice.

Lipschitz-Killing curvatures

[Marinucci-Wigman, 2014], [Marinucci-Rossi, 2015], [C.-Marinucci, 2018] show that the three Lipschitz-Killing curvatures are asymptotically fully correlated to $h_{2;\ell}$ for all $u_1, u_2 \neq 0$ (and $u \neq 1, -1$ for the Euler characteristic) and then

$$\lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{L}_j(u_1,\ell), \mathcal{L}_k(u_2,\ell)) = 1, \qquad j,k = 0,1,2.$$

The number of critical values is then perfectly correlated, as $\ell \to \infty$, with the area, the Euler characteristic and the boundary length at any nonzero level u

$$\lim_{\ell\to\infty} \mathrm{Corr}(\mathcal{L}_k(u,\ell), \mathcal{N}_\ell(u,\infty)) = 1, \qquad k = 0,1,2.$$

Nodal lines, level curves and critical points

The leading term corresponding to $h_{2;\ell}$ of all these geometrical functionals vanishes and the asymptotic behaviour is different: for $u \neq 0$

$$\lim_{\ell\to\infty} \text{Corr}(\mathcal{L}_1(0,\ell), h_{4;\ell}) = 1, \quad \lim_{\ell\to\infty} \text{Corr}(\mathcal{L}_1(0,\ell), \mathcal{L}_1(u,\ell)) = 0,$$

after removing the effect of the norm [Marinucci-Rossi, 2021] for any $u \in \mathbb{R}$, it holds that

$$\lim_{\ell\to\infty} \text{Corr}_{||f_\ell(x)||_{L^2(\mathbb{S}^2)}}(\mathcal{L}_1(0,\ell), \mathcal{L}_1(u,\ell)) = 1.$$

Critical values and critical points are asymptotically independent, hence critical points carry no information about the other geometrical functionals at any non-zero levels, for $u \neq 0$,

$$\lim_{\ell\to\infty} \text{Corr}(\mathcal{L}_k(u,\ell), \mathcal{N}_\ell^c) = 0, \qquad k = 0,1,2,$$

the sample norm dominates the Lipschitz-Killing curvatures of the excursion sets at non-zero levels. When its effect is adequately removed, the behaviour of $\mathcal{L}_1(u,\ell)$ at any level is fully explained by the total number of critical points

$$\lim_{\ell\to\infty} \text{Corr}(\mathcal{L}_1(0,\ell), \mathcal{N}_\ell^c) = 1, \qquad \lim_{\ell\to\infty} \text{Corr}_{||f_\ell(x)||_{L^2(\mathbb{S}^2)}}(\mathcal{L}_1(u,\ell), \mathcal{N}_\ell^c) = 1 \ u \neq 0.$$

Euler characteristic and critical points

[C.-Marinucci-Wigman, 2016] shows that

$$\text{Var}(\mathcal{L}_0(u,\ell)) = \frac{\ell^3}{4}\left[H_1(u)H_2(u)\frac{e^{-u^2/2}}{\sqrt{2\pi}}\right]^2 + O(\ell^2\log^2\ell),$$

and [C.-Marinucci 2018] shows that the high frequency behaviour of $\mathcal{L}_0(u,\ell)$ is dominated by the projection onto the second order chaos

$$\mathcal{L}_0(u,\ell) - \mathbb{E}[\mathcal{L}_0(u,\ell)] = \mathcal{L}_0(u,\ell)[2] + o_{\mathbb{P}}(\sqrt{\text{Var}(\mathcal{L}_0(u,\ell))}),$$

with

$$\mathcal{L}_0(u,\ell)[2] = \frac{\ell^2}{2}\left[H_1(u)H_2(u)\frac{e^{-u^2/2}}{\sqrt{2\pi}}\right]h_{\ell,2} + R(\ell)$$

where $\mathbb{E}[R^2(\ell)] = O(\ell^2\log\ell)$. The projection onto the second order chaos term disappears in the nodal case. However, differently from what happens with nodal length and critical points, the fourth chaos vanishes as well

Theorem

$$\mathcal{L}_0(0,\ell)[4] = 0.$$

Thank you!