Geometry of random nodal domains: Universality of the outliers in weakly confined Coulomb systems.

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Random polynomials

Two ingredients:

- $(a_k)_{k\in\mathbb{N}}$ a family of independent $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables
- $\nu \in \mathcal{P}(\mathbb{C})$ and its logarithmic potential $V^{\nu}(z) = \int \log|z-w| d\nu(w)$.

Définition

If $(R_{k,N})_{k\leq N}$ is an O.N.B. of $\mathbb{C}_N[X]$ for $\langle P, Q\rangle = \int P\overline{Q}e^{-2NV^{\nu}} d\nu$

$$P_N(z) = \sum_{k=0}^{N} a_k R_{k,N}(z)$$

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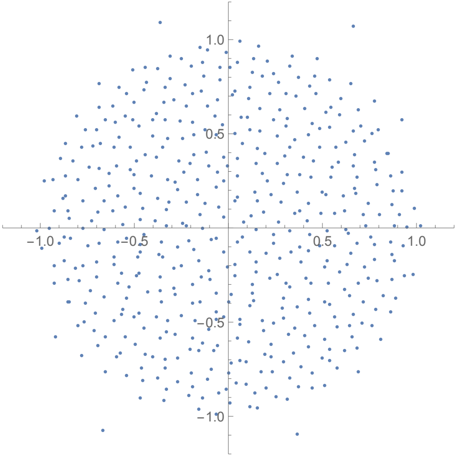
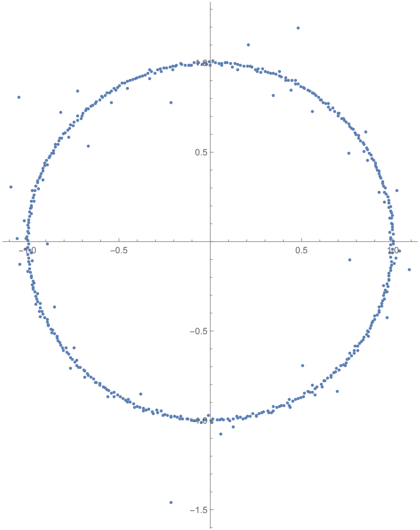
This model was initially introduced by Zeitouni and Zelditch with a geometric point of view. It covers all known model of random polynomials for good choices of ν.

Zeros of random polynomials.



ν = uniform on $\mathbb{D}$.



ν uniform on $\mathbb{S}^1$.

Outliers?

For this model of random polynomials, the behavior of the empirical measures is well understood, $\frac{1}{N}\sum_{k=1}^{N}\delta_{z_k} \xrightarrow[N\to\infty]{} \nu$

- For a very large class of measures ν (Zeitouni–Zelditch)
- and for most distributions of coefficients with a nice density, with a large deviations principle (B.–Zeitouni)

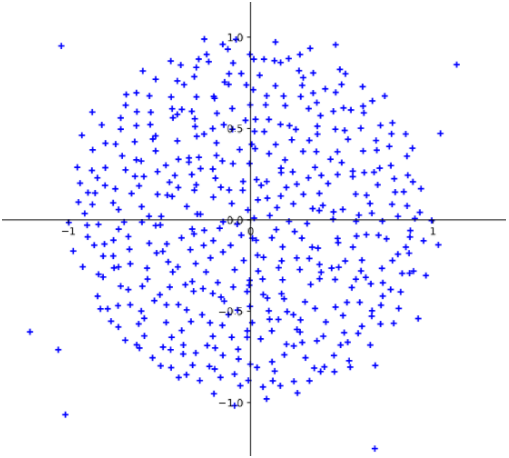
Universality for the convergence of empirical measures is a very wide topic.

Outliers

For a given connected component Ω of $\mathbb{C} \setminus \operatorname{supp}\nu$, we want to understand the behavior of the outliers in Ω

$$\Phi_{\Omega,N} = \{z \in \Omega, P_N(z) = 0\}.$$

Jellium on the unit circle and disk, $\beta = 2$



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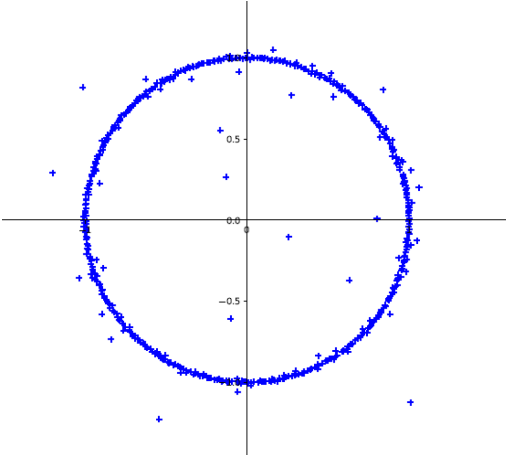

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Coulomb gas in dimension 2

N electrons with charge -1, under an electrical field $(N+1)V$

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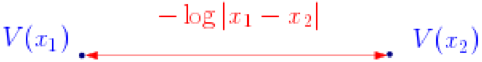

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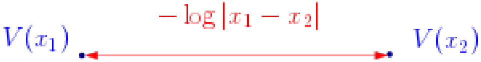

For 2 electrons, at the locations $x_1, x_2 \in \mathbb{C}$, the total energy is

- Coulombian repulsion between electrons: $-\log|x_1 - x_2|$
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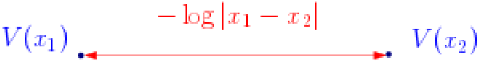

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Attention: We consider an electrical field $(N+1)V$ proportional to the number of electrons. For N electrons, the total energy becomes

$$H_N(x_1, \ldots, x_N) = \sum_{i<j} -\log|x_i - x_j| + (N+1)\sum_{k=1}^{N} V(x_k)$$

The "Jellium"

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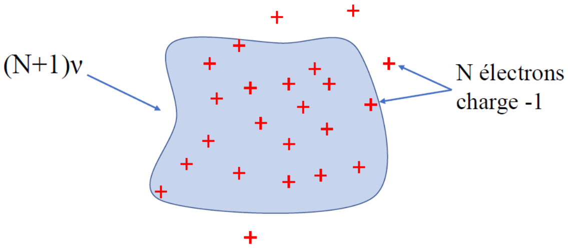
The jellium model corresponds to $V(z) = V^{\nu}(z) = \int \log|z-w| d\nu(w)$, where $\nu \in \mathcal{P}(\mathbb{C})$.



A positive, continuous distribution of charges attracts the electrons. The system is nearly neutral, the charge of ν is $(N+1)$. The system cannot be charge neutral in infinite volume.

Determinantal jellium

Définition

The determinantal jellium with N particles and background $(N+1)\nu$ is a random vector in $\mathbb{C}^N$, with distribution

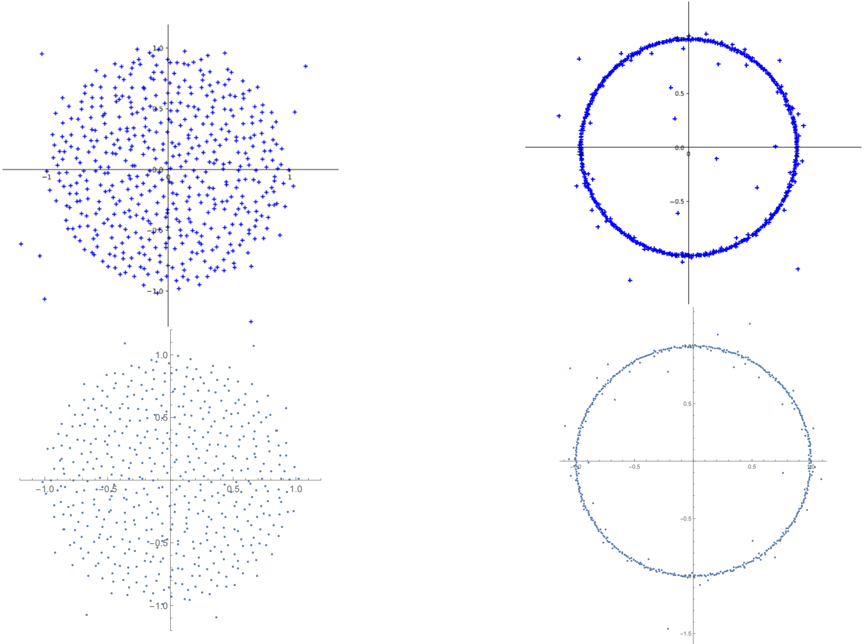
$$(x_1, \dots, x_N) \sim \frac{1}{Z_N} e^{-2\left(\sum_{i<j} -\log|x_i - x_j| + (N+1)\sum_{k=1}^{N} V^{\nu}(x_k)\right)} d\lambda_{\mathbb{C}^N}$$

where $Z_N(V, \beta)$ is a normalizing constant.

This model is expected to "look like" the zeros of a random polynomials with Gaussian coefficients, associated to ν (because the joint distributions look alike).
The macroscopic convergence is already known for this model

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{x_k} \xrightarrow[N\to\infty]{a.s.} \nu$$

Comparison



Outliers for random polynomials

Theorem (B.– García-Zelada–Nishry–Wenmann (2021))

Let $\nu \in \mathcal{P}(\mathbb{C})$, supported on a closed analytic Jordan curve Γ, with a real analytic density on this curve.
Let Ω be a any of two components of $\mathbb{C} \setminus \operatorname{supp}\nu$.

$$\Phi_{\Omega,N} \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{B}_\Omega$$

where $\mathcal{B}_\Omega$ is the Bergman process on Ω. In addition, the two limiting processes are independent.

The limiting point process is the zero set of the Szegő random function of Ω

$$f_\Omega(z) = \sum_{k=0}^{+\infty} a_k \psi_k(z) \text{ where the } a_k' s \text{ are i.i.d. } \mathcal{N}_\mathbb{C}(0,1)$$

and ψ_k's O.N.B of $H^2(\Gamma)$ for $< f, g >= \int_\Gamma f\bar{g} d\sigma_\Gamma$, where σ_Γ is the arclenght measure.

Remarks

Results from B.–García-Zelada

- This result holds without any regularity condition on ν if it is radial.
- In the radial case, the result holds for any distribution of the coefficients.

In the case where ν is supported on the unit circle, the Szegő function is

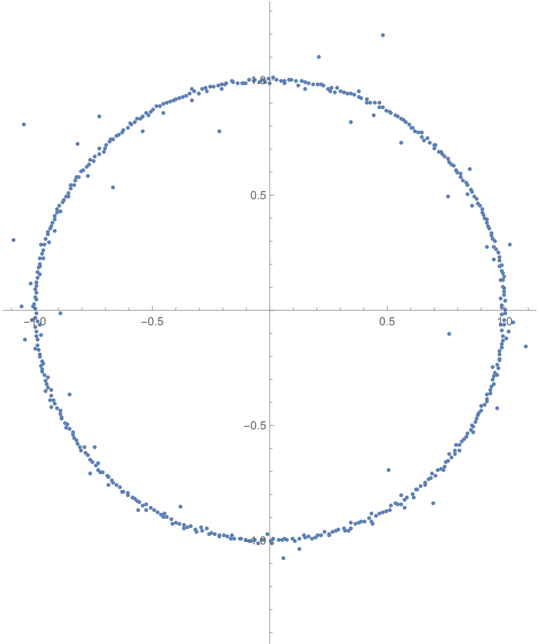
$$f(z) = \sum_{k=0}^{+\infty} a_k z^k, \text{ where the } a_k's \text{ are i.i.d. } \mathcal{N}_{\mathbb{C}}(0,1).$$

Conjecture

The result should hold for any distribution on the coefficients, and for reasonable supports as in the jellium case.

We lack evidence to support this conjecture apart from the link with the jellium.

Every point is an outlier



Outliers of the jellium

Theorem (B.– García-Zelada–Nishry–Wenmann (2021))

Let $\nu \in \mathcal{P}(\mathbb{C})$*, be a "nice" measure.*
Let Ω *be a simply connected component* $\mathbb{C} \setminus \mathrm{supp}\,\nu$*.*

$$X_N \cap \Omega \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{B}_\Omega$$

where $\mathcal{B}_\Omega$ *is the Bergman process on* Ω *(DPP associated to the Bergman kernel of* Ω*).*

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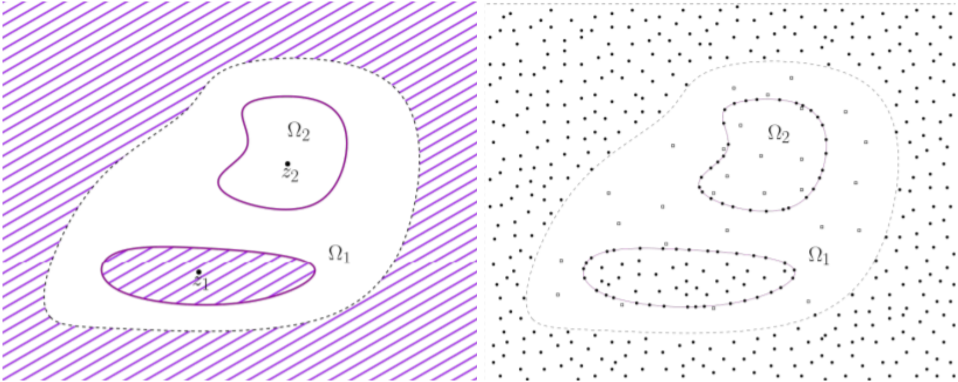
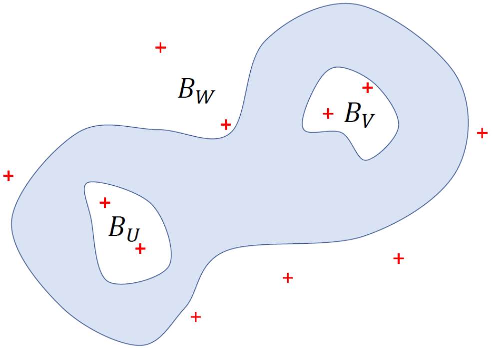
Radial case, without regularity condition: B.–García-Zelada.

Nice measures, illustration by Alon Nishry



On the left, on admissible measure, on the right, an illustration of what the jellium should look like (not a simulation). The boundary of the components are C^ω curves, and the densities with respect to Lebesgue or arclenght are smooth.

Illustration



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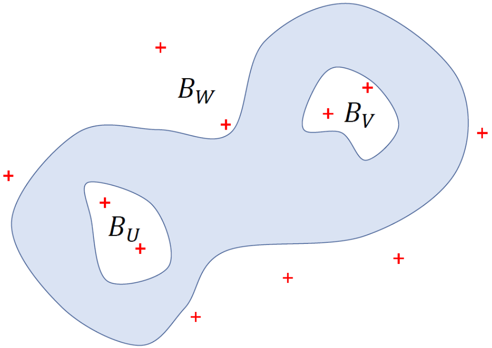


$\phi_U : \mathbb{D} \to U$ conformal map $\phi_U(\mathcal{B}_\mathbb{D}) = \mathcal{B}_U$

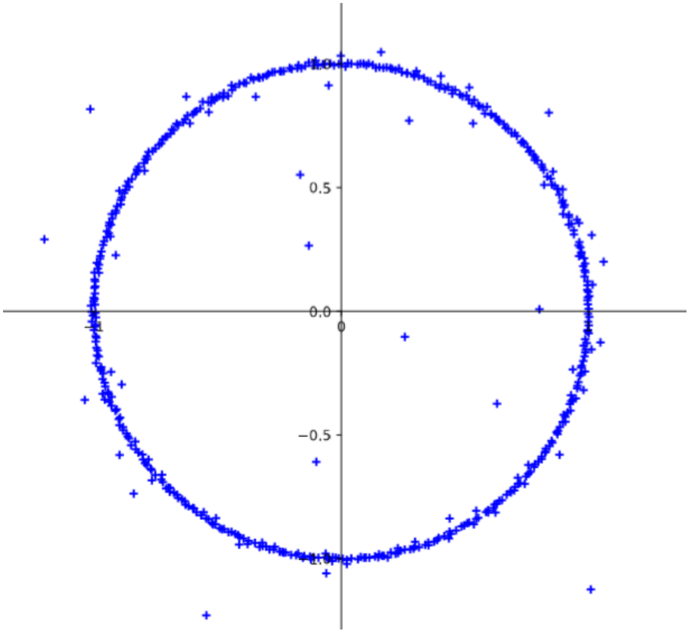
$\mathcal{B}_\mathbb{D}$ determinantal with kernel

$$B_\mathbb{D}(z,w) = \frac{1}{\pi(1-z\bar{w})^2}.$$

In particular, for any $A \subset \mathbb{D}$

$$\mathbb{E}(\mathcal{B}_\mathbb{D} \cap A) = \int_A \frac{1}{\pi(1-|z|^2)^2} d\ell(z).$$

Bergman process in the unit disk



Convergence of kernels: random polynomials

Random polynomials give a Gaussian field

$$(P_N(z))_{z\in\Omega} \text{ Gaussian field, with correlation kernel } C_N(z,w) = \sum_{k=0}^{N} R_{k,N}(z)\overline{R_{k,N}(w)}.$$

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It is sufficient to find a sequence of non-vanishing functions h_n such that

$$h_n(z)\overline{h_n(w)}C_N(z,w) \xrightarrow[N\to\infty]{\text{loc unif}} S_\Omega(z,w)$$

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The zero set of the GAF with covariance kernel S_Ω is the Bergman point process of Ω (Peres–Virág)

Determinantal Jellium: $\beta = 2$.

Theorem

The Coulomb gas with N particles, potential V and inverse temperature $\beta = 2$ is a determinantal point process on $\mathbb{C}$, with kernel associated to the Lebesgue measure

$$\forall z, w \in \mathbb{C} \quad K_N(z,w) = \sum_{k=0}^{N-1} Q_{k,N}(z)\overline{Q_{k,N}}(w) e^{-(N+1)(V(z)+V(w))}$$

where the $Q_{k,n}$'s are an orthonormal basis of $L^2(e^{-(N+1)V}) \cap \mathbb{C}_{N-1}[X]$.

The kernel is not unique, for instance, one can add to V a complex phase of the form iV_1.

The $Q_{k,N}$ are not the $R_{k,N}$! The inner products are

$$\langle P, Q \rangle = \int_{\mathbb{C}} P\bar{Q} e^{-2(N+1)V^{\nu}} d\ell \text{ and } \langle P, Q \rangle = \int_{\Gamma} P\bar{Q} e^{-2NV^{\nu}} d\nu$$

Why are DPPs so convenient?

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Local convergence of the kernels $\implies$ Weak convergence of point processes (law of number of points in compact sets)
We have some freedom in the choice of the kernel K_N (for the complex phase), and we want to find a good sequence such that, locally uniformly in $\Omega \times \Omega$

$$K_N(z,w) \xrightarrow[N\to\infty]{\text{loc unif}} B_\Omega(z,w).$$

Sketch of proof

Step 1: the key inequality Using reproducing kernel properties, one can show that, if Ω is simply connected

$$\forall z \in \Omega, \forall N \in \mathbb{N}, \quad K_N(z,z) \leq B_\Omega(z,z)$$

and for random polynomials, if ν is **supported on a curve** and h_N is nonvanishing and holomorphic

$$\forall z \in \Omega, \forall N, \quad |h_N(z)|^2 C_N(z,z) \leq S(z,z).$$

If you chose the kernels to be holomorphic in z and $\bar{w}$, Montel's theorem implies that the kernels form a normal family. Hence you only have to show pointwise convergence on the diagonal to identify the local uniform limit.

Making kernels holomorphic

Jellium: As V^γ is harmonic on Ω (simply connected), there exists an harmonic conjugate iV_1 such that $V = V^\gamma + iV_1$ is holomorphic.

Random polynomials: By analogy with the jellium, we choose $h_N(z) = e^{NV(z)}$

Step 2: Pointwise convergence of orthogonal polynomials.
For random polynomials: The kernel does not depend on the choice of the orthonormal basis. Our goal is to find a good basis of the Hardy space $H^2(\Gamma)$, $(\psi_k)_{k\in\mathbb{N}}$ and orthonormal bases of $\mathbb{C}_N[X] \cap L^2(e^{-2NV^\nu} d\nu)$ such that for fixed k

$$|R_{k,N}(z)|^2 e^{-2NV^\nu(z)} \underset{N\to\infty}{\longrightarrow} |\psi_k(z)|^2.$$

For the Jellium, the idea is the same but the scalar products are different. If $(\psi_k)_{k\in\mathbb{N}}$ is a "good" orthonormal basis of $L^2(\Omega) \cap \mathcal{H}$ and $Q_{k,N}$ is an O.N.B. of $\mathbb{C}_N[X] \cap L^2(e^{-2(N+1)V^\nu} d\ell, \Omega)$

$$|Q_{k,N}(z)|^2 e^{-2(N+1)V^\nu(z)} \underset{N\to\infty}{\longrightarrow} |\phi_k(z)|^2.$$

How do we obtain the asymptotic?

Starting point: $(\phi_k)_{k\in\mathbb{N}}$ orthonormal in $L^2(\Omega) \cap \mathcal{H}$. For simplicity, imagine that Ω is the unbounded component.

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2. the $F_{k,N}$ are orthonormal in $L^2(e^{-2(N+1)V^\vee} d\ell, \Omega)$, but they are not polynomials and are not holomorphic.
3. $\bar{\partial}$ magic: there exists a correction $v_{k,N}$ with small L^2 norm such that $F_{k,N} - v_{k,N}$ is holormorphic on $\mathbb{C}$
   + Liouville Theorem (f entire, $|f(z)| \leq C|z|^k \implies$ f polynomial or degree $\leq$ k.)

$\bar\partial$ magic

Theorem ($\bar\partial$ Hörmander estimates)

Under some regularity assumptions on the weight ϕ, for any $f \in L^\infty(\mathbb{C})$, there exists a solution $v : \mathbb{C} \to \mathbb{C}$ to the equation $\bar\partial v = f$ such that

$$\int_{\mathbb{C}} |v(z)|^2 e^{-\phi(z)} dm(z) \leq \int |f(z)|^2 \frac{e^{-\phi(z)}}{\Delta\phi(z)} dm(z).$$

We apply this theorem with $\phi = 2\kappa_N V^\nu$ and $f = \bar\partial F_{k,N}$, so we get $\bar\partial \left(F_{k,N} - v \right) = 0$.

We cannot apply Hörmander's theorem directly, but we can make it work after some mollifications.

For random polynomials, the ideas are the same, but one need to show that Γ can be thickened a little to apply the cutoff.

End of proof

The polynomials $\tilde{Q}_{k,N} = F_{k,N} - \nu k, N$ differ from an orthonormal system by a quantity with norm going to zero.

We apply the Gram–Schmidt procedure to this family and we obtain an orthonormal basis $Q_{k,N}$ which has the same convergence properties as the $F_{k,N}$.

The continuity of the Gram-Schmidt algorithm gives

$$|Q_{k,N}(z)|^2 e^{-2(N+1)V^{\nu}(z)} \underset{N\to\infty}{\longrightarrow} |\phi_k(z)|^2$$

Jellium in the non-simply connected case

Théorème (B.–García-Zelada–Nishry–Wenmann)

Let ν be a nice measure, and let Ω be a l-connected component of $\mathbb{C} \setminus \operatorname{supp} \nu$. We fix $z_1, \ldots, z_l$ in each of the hole in Ω and we write $q_1, \ldots, q_l$ the charge of this components. Assume that

$$(e^{2i\pi(N+1)q_1}, \ldots, e^{2i\pi(N+1)q_l}) \underset{subsequence}{\longrightarrow} (e^{2i\pi Q_1}, \ldots, e^{2i\pi Q_l})$$

then

$$\{x_1, \ldots, x_n\} \cap \Omega \underset{subsequence}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{B}_{\Omega,Q}$$

where $\mathcal{B}_{\Omega,Q}$ is the Bergman process on Ω associated to the weight $\prod_{k=1}^{l} |z - z_k|^{-2q_k}$.

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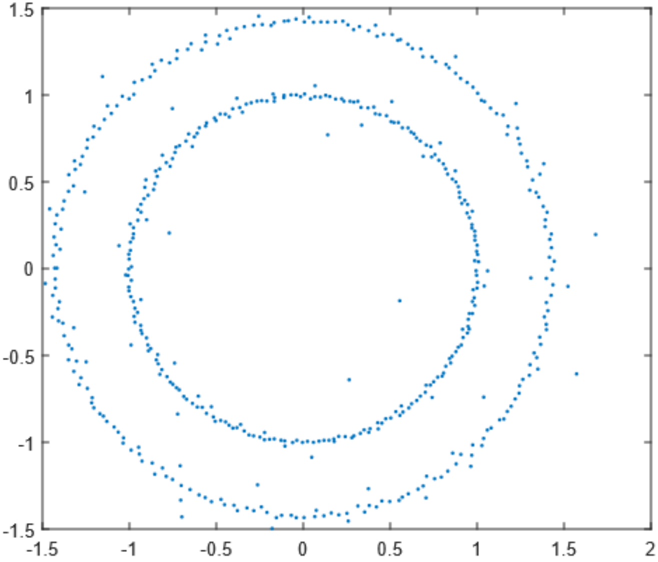
The outliers process does not converge (in general), but we know all the possible limits of subsequences. The limiting process is a DPP associated to the kernel of the projection $L^2(\prod_{k=1}^{l} |z - z_k|^{-2q_k} 1_\Omega) \longrightarrow L^2(\prod_{k=1}^{l} |z - z_k|^{-2Q_k} 1_\Omega) \cap \mathcal{H}$.

Two circles with charge $(N + 1)q$ and $(N + 1)(1 - q)$



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1. For the inner disk, the theorem applies directly as it is simply connected.
2. For the middle annulus, the convergence of the outlier process will depend on the convergence of $e^{2i\pi(N+1)q} \to e^{2i\pi Q}$.

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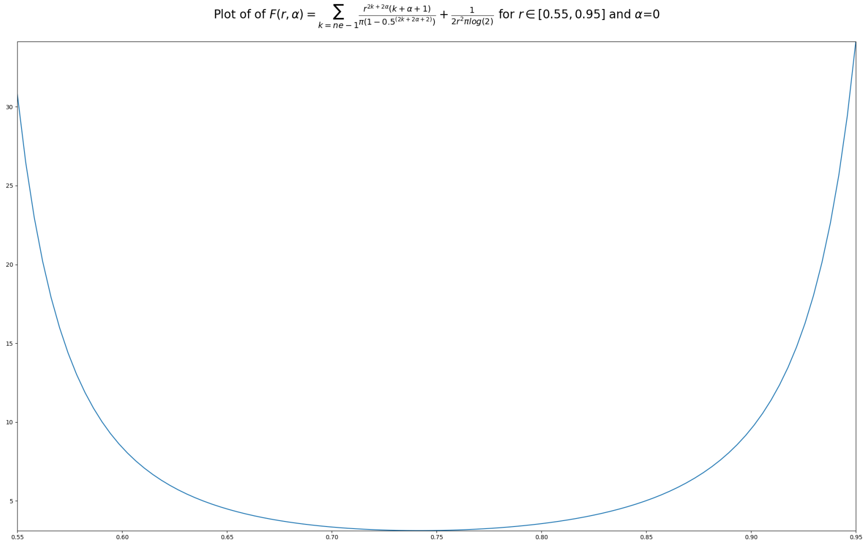
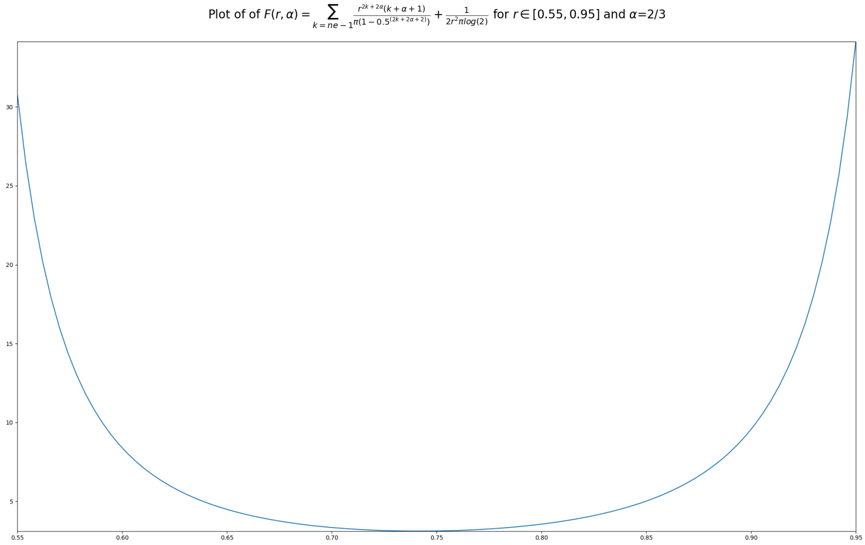
For the annulus, we can compute the possible intensities in the limit

$$\begin{cases} Q = 0 & B(z,z) = \frac{1}{2|z|^2\pi \log 2} + \sum_{k\in\mathbb{Z}\setminus\{-1\}} \frac{k+Q+1}{\pi(1-0.5^{2k+2Q+2})}|z|^{2(k+Q)} \\ Q \in (0,1) & B_Q(z,z) = \sum_{k\in\mathbb{Z}} \frac{k+Q+1}{\pi(1-0.5^{2k+2Q+2})}|z|^{2(k+Q)} \end{cases}$$

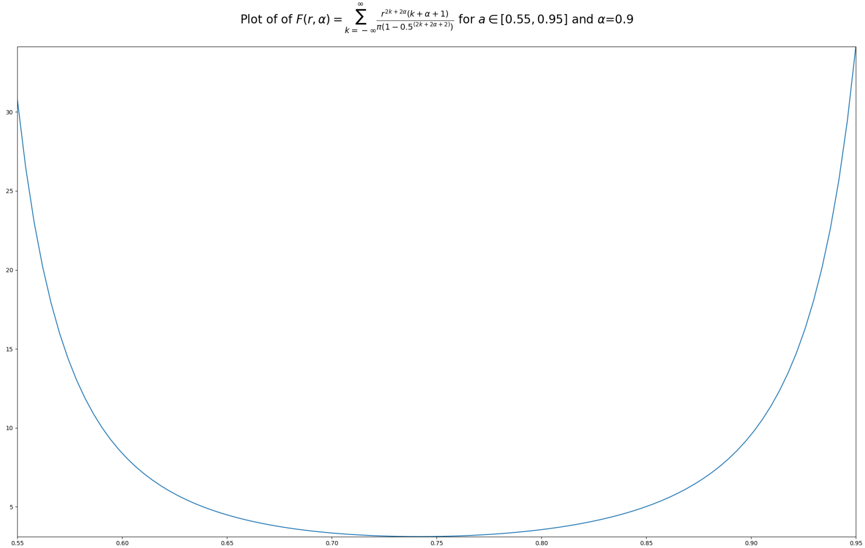
Can you spot the difference? Q = 0



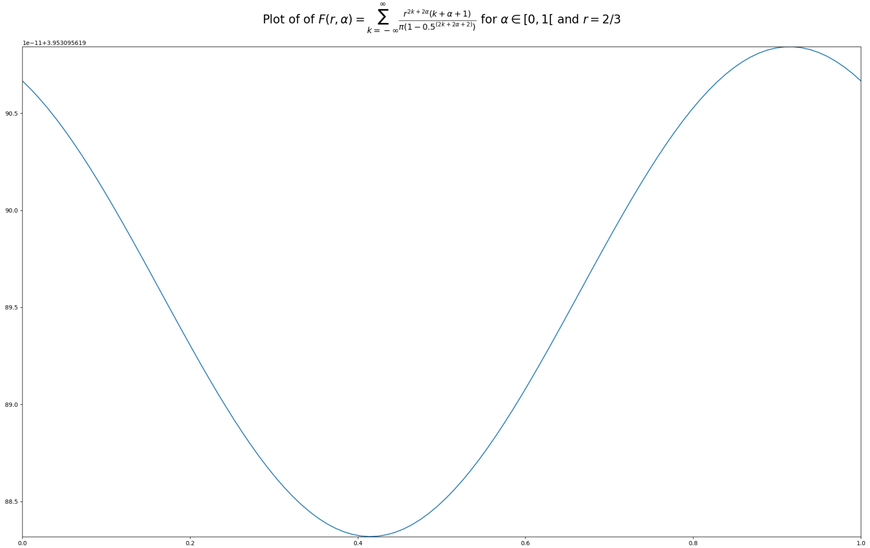
Can you spot the difference? Q = 2/3



Can you spot the difference? Q = 0.9



It was a trap. The difference is of order 10^{-11}.



Thank you for your attention.

How do we get the key inequality?

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One can also easily show that

$$K_N(z,z) = \sum_{k=0}^{N-1} Q_{k,N}(z)\overline{P_{k,N}}(z)e^{-\kappa_N(V_0(z)+\overline{V_0(z)})} = \sup_{C_{N-1}[X]} \frac{|P(z)|^2 e^{-2\kappa_N V(z)}}{\int_{\mathbb{C}} |P(z)|^2 e^{-2\kappa_N V^\nu(z)} dz}$$

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Chose $\rho(z) = e^{-\kappa_N V_0(z)}$ and you get

$$\forall z \in U, \forall N \in \mathbb{N}, \quad K_N(z,z) \leq B_U(z,z).$$

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And we can get the corresponding inequality

$$\forall z \in \Omega, \forall N \in \mathbb{N}, K_N(z,z) \leq B_{\Omega,[\kappa_N q]}(z,z).$$