Almost sure asymptotics for the number of zeros of random trigonometric polynomials

Jürgen Angst

Université de Rennes 1

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Joint work with Guillaume Poly and Thibault Pautrel

We consider random trigonometric polynomials of the form

$$f_n(t) := \frac{1}{\sqrt{n}} \sum_{1\le k\le n} a_k \cos(kt) + b_k \sin(kt), \quad t \in \mathbb{R},$$

where $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are two independent sequences of random variables defined a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We are interested in the number of zeros of f_n in an interval $[a, b] \subseteq [0, 2\pi]$, which will be denoted by

$$\mathcal{H}^0(\{f_n = 0\} \cap [a, b])$$

Theorem (Dunnage, 1966)

If the $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ independent standard Gaussian variables, then

$$\lim_{n\to+\infty} \frac{\mathbb{E}[\mathcal{H}^0(\{f_n=0\}\cap[a,b])]}{n} = \frac{(b-a)}{\pi\sqrt{3}}.$$

Theorem (Flasche, 2016)

If the a_k and b_k are i.i.d., centered with unit variance, then as n goes to infinity

$$\lim_{n\to+\infty} \frac{\mathbb{E}[\mathcal{H}^0(\{f_n = 0\} \cap [a,b])]}{n} = \frac{(b-a)}{\pi\sqrt{3}}.$$

Suppose now that $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are Gaussian variables with correlation function $\rho : \mathbb{N} \to \mathbb{R}$, i.e.

$$\begin{aligned} \mathbb{E}[a_k a_\ell] &= \mathbb{E}[b_k b_\ell] =: \rho(|k-\ell|), \\ \mathbb{E}[a_k b_\ell] &= 0, \quad \forall k, \ell \in \mathbb{N}^*, \end{aligned}$$

such that

$$\psi_\rho(x) := \sum_{k\in\mathbb{Z}} \rho(|k|) e^{ikx}, \quad x \in]0, 2\pi[,$$

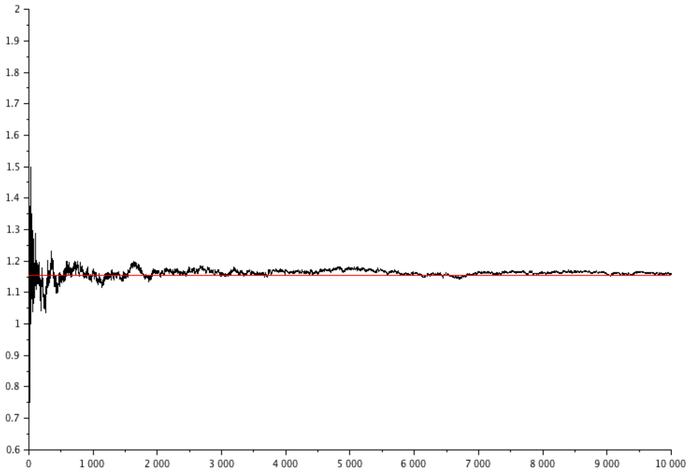
is well defined and satisfies

- $\psi_\rho \in \mathbb{L}^1([0, 2\pi], dx)$
- ψ_ρ is continuous on $]0, 2\pi[$
- $\gamma_\rho := \inf_{t\in[0,2\pi]} \psi_\rho(t) > 0.$

Theorem (A.–Dalmao–Poly, 2017)

In the above dependent Gaussian framework, as n goes to infinity, we have

$$\lim_{n\to+\infty} \frac{\mathbb{E}[\mathcal{H}^0(\{f_n=0\}\cap[a,b])]}{n} = \frac{(b-a)}{\pi\sqrt{3}}.$$



2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0
1 000
2 000
3 000
4 000
5 000
6 000
7 000
8 000
9 000
10 000

Independent model

Theorem (A.–Poly, 2019)

If the $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are i.i.d., symmetric variables with a fourth moment, then $\mathbb{P}-$almost surely as n goes to infinity, we have

$$\lim_{n\to+\infty} \frac{\mathcal{H}^0(\{f_n = 0\} \cap [a,b])]}{n} = \frac{(b-a)}{\pi\sqrt{3}}.$$

Gaussian dependent model

Theorem (A.–Pautrel–Poly, 2021)

In the above dependent Gaussian framework, $\mathbb{P}-$almost surely as n goes to infinity, we have

$$\lim_{n\to+\infty} \frac{\mathcal{H}^0(\{f_n = 0\} \cap [a, b])]}{n} = \frac{(b-a)}{\pi\sqrt{3}}.$$

Monochromatic Random Wave model

On a "generic" Riemannian manifold, define this time

$$k(\lambda) := \#\{n \in \mathbb{N},\, \lambda \leq \lambda_n \leq \lambda + 1\}.$$

$$f_\lambda : x \mapsto \frac{1}{\sqrt{k(\lambda)}} \sum_{\lambda \leq \lambda_n \leq \lambda+1} a_n \varphi_n(x).$$

Theorem (Gass, 2020)

Almost surely with respect to the sequence $(a_k)_{k \geq 0}$,

$$\lim_{\lambda \to +\infty} \frac{\mathcal{H}^{d-1}(\{f_\lambda = 0\})}{\lambda} = \frac{1}{\sqrt{\pi\, d}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

Recall that

$$f_n(t) := \frac{1}{\sqrt{n}} \sum_{1 \leq k \leq n} a_k \cos(kt) + b_k \sin(kt), \quad t \in \mathbb{R},$$

and suppose that $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are i.i.d.symmetric variables with a fourth moment.

Let X be an independent random variable with values in $[0, 2\pi]$, denote by $\mathbb{P}_X$ its law and $\mathbb{E}_X$ the associated expectation.

Theorem (Salem–Zygmund, 1954)

If X is uniform, then $\mathbb{P}$ almost surely, as n goes to infinity, the law of $f_n(X)$ under $\mathbb{P}_X$ converges to the one of a standard Gaussian, i.e. $\forall t \in \mathbb{R}$

$$\mathbb{E}_X\left[e^{itf_n(X)}\right] = \frac{1}{2\pi}\int_0^{2\pi} e^{itf_n(x)}dx \to e^{-t^2/2}.$$

Quantification of Salem–Zygmund CLT

We consider the distance

$$d^X_{\mathcal{C}^3}\left(U,V\right) := \sup_{\substack{\|\phi^{(k)}\|_\infty \leq 1 \\ 0\leq k\leq 3}} \mathbb{E}_X\left[\phi\left(U\right) - \phi\left(V\right)\right].$$

Theorem (A.–Poly, 2019)

If X is uniform, then for all $\beta < \frac{1}{6}$, $\mathbb{P}$ almost surely, there exists a constant $C = C(\omega)$ such that

$$d^X_{\mathcal{C}^3}\left(f_n(X), \mathcal{N}(0,1)\right) \leq \frac{C}{n^\beta}.$$

Functional version of Salem–Zygmund CLT

We consider the process $(g_n(t))_{t\in[0,2\pi]}$ defined by

$$g_n(t) := f_n\left(X + \frac{t}{n}\right).$$

Theorem (A.–Poly, 2019)

If X is uniform, then $\mathbb{P}$ almost surely, as n goes to infinity, the process $(g_n(t))_{t\in[0,2\pi]}$ converges in distribution for the $\mathcal{C}^1$ topology, to a stationary Gaussian process $(g_\infty(t))_{t\in[0,2\pi]}$ with covariance

$$\mathbb{E}_X[g_\infty(t)g_\infty(s)] = \frac{\sin(t-s)}{t-s}.$$

Other extensions

Theorem (A.–Poly, 2019)

If X is uniform, then $\mathbb{P}$ almost surely, as n goes to infinity

$$\lim_{n\to+\infty} \mathrm{d}_{\mathrm{VT}}^{\mathrm{X}}(f_n(X), \mathcal{N}(0,1)) = 0.$$

Theorem (A.–Poly, 2019)

If X is such that there exists $\exists \alpha > 0, C > 0$, $\forall k \neq 0$

$$|\widehat{\mathbb{P}}_X(k)| \leq \frac{C}{|k|^\alpha},$$

and if the a_k, b_k admit a moment of order $\beta > 2/\min(\alpha, 1/2)$ then $\mathbb{P}$ almost surely, as n goes to infinity

$$f_n(X) \xrightarrow{d} \mathcal{N}(0,1), \quad \text{under} \quad \mathbb{P}_X.$$

A simple representation lemma

Lemma

If f is a $2\pi-$periodic function with a finite number of zeros, then for any $0 < h < 2\pi$, we have

$$\frac{h}{2\pi} \times \mathcal{H}^0(\{f=0\} \cap [0,2\pi]) = \mathbb{E}_X \left[\mathcal{H}^0\left(\{f=0\} \cap [X, X+h]\right)\right],$$

where X is a random variable, with uniform distribution in $[0,2\pi]$.

- By the above stochastic representation with $h = 2\pi/n$

$$\frac{\mathcal{H}^0(\{f_n = 0\} \cap [0, 2\pi])}{n} = \mathbb{E}_X \left[\mathcal{H}^0 \left(\{f_n = 0\} \cap \left[X, X + \frac{2\pi}{n} \right] \right) \right]$$

$$= \mathbb{E}_X \left[\mathcal{H}^0 \left(\{g_n = 0\} \cap [0, 2\pi] \right) \right].$$

- By the above functional CLT, $\mathbb{P}$ almost surely, we have the convergence is distribution in the $\mathcal{C}^1$ topology

$$(g_n(t))_{t \in [0, 2\pi]} \xrightarrow{d} (g_\infty(t))_{t \in [0, 2\pi]}.$$

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$$\frac{\mathcal{H}^0(\{f_n = 0\} \cap [0, 2\pi])}{n} = \mathbb{E}_X \left[\mathcal{H}^0 \left(\{f_n = 0\} \cap \left[X, X + \frac{2\pi}{n} \right] \right) \right]$$

$$= \mathbb{E}_X \left[\mathcal{H}^0 \left(\{g_n = 0\} \cap [0, 2\pi] \right) \right].$$

- By the above functional CLT, $\mathbb{P}$ almost surely, we have the convergence is distribution in the $\mathcal{C}^1$ topology

$$(g_n(t))_{t \in [0, 2\pi]} \xrightarrow{d} (g_\infty(t))_{t \in [0, 2\pi]}.$$

- Since the limit process g_∞ is non-degenerate, we deduce that $\mathbb{P}$ almost surely, under $\mathbb{P}_X$

$$\mathcal{H}^0\left(\{g_n = 0\}, [0, 2\pi]\right) \xrightarrow{d} \mathcal{H}^0\left(\{g_\infty = 0\} \cap [0, 2\pi]\right).$$

- To conclude that $\mathbb{P}$ almost surely

$$\begin{array}{rcl} \dfrac{\mathcal{H}^0(\{f_n = 0\} \cap [0, 2\pi])}{n} & = & \mathbb{E}_X\left[\mathcal{H}^0\left(\{g_n = 0\}, [0, 2\pi]\right)\right] \\ & & \downarrow \\ \dfrac{2}{\sqrt{3}} & = & \mathbb{E}_X\left[\mathcal{H}^0\left(\{g_\infty = 0\} \cap [0, 2\pi]\right)\right] \end{array}$$

we are left to prove some uniform integrability for the family of variables $\left(\mathcal{H}^0\left(\{g_n = 0\}, [0, 2\pi]\right)\right)_{n \geq 1}$ under $\mathbb{P}_X$.

We establish that for some $p > 1$

$$\sup_{n \geq 1} \mathbb{E}_X \left[\mathcal{H}^0 \left(\{g_n = 0\}, [0, 2\pi] \right)^p \right] < +\infty.$$

By a classical routine, this reduces to a small ball estimate

$$\mathbb{P}_X(|g_n(0)| \leq \varepsilon) = \mathbb{P}_X(|f_n(X)| \leq \varepsilon) \leq \dots$$

We conclude using the quantitative CLT associated with a powerful estimate by Nazarov–Nishry–Sodin, for $\forall p \geq 1$

$$\mathbb{E}_X \left[|\log(|f_n(X)|)|^p \right] < +\infty,$$