Spectral analysis of sub-Riemannian Laplacians, Weyl measures

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Sub-Riemannian Laplacian

(M, D, g) sub-Riemannian (sR) structure:

- M smooth connected manifold of dimension n
- $m \in \mathbb{N}^*$, $D = \mathrm{Span}(X_1, \ldots, X_m) \subset TM$ (*horizontal distribution*: sub-sheaf)
- sR metric defined by

$$\forall q \in M \qquad \forall v \in D_q \qquad g_q(v, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 \;\middle|\; v = \sum_{i=1}^{m} u_i X_i(q) \right\}$$

Examples:

$n = 3,\ m = 2$

- (flat) 3D contact case: $X_1 = \partial_x, \quad X_2 = \partial_y + x\,\partial_z$ (also called 3D Heisenberg)
- (flat) Martinet case: $X_1 = \partial_x, \quad X_2 = \partial_y + \frac{x^2}{2}\partial_z$

$n = 2,\ m = 2$ ("almost-Riemannian" case)

- (flat) Baouendi-Grushin case: $X_1 = \partial_x, \quad X_2 = x\,\partial_y$
- (flat) p-Grushin case: $X_1 = \partial_x, \quad X_2 = x^p \partial_y$

Sub-Riemannian Laplacian

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$$\forall q \in M \qquad \forall v \in D_q \qquad g_q(v, v) = \inf \left\{ \sum_{i=1}^m u_i^2 \;\middle|\; v = \sum_{i=1}^m u_i X_i(q) \right\}$$

μ: arbitrary smooth measure on M

$$\boxed{\triangle = -\sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m \left(X_i^2 + \mathrm{div}_\mu(X_i) X_i \right)}$$

(X_i^*: adjoint in $L^2(M, \mu)$)

- 3D contact: $\triangle = \partial_x^2 + (\partial_y + x\,\partial_z)^2$
- Martinet: $\triangle = \partial_x^2 + (\partial_y + \frac{x^2}{2}\partial_z)^2$
- Grushin: $\triangle = \partial_x^2 + x^2 \partial_y^2$

Sub-Riemannian Laplacian

Equivalent definitions:

- $-\triangle =$ selfadjoint nonnegative operator on $L^2(M,\mu)$, Friedrichs extension of the Dirichlet integral

$$Q(\phi) = \int_M \|d\phi\|_{g^*}^2 \, d\mu \qquad \phi \in C_c^\infty(M)$$

$$\left(g^*(\xi,\xi) = \max_{v\in D_q\setminus\{0\}} \frac{\langle \xi, v\rangle^2}{g_q(v,v)} \text{ cometric associated with } g \right)$$

- $\boxed{\triangle\phi = \mathrm{div}_\mu\left(\nabla_{sR}\phi\right)}$ where:

div_μ defined by $L_X \, d\mu = \mathrm{div}_\mu(X)\, d\mu \quad \forall X$ vector field on M

∇_{sR} horizontal gradient defined by $g_q(\nabla_{sR}\phi(q), v) = d\phi(q).v \quad \forall v \in D_q$

note that $\|d\phi\|_{g^*} = \|\nabla_{sR}\phi\|_g$

Hörmander operators

More generally:

X_0 smooth vector field on M, c smooth function on M, bounded above

$$\boxed{\triangle = \sum_{i=1}^{m} X_i^2 + X_0 + c\,\mathrm{id}}$$

$\rightarrow$ operator on $L^2(M,\mu)$

Remark: $\triangle$ symmetric $\Leftrightarrow$ $X_0 = \sum_{i=1}^{m} \mathrm{div}_\mu(X_i) X_i$

Hörmander operators

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X_0 smooth vector field on M, c smooth function on M, bounded above

$$\boxed{\triangle = \sum_{i=1}^{m} X_i^2 + X_0 + c\,\mathrm{id}}$$

Under Hörmander's assumption[a]

$$\mathrm{Lie}(D) = \mathrm{Lie}(X_1, \ldots, X_m) = \mathrm{Span}(X_i,\ [X_i, X_j],\ [X_i, [X_j, X_k]], \ldots) = TM$$

the operator $-\triangle$ is locally subelliptic:

$$\|u\|_{H^{2/r}} \leqslant C(\|\triangle u\|_{L^2} + \|u\|_{L^2}) \qquad \text{(local subellipticity estimate)}$$

i.e., gain of Sobolev regularity ($r = 2$ for 3D contact and Grushin, $r = 3$ for Martinet).

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[a] The weakest condition $\mathrm{Lie}(X_0, X_1, \ldots, X_m) = TM$ is enough for subellipticity.

Hörmander operators

More generally:

X_0 smooth vector field on M, c smooth function on M, bounded above

$$\boxed{\triangle = \sum_{i=1}^{m} X_i^2 + X_0 + c\,\mathrm{id}}$$

Here, $r(q) =$ degree of nonholonomy at q, defined by:

- $D^0 = \{0\}$, $D^1 = D = \mathrm{Span}(X_1, \ldots, X_m)$
- $D^{k+1} = D^k + [D, D^k]$ for $k \geqslant 1$ (sequence of sub-sheafs $D^k \subset TM$)

<u>sR flag at q</u>: $\boxed{\{0\} = D_q^0 \subset D_q = D_q^1 \subset D_q^2 \subset \ldots \subset D_q^{r(q)-1} \subsetneq D_q^{r(q)} = T_q M}$

<u>Example</u>: 3D contact

$X_1 = \partial_x$, $X_2 = \partial_y + x\partial_z$

$[X_1, X_2] = \partial_z$

$\to r = 2$

<u>Example</u>: Martinet

$X_1 = \partial_x$, $X_2 = \partial_y + \frac{x^2}{2}\partial_z$

$[X_1, X_2] = x\partial_z$, $[X_1, [X_1, X_2]] = \partial_z$

$$\to r = \begin{cases} 3 & \text{along } x = 0 \\ 2 & \text{outside (contact)} \end{cases}$$

<u>Example</u>: Baouendi-Grushin

$X_1 = \partial_x$, $X_2 = x\partial_y$

$[X_1, X_2] = \partial_y$

$$\to r = \begin{cases} 2 & \text{along } x = 0 \\ 1 & \text{outside (Riemannian)} \end{cases}$$

sR flag

sR flag at q: $\boxed{\{0\} = D_q^0 \subset D_q = D_q^1 \subset D_q^2 \subset \ldots \subset D_q^{r(q)-1} \subsetneq D_q^{r(q)} = T_q M}$

$r(q)$: degree of nonholonomy at q

$n_i(q) = \dim D_q^i$ $\boxed{\mathcal{Q}(q) = \sum_{i=1}^{r} i(n_i(q) - n_{i-1}(q)) = \sum_{i=1}^{n} w_i(q)}$ "homogeneous dimension" at q

$=$ Hausdorff dimension around q if q regular

q is regular if all $\dim D^i$ are locally constant. The sR structure is equiregular if all points are regular.

SR weights at q:

$$w_1(q) = \cdots = w_{n_1}(q) = 1$$
$$w_{n_1+1}(q) = \cdots = w_{n_2}(q) = 2$$
$$\vdots$$
$$w_{n_{r-1}+1}(q) = \cdots = w_{n_r}(q) = r$$

Example: 3D contact case

$X_1 = \partial_x \qquad X_2 = \partial_y + x\partial_z$

$[X_1, X_2] = \partial_z$

$\to$ equiregular

$w_1 = w_2 = 1, \quad w_3 = 2$

$\mathcal{Q} = 4$

Example: 3D Martinet case

$X_1 = \partial_x \qquad X_2 = \partial_y + \frac{x^2}{2}\partial_z$

$[X_1, X_2] = x\partial_z \qquad [X_1, [X_1, X_2]] = \partial_z$

$\to$ singular at $x = 0$ (and contact outside)

$$w_1 = w_2 = 1, \quad w_3 = \begin{cases} 3 & \text{on } x = 0 \\ 2 & \text{outside} \end{cases}$$

$$\mathcal{Q} = \begin{cases} 5 & \text{on } x = 0 \\ 4 & \text{outside} \end{cases}$$

Heat kernel

Heat kernel $e = e_{\triangle,\mu} : (0, +\infty) \times M \times M \to (0, +\infty)$

(density of the Schwartz kernel of $e^{t\triangle}$ w.r.t. μ)

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Heat kernel $e = e_{\triangle,\mu} : (0,+\infty) \times M \times M \to (0,+\infty)$

(density of the Schwartz kernel of $e^{t\triangle}$ w.r.t. μ)

Lower and upper exponential estimates are known (on any compact):

$$\frac{\mathrm{C}_1(\mathrm{q})}{t^{\mathcal{Q}(q)/2}} e^{-\mathrm{d}_{\mathrm{sR}}(q,q')^2/(4-\varepsilon)t} \leqslant e(t,q,q') \leqslant \frac{\mathrm{C}_2(\mathrm{q})}{t^{\mathcal{Q}(q)/2}} e^{-\mathrm{d}_{\mathrm{sR}}(q,q')^2/(4+\varepsilon)t}$$

(Varopoulos, Kusuoka Stroock, Jerison Sanchez-Calle, Cheeger Gromov Taylor, Saloff-Coste, Coulhon Sikora, Grigor'yan)

First objective

- Establish small-time expansions for the heat kernel near the diagonal.

Spectral properties of sR Laplacians

In the selfadjoint case:

$$\boxed{\triangle \;=\; -\sum_{i=1}^{m} X_i^* X_i \;=\; \sum_{i=1}^{m}\Big(X_i^2 + \mathrm{div}_\mu(X_i)X_i\Big)}$$

On M compact, under Hörmander's assumption, $-\triangle$ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty$$

Let $(\phi_j)_{j\in\mathbb{N}}$ be an orthonormal eigenbasis of $L^2(M,\mu)$.

Second objective

- Derive (micro-)local Weyl laws, i.e., compute an expansion for $t > 0$ small of

$$\mathrm{Tr}(f\, e^{t\triangle}) = \int_M f(q)\, e(t,q,q)\, d\mu(q) = \sum_{j=0}^{+\infty} e^{-\lambda_j t} \int_M f\phi_j^2\, d\mu$$

  (or, in microlocal version, replace f with $\mathrm{Op}(a)$)
  and infer the asymptotics of the spectral counting function $N(\lambda) = \#\{j \mid \lambda_j \leqslant \lambda\}$

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Note that (Fefferman Phong 1981)

$$C_1 \int_M \lambda^{\mathcal{Q}(q)} \, d\mu(q) \quad \leqslant \quad N(\lambda) \quad \leqslant \quad C_2 \int_M \lambda^{\mathcal{Q}(q)} \, d\mu(q)$$

hence in the equiregular case $C_1 \lambda^{\mathcal{Q}} \leqslant N(\lambda) \leqslant C_2 \lambda^{\mathcal{Q}}$. Actually by Métivier 1976, $N(\lambda) \sim \mathrm{Cst} \lambda^{\mathcal{Q}}$.

Spectral properties of sR Laplacians

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Let $(\phi_j)_{j \in \mathrm{N}}$ be an orthonormal eigenbasis of $L^2(M, \mu)$.

Second objective

- Derive (micro-)local Weyl laws.
- Establish Quantum Ergodicity (QE) properties, i.e., behavior of $\mu_j = |\phi_j|^2 \, d\mu$ for highfrequencies.

Nilpotentization

Nilpotentization of the sR structure (M, D, g) at $q \in M$:

$$\boxed{(\widehat{M}^q, \widehat{D}^q, \widehat{g}^q) = \text{Gromov-Hausdorff tangent space}}$$

$\rightarrow$ this is the good notion of tangent space in sR geometry.

Thanks to a chart of privileged coordinates at q (exponential coordinates):

- $\widehat{M}^q$ is identified with $\mathbb{R}^n$ endowed with dilations

$$\delta_\varepsilon(x) = \left(\varepsilon^{w_1(q)} x_1, \ldots, \varepsilon^{w_n(q)} x_n\right)$$

- $\widehat{D}^q = \mathrm{Span}(\widehat{X}_1^q, \ldots, \widehat{X}_m^q)$ with

$$\widehat{X}_i^q = \lim_{\varepsilon \to 0} \varepsilon \delta_\varepsilon^* X_i$$

("nonholonomic first-order approximation")

- $\widehat{\mu}^q = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\mathcal{Q}(q)}} \delta_\varepsilon^* \mu = \mathrm{Cst}(q)\, dx$

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Nilpotentized sR Laplacian: $$\boxed{\widehat{\triangle}^q = \sum_{i=1}^{m} \left(\widehat{X}_i^q\right)^2}$$

$\rightarrow$ heat kernel: $\widehat{e}^q = e_{\widehat{\triangle}^q, \widehat{\mu}^q} : (0, +\infty) \times \widehat{M}^q \times \widehat{M}^q \rightarrow \mathbb{R}$

Remark: Homogeneity

$$\widehat{e}^q(t, x, x') = \varepsilon^{\mathcal{Q}(q)}\, \widehat{e}^q(\varepsilon^2 t, \delta_\varepsilon(x), \delta_\varepsilon(x')) \quad \forall \varepsilon \in \mathbb{R}$$

Heat kernel asymptotics

Theorem (Colin de Verdière Hillairet Trélat, Ann. H. Leb. 2021)

In local privileged coordinates at $q \in M$ arbitrary, for every $N \in \mathbb{N}^*$:

$$t^{\mathcal{Q}(q)/2}\, e\left(t, \delta_{\sqrt{t}}(x), \delta_{\sqrt{t}}(x')\right) = \widehat{e}^q(1, x, x') + \sum_{i=1}^{N} a_i(x, x') t^{i/2} + \mathrm{o}(t^N)$$

as $t \to 0^+$, in $C^\infty(M \times M)$ topology, with a_j smooth and $a_{2j-1}(0,0) = 0$.

- q need not be regular.
- If q is regular then the asymptotic expansion is locally uniform wrt q.
- Still valid for $\triangle = \sum_{i=1}^{m} X_i^2 + X_0 + c\,\mathrm{id}$, provided that:
  - either X_0 smooth section of D;
  - or X_0 smooth section of D^2, and then replace $\widehat{\triangle}^q$ with $\widehat{\triangle}^q + \widehat{X}_0^q$.

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as $t \to 0^+$, in $C^\infty(M \times M)$ topology, with a_j smooth and $a_{2j-1}(0,0) = 0$.

- $x = x' = 0 \Rightarrow$ expansion of the kernel along the diagonal, and

  $$e(t, q, q) \sim \frac{\widehat{e}^q(1, 0, 0)}{t^{\mathcal{Q}(q)/2}} = \widehat{e}^q(t, 0, 0)$$

  $\rightarrow$ useful to derive the local Weyl law.
  Generalization of results by Métivier (1976), Ben Arous (1989).

- estimations **near** the diagonal $\rightarrow$ microlocal Weyl law and singular sR structures.

Heat kernel asymptotics

Idea of the proof: (in a chart) $X_i^\varepsilon = \varepsilon\delta_\varepsilon^* X_i \to \widehat{X}_i^q$

$$\triangle^\varepsilon = \varepsilon^2\delta_\varepsilon^*\triangle(\delta_\varepsilon)_* = -\sum_{i=1}^m (X_i^\varepsilon)^* X_i^\varepsilon = \widehat{\triangle}^q + \varepsilon\mathcal{A}_1 + \varepsilon^2\mathcal{A}_2 + \cdots$$

$$\Rightarrow \quad e^{t\triangle^\varepsilon} \to e^{t\widehat{\triangle}^q} \quad \text{pointwise (Trotter-Kato)}$$

$$\Rightarrow \quad e^\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} e^q \quad \text{in} \quad C^{-\infty}([t_0,t_1]\times K\times K)$$

Note that $e^\varepsilon(s,x,x') = \varepsilon^{\mathcal{Q}(q)} e(\varepsilon^2 s, \delta_\varepsilon(x), \delta_\varepsilon(x'))$.

- By uniform local subelliptic estimates: $e^{t\triangle^\varepsilon}$ is locally uniformly smoothing for $t\in[t_0,t_1]$ ($t_0>0$), i.e., it maps any local Sobolev space to any local Sobolev space, uniformly wrt ε.
- Then $(e^\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ is bounded in $C^\infty((0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n)$

$$\Rightarrow \quad e^\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} e^q \quad \text{in} \quad C^{\infty}((0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n) \quad \text{(Montel space)}$$

- Hypoelliptic version of the Kac principle: asymptotics of heat kernels is purely local ("not feeling the boundary")

Heat kernel asymptotics

Asymptotic expansion in ε: as in [Barilari, JMS 2013]

$$
\begin{aligned}
e^{t\triangle^\varepsilon} &= e^{t\widehat{\triangle}^q} + \int_0^t e^{(t-s)\triangle^\varepsilon}(\triangle^\varepsilon - \widehat{\triangle}^q)e^{s\widehat{\triangle}^q}\,ds \\
&= e^{t\widehat{\triangle}^q} + e^{t\triangle^\varepsilon} \star \Big((\triangle^\varepsilon - \widehat{\triangle}^q)e^{t\widehat{\triangle}^q}\Big) \\
&= e^{t\widehat{\triangle}^q} + \varepsilon\, \underbrace{e^{t\widehat{\triangle}^q} \star \mathcal{A}_1 e^{t\widehat{\triangle}^q}}_{\mathcal{C}_1(t)} + \varepsilon^2\, \underbrace{e^{t\widehat{\triangle}^q} \star \Big(\mathcal{A}_2 e^{t\widehat{\triangle}^q} + \mathcal{A}_1\mathcal{C}_1(t)\Big)}_{\mathcal{C}_2(t)} + \cdots \\
&= e^{t\widehat{\triangle}^q} + \sum_{i=1}^{N} \varepsilon^i \mathcal{C}_i(t) + \mathrm{o}(\varepsilon^N)
\end{aligned}
$$

and then take Schwartz kernels.

Main difficulty here: proving that $C_i(t)$ is smoothing requires to establish **global** smoothing properties of $e^{t\widehat{\triangle}^q}$ in Sobolev spaces with polynomial weights, and global continuous embeddings. $\rightarrow$ difficult, long and technical

An important tool is the Kannai transform: Cheeger Gromov Taylor, Coulhon Sikora. Cf also Eckmann Hairer.

(Micro-)local Weyl measure

M compact

Local Weyl measure = probability measure $w_\triangle$ on M defined (if the limit exists) by

$$\int_M f\,dw_\triangle = \lim_{t\to 0+} \frac{\mathrm{Tr}\left(f\,e^{t\triangle}\right)}{\mathrm{Tr}\left(e^{t\triangle}\right)} = \lim_{t\to 0+} \frac{\int_M e(t,q,q) f(q)\,d\mu(q)}{\int_M e(t,q,q)\,d\mu(q)} \qquad \forall f \in C^0(M)$$

i.e.,

$$\boxed{w_\triangle = \mathrm{weak}\lim_{t\to 0+} \frac{e(t,q,q)}{\int_M e(t,q',q')\,d\mu(q')}\,\mu}$$

Microlocal Weyl measure = probability measure $W_\triangle$ on $S^\star M$ defined (if the limit exists) by

$$\int_{S^\star M} a\,dW_\triangle = \lim_{t\to 0+} \frac{\mathrm{Tr}\left(\mathrm{Op}(a) e^{t\triangle}\right)}{\mathrm{Tr}\left(e^{t\triangle}\right)} \qquad \forall a \in \mathcal{S}^0(S^\star M)$$

(Micro-)local Weyl measure

Equivalent definition (by the Karamata tauberian theorem):

$-\triangle\phi_j = \lambda_j\phi_j$, $(\phi_j)_{j\in\mathbf{N}^*}$ orthonormal eigenbasis of $L^2(M,\mu)$, $0=\lambda_0<\lambda_1\leqslant\cdots\leqslant\lambda_j\leqslant\cdots\to+\infty$

Spectral counting function: $N(\lambda)=\#\{k \mid \lambda_j\leqslant\lambda\}$

Local Weyl measure = probability measure $w_\triangle$ on M defined (if the limit exists) by

$$\int_M f\,dw_\triangle = \lim_{\lambda\to+\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\leqslant\lambda}\int_M f|\phi_j|^2\,d\mu \qquad \forall f\in C^0(M)$$

i.e.,

$$\boxed{w_\triangle = \text{weak}\lim_{\lambda\to+\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\leqslant\lambda}|\phi_j|^2\,\mu} \qquad \text{(Cesàro mean)}$$

Microlocal Weyl measure = probability measure $W_\triangle$ on $S^\star M$ defined (if the limit exists) by

$$\int_{S^\star M} a\,dW_\triangle = \lim_{\lambda\to+\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\leqslant\lambda}\langle \mathrm{Op}(a)\phi_j,\phi_j\rangle_{L^2(M,\mu)} \qquad \forall a\in\mathcal{S}^0(S^\star M)$$

Local Weyl law in the equiregular case

Theorem

In the equiregular case, the local Weyl measure $w_\triangle$ exists, is smooth, and

$$\boxed{\frac{dw_\triangle}{d\mu}(q) = \frac{\widehat{e}^q(1,0,0)}{\int_M \widehat{e}^{q'}(1,0,0)\, d\mu(q')}}$$

Proof: Along the diagonal, $t^{\mathcal{Q}/2} e(t,q,q) \longrightarrow \widehat{e}^q(1,0,0)$ as $t \to 0^+$.

Remark: Since $w_\triangle$ is smooth, it differs in general from $\mathcal{H}_S$ (which is not smooth in general for $n \geqslant 5$, see [Agrachev Barilari Boscain 2012])

Consequence: $\boxed{N(\lambda) \sim \frac{\int_M \widehat{e}^q(1,0,0)\, d\mu(q)}{\Gamma(\mathcal{Q}/2+1)} \lambda^{\mathcal{Q}/2}}$ as $\lambda \to +\infty$ ($\mathcal{Q}$: Hausdorff dim)

This asymptotics was already known by Métivier 1976.

Example: 3D contact case, $N(\lambda) \sim \frac{1}{32}\lambda^2$.

Microlocal Weyl law: we can compute it explicitly.

Singular sR structures

The *singular set* is the closed subset of M defined by

$$\mathscr{S} = \{q \in M \mid \mathcal{Q}(q) > \inf_{q' \in M} \mathcal{Q}(q')\}.$$

In addition to the sR flag $\{0\} = D_q^0 \subset D_q = D_q^1 \subset D_q^2 \subset \ldots \subset D_q^{r(q)-1} \subsetneq D_q^{r(q)} = T_q M$, we now also consider the sR flag restricted to $\mathscr{S}$:

$$\{0\} \subset \left(D^1(q) \cap T_q\mathscr{S}\right) \subset \cdots \subset \left(D^{r(q)-1}(q) \cap T_q\mathscr{S}\right) \subset \left(D^{r(q)}(q) \cap T_q\mathscr{S}\right) = T_q\mathscr{S}$$

Definition (following Gromov): $\mathscr{S}$ is an equisingular smooth submanifold of M if all integers $n_i(q) = \dim D_q^i$ and $n_i^{\mathscr{S}}(q) = \dim\left(D_q^i \cap T_q\mathscr{S}\right)$ are constant as $q \in \mathscr{S}$. In particular:

$$\boxed{\mathcal{Q}^{\mathscr{S}} = \sum_{i=1}^{r} i(n_i^{\mathscr{S}} - n_{i-1}^{\mathscr{S}})}$$

is the Hausdorff dimension of $\mathscr{S}$. (Ghezzi Jean, 2015)

Two simple singular sR structures

Baouendi-Grushin case (with no tangency points):

- Local model: $X = \partial_x$, $Y = x\partial_y$, $\mathscr{S} = \{x = 0\}$.
- $\mathcal{Q}^{\mathscr{S}} = \mathcal{Q}^{M\setminus\mathscr{S}} = 2$

Regular Martinet case:

- Local model: $X = \partial_x$, $Y = \partial_y + \frac{x^2}{2}\partial_z$, $\mathscr{S} = \{x = 0\}$.
- $\mathcal{Q}^{\mathscr{S}} = \mathcal{Q}^{M\setminus\mathscr{S}} = 2$

In both cases, there is a smooth measure ν on $\mathscr{S}$, canonically inferred from μ.

Two simple singular sR structures

Small-time expansion of the local Weyl law at any order:

- Baouendi-Grushin:

$$\mathrm{Tr}(f\, e^{t\triangle}) = \int_M f(q)\, e(t,q,q)\, d\mu(q) = \frac{\ln\frac{1}{t}}{t} F_1(t) + \frac{1}{t} F_0(\sqrt{t}) \qquad \forall t>0$$

$$= \left(\frac{1}{4\pi}\int_{\mathscr{S}} f\, d\nu\right)\frac{\ln\frac{1}{t}}{t} + \frac{1}{4\pi}\left(\mathrm{p.f.}\int_{M\setminus\mathscr{S}} f\, dP + (\gamma + 4\ln 2)\int_{\mathscr{S}} f\, d\nu\right)\frac{1}{t} + \mathrm{o}\left(\frac{1}{t}\right)$$

(intrinsic two-terms expansion)

- Martinet:

$$\mathrm{Tr}(f\, e^{t\triangle}) = \frac{\ln\frac{1}{t}}{t^2} F_1(t) + \frac{1}{t^2} F_0(\sqrt{t}) = \left(\frac{1}{16}\int_{\mathscr{S}} f\, d\nu\right)\frac{\ln\frac{1}{t}}{t^2} + \mathrm{o}\left(\frac{\ln\frac{1}{t}}{t^2}\right)$$

Consequence: $w_\triangle = \frac{\nu}{\nu(\mathcal{S})}$ and Weyl law:

Baouendi-Grushin : $N(\lambda) \sim \dfrac{\nu(\mathcal{S})}{4\pi}\lambda \ln\lambda$ Martinet : $N(\lambda) \sim \dfrac{\nu(\mathcal{S})}{32}\lambda^2 \ln\lambda$

$\Rightarrow$ spectral concentration on the singular manifold $\mathcal{S}$

In the Baouendi-Grushin case the asymptotics of the Weyl law was known by Menikoff Sjöstrand 1978.

Generalization (equisingular case)

Theorem:

If $\mathscr{S}$ is an equisingular smooth submanifold of M and if the horizontal distribution D is $\mathscr{S}$-nilpotentizable (i.e., $D \sim \widehat{D}^q$ for every $q \in \mathscr{S}$) then

$$\boxed{\mathrm{Tr}(f\, e^{t\triangle}) = \underbrace{\frac{1}{t^{\mathcal{Q}^{M\setminus\mathscr{S}}/2}} F_0(t)}_{\text{“equiregular part”}} + \frac{1}{t^{\mathcal{Q}^{\mathscr{S}}/2}} F_1(\sqrt{t}) + \frac{\ln\frac{1}{t}}{t^{\min(\mathcal{Q}^{M\setminus\mathscr{S}},\mathcal{Q}^{\mathscr{S}})/2}} F_2(\sqrt{t}) \qquad \forall t > 0}$$

- If $\mathcal{Q}^{\mathscr{S}} > \mathcal{Q}^{M\setminus\mathscr{S}}$ then dominating term in $\frac{1}{t^{\mathcal{Q}^{\mathscr{S}}/2}}$, smooth Weyl measure supported on $\mathscr{S}$, of density a “transverse trace” of $e^{t\widehat{\triangle}^q}$, and $N(\lambda) \sim \mathrm{Cst}\, \lambda^{\mathcal{Q}^{\mathscr{S}}/2}$ with an explicit Cst.
- If $\mathcal{Q}^{\mathscr{S}} = \mathcal{Q}^{M\setminus\mathscr{S}}$ then dominating term in $\frac{\ln\frac{1}{t}}{t^{\mathcal{Q}^{\mathscr{S}}/2}}$, smooth Weyl measure supported on $\mathscr{S}$, of density given in terms of a “double nilpotentization” of the heat kernel (one nilp. in $\mathscr{S}$, one nilp. in $M \setminus \mathscr{S}$), and $N(\lambda) \sim \mathrm{Cst}\, \lambda^{\mathcal{Q}^{\mathscr{S}}/2} \ln \lambda$ with an explicit Cst.
- If $\mathcal{Q}^{\mathscr{S}} < \mathcal{Q}^{M\setminus\mathscr{S}}$ then dominating term in $\frac{1}{t^{\mathcal{Q}^{M\setminus\mathscr{S}}/2}}$: the equiregular part dominates, smooth Weyl measure not concentrated, and $N(\lambda) \sim \mathrm{Cst}\, \lambda^{\mathcal{Q}^{M\setminus\mathscr{S}}/2}$ with an explicit Cst.

Strategy of proof:

"$(J+K)$-decomposition" of $I(t) = \mathrm{Tr}(f\, e^{t\triangle}) = \int_M f(t,q)\, e(t,q,q)\, dq$:

Write $I(t) = J(t) + K(t)$ with

$$J(t) = \int_{\mathcal{B}(\mathscr{S},\sqrt{t})} f(q')\, e(t,q',q')\, dq' \qquad K(t) = \int_{M\setminus\mathcal{B}(\mathscr{S},\sqrt{t})} f(q')\, e(t,q',q')\, dq'$$

- Setting $q' = \delta^{\mathscr{S}}_{\sqrt{t}}(y)$,

$$J(t) = \frac{1}{t^{\mathcal{Q}^{\mathscr{S}}/2}} \int_{\mathscr{S}\times\mathcal{B}^{n-k}} f(\delta^{\mathscr{S}}_{\sqrt{t}}(y))\, \underbrace{(\sqrt{t})^{\mathcal{Q}^{M}(\mathscr{S})}\, e(t, \delta^{\mathscr{S}}_{\sqrt{t}}(y), \delta^{\mathscr{S}}_{\sqrt{t}}(y))}_{=\widehat{e}^{q}(1,y,y)+\cdots\ \text{by the fundamental lemma}}\, dy = \frac{F_J(\sqrt{t})}{t^{\mathcal{Q}^{\mathscr{S}}/2}}$$

- Expanding $K(t)$ is much more difficult and requires to perform a "double nilpotentization" of e: one on $\mathscr{S}$ and the other outside of $\mathscr{S}$.

  Nilpotentizability ensures that the double limit is well defined.

Generalization (equisingular stratified case)

Theorem

If $\mathscr{S}$ is Whitney stratifiable, with strata $\mathscr{S}_i$ that are equisingular smooth submanifolds of M and if D is $\mathscr{S}$-nilpotentizable then

$$\boxed{\mathrm{Tr}(f\,e^{t\triangle}) = \underbrace{\frac{1}{t^{\mathcal{Q}^{M\setminus\mathscr{S}}/2}} F_0(t)}_{\text{“equiregular part”}} + \sum_{p=0}^{s} \frac{\ln^p \frac{1}{t}}{t^{\mathcal{Q}^p/2}} F_p(\sqrt{t}) \qquad \forall t > 0}$$

where $\mathcal{Q}^0 < \cdots < \mathcal{Q}^s$ are the Hausdorff dimensions of the stratification (including $M \setminus \mathscr{S}$), and

$$\boxed{\mathrm{Tr}(f\,e^{t\triangle}) = \left(\int_M f\,d\nu\right) \frac{\ln^{\ell-1} \frac{1}{t}}{t^{\mathcal{Q}^s/2}} + \mathrm{o}\left(\frac{\ln^{\ell-1} \frac{1}{t}}{t^{\mathcal{Q}^s/2}}\right)} \quad \text{and} \quad \boxed{N(\lambda) \sim \lambda^{\mathcal{Q}^s} \ln^{\ell-1} \lambda}$$

where ℓ is the number of Hausdorff dimensions $\mathcal{Q}^{\mathscr{S}_i}$ equal to the maximum $\mathcal{Q}^s$.

The measure ν is supported on $\mathscr{S}_1 \cup \cdots \cup \mathscr{S}_i$ if $\mathcal{Q}^{\mathscr{S}_i} = \mathcal{Q}^s > \max(\mathcal{Q}^{\mathscr{S}_1}, \ldots, \mathcal{Q}^{\mathscr{S}_{i-1}}) = \mathcal{Q}^{s-1}$. Its density is expressed in terms of ”multiple nilpotentizations” of the heat kernel.

Consequence: Quantum Ergodicity (QE) properties

If $\mathcal{Q}^s \geqslant \mathcal{Q}^{M \setminus \mathscr{S}}$ then "almost all" (density-one) probability measures $\mu_j = |\phi_j|^2 \, d\mu$ concentrate on $\mathscr{S}$ for highfrequencies (i.e., their "essential" weak limits are supported on $\mathscr{S}$).

QE property in the Baouendi-Grushin case

In the Baouendi-Grushin case, if $\mathscr{S}$ is connected with at most one tangency point, there is only one "essential" weak limit, which is the Weyl measure.

$\longrightarrow$ First example in sR geometry of a QE result with a limit measure that is singular.

$\longrightarrow$ In the 3D contact, we had already established the QE property, under the assumption that the Reeb flow be ergodic (cf Colin de Verdière Hillairet Trélat, Duke 2018).

When nilpotentizability fails

Ongoing work: when $\mathscr{S}$ is Whitney stratifiable with polynomial singularities but D fails to be $\mathscr{S}$-nilpotentizable,

$$\boxed{\mathrm{Tr}(f\, e^{t\triangle}) \underset{t\to 0^+}{\sim} \mathrm{Cst}\,\frac{\ln^k \frac{1}{t}}{t^r} \qquad \text{and} \qquad N(\lambda) \underset{\lambda\to+\infty}{\sim} \lambda^r \ln^k \lambda}$$

for some $k \in \{0, 1, \dots, n\}$ and $r \in \mathbb{Q}$ s.t. $r \geqslant \frac{\mathcal{Q}^{M\setminus\mathscr{S}}}{2}$.

But the geometric characterization of r remains to be found as well as the measure concentration rule.

Some examples of singular sR structures

name	definition	asymptotics	concentration on N
k-Grushin	$X_1 = \partial_1, \quad X_2 = x_1^k \partial_2 \qquad (k \geqslant 1)$	$\frac{\ln \frac{1}{t}}{t}$ if $k = 1$ $\frac{1}{t^{k+1}}$ if $k \geqslant 2$	$N = \mathcal{S} = \{x_1 = 0\}$
Sing. k-Grushin	$X_1 = \partial_1, \quad X_2 = (x_1^k - x_2)\partial_2 \quad (k \geqslant 2)$	$\frac{\ln \frac{1}{t}}{t} \quad \forall k \geqslant 2$	$N = \mathcal{S} = \{x_2 = x_1^k\}$
	$X_1 = \partial_1, \quad X_2 = (x_1^{2p} + x_1 y_1^k)\partial_2$ $p, k \in \mathbb{N}^*$	$\frac{\ln^2 \frac{1}{t}}{t}$ if $k = 1$ $\frac{1}{t^{p+\frac{1}{2}-\frac{2p-1}{2k}}}$ if $k \geqslant 2$	$N = \{(0,0)\}$ $\subset \mathcal{S} = \{x_1^{2p} + x_1 y_1^k = 0\}$
	$X_1 = \partial_1, \quad X_2 = (x_1^2 - x_2^3)\partial_2$	$\frac{1}{t^{7/6}}$	$N = \{(0,0)\} \subsetneq \mathcal{S} = \{x_1^2 = x_2^3\}$
Martinet	$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1^2 \partial_3$	$\frac{\ln \frac{1}{t}}{t^2}$	$N = \mathcal{S} = \{x_1 = 0\}$
Nilp. tang. hyp.	$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1^2 x_2 \partial_3$	$\frac{\ln^2 \frac{1}{t}}{t^2}$	$N = \{x_1 = x_2 = 0\}$ $\subsetneq \mathcal{S} = \{x_1 x_2 = 0\}$
Ghezzi Jean	$X_1 = \partial_1$ $X_2 = \partial_2 + x_1 \partial_3 + x_1^2 \partial_5$ $X_3 = \partial_4 + (x_1^k + x_2^k)\partial_5 \quad (k \geqslant 2)$	$\frac{1}{t^{7/2}}$ if $k = 2$ $\frac{\ln \frac{1}{t}}{t^{7/2}}$ if $k = 3$ $\frac{1}{t^{2+\frac{k}{2}}}$ if $k \geqslant 4$	$N = \mathbb{R}^5 \supsetneq \mathcal{S} = \{x_1 = x_2 = 0\}$ $N = \mathcal{S} = \{x_1 = x_2 = 0\}$ $N = \mathcal{S} = \{x_1 = x_2 = 0\}$

Even more exotic Weyl laws

Consider the local model

$$\boxed{X = \partial_x \qquad Y = \left(x^2 + g(y)\right)\partial_y}$$

with g smooth, $g(0) = 0$ and $g(y) > 0$ if $y \neq 0$. We compute

$$\mathrm{Tr}(f\, e^{t\triangle}) \sim \frac{\mathrm{Cst}}{t^{3/2}} g^{-1}(t) + \frac{\mathrm{Cst}}{t}\int_t^1 \frac{du}{\sqrt{u}g'(g^{-1}(u))} + \frac{\mathrm{Cst}}{\sqrt{t}}\int_t^1 \frac{du}{ug'(g^{-1}(u))}$$

We obtain interesting examples by taking g flat at 0
$\rightarrow$ kind of flat perturbation of the 2-Grushin case.

$g(y)$	$\mathrm{Tr}(f\, e^{t\triangle}) \sim \mathrm{Cst} \times$	$N(\lambda) \sim \mathrm{Cst} \times$
$\dfrac{1}{e^{1/\lvert y\rvert^\alpha}}, \quad \alpha > 0$	$\dfrac{1}{t^{3/2}\left(\ln \frac{1}{t}\right)^{1/\alpha}}$	$\dfrac{\lambda^{3/2}}{(\ln \lambda)^{1/\alpha}}$
$\dfrac{1}{e^{\beta e^{1/\lvert y\rvert^\alpha}}}, \quad \alpha, \beta > 0$	$\dfrac{1}{t^{3/2}\left(\ln\ln \frac{1}{t^{1/\beta}}\right)^{1/\alpha}}$	$\dfrac{\lambda^{3/2}}{\sqrt{\left(\ln\ln \lambda^{1/\beta}\right)^{1/\alpha}}}$
$\dfrac{1}{\exp^{[k]}\lvert y\rvert} = \dfrac{1}{e^{e^{\cdots e^{1/\lvert y\rvert}}}}$	$\dfrac{1}{t^{3/2}\ln^{[k]} \frac{1}{t}}$	$\dfrac{\lambda^{3/2}}{\ln^{[k]}\lambda} = \dfrac{\lambda^{3/2}}{\ln\cdots\ln\lambda}$
$e^{-\frac{\ln^2 y}{y}}$	$\dfrac{\ln^2 \ln \frac{1}{t}}{t^{3/2}\ln \frac{1}{t}}$	$\dfrac{\lambda^{3/2}\ln^2\ln\lambda}{\ln\lambda}$

Even more exotic Weyl laws

Consider the local model

$$\boxed{X_1 = \partial_1 \qquad X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_5 \qquad X_3 = \partial_4 + e^{-1/(x_1^2+x_2^2)}\partial_5}$$

We compute

$$\boxed{\mathrm{Tr}(f\, e^{t\triangle}) \sim \mathrm{Cst}\, \frac{e^{1/t}}{t} \qquad\qquad N(\lambda) \sim \mathrm{Cst}\, \frac{e^{2\sqrt{\lambda}}}{\lambda^{1/4}}}$$

Non-standard Weyl law.

Perspectives: spectral issues in sR geometry

- Can we find a sR case whose Weyl law has an “arbitrary” asymptotics? (inverse problem)
- Does there exist an intrinsic interpretation of the coefficients of the local Weyl law, in terms of curvatures, like in the Riemannian case?
- Find spectral invariants in sR geometry (Reeb periods in the 3D contact case).
- Quantum Ergodicity properties for more general sR cases?
- Application to controllability, observability:
  - Subelliptic wave equations are **never** observable (Letrouit, 2021).
  - Subelliptic heat/Schrödinger equations can be observable, with a minimal time (Beauchard, Cannarsa 2014 ; Duprez Koenig 2020 ; Burq Sun 2020), but still no geometric picture.
- Trace formulas in sR geometry (Melrose 1984, Savale 2020, Letrouit ongoing)