# Large deviations and entropy production in viscous fluid flows

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# Outline

# Large deviation principle

Suppose we are observing a stationary process $\{u_k\}_{k\geq 1}$ in a phase space $X$. The starting point in the investigation of entropic fluctuations is the large deviation principle (LDP). Setting $\boldsymbol{X} = X^{\mathbb{N}}$, define the empirical measures

$$\boldsymbol{\nu}_t = t^{-1}\sum_{k=1}^{t}\delta_{\boldsymbol{u}_k}, \quad t \geq 1,$$

where $\boldsymbol{u}_k = (u_j, j \geq k)$. Thus, $\{\boldsymbol{\nu}_t = \boldsymbol{\nu}_t^{\omega}\}$ is a sequence of random probability measures on $\boldsymbol{X}$. Suppose the LDP holds: $\exists$ l.s.c. function $\boldsymbol{I} : \mathcal{P}(\boldsymbol{X}) \to [0, +\infty]$ called rate function s.t.

$$\begin{aligned}-\boldsymbol{I}(\dot{\Gamma}) &\leq \liminf_{t\to\infty} t^{-1}\log \mathbb{P}\{\boldsymbol{\nu}_t \in \Gamma\}\\ &\leq \limsup_{t\to\infty} t^{-1}\log \mathbb{P}\{\boldsymbol{\nu}_t \in \Gamma\} \leq -\boldsymbol{I}(\overline{\Gamma}),\end{aligned}$$

where $\overline{\Gamma}/\dot{\Gamma}$ is the closure/interior of $\Gamma \in \mathcal{B}(\mathcal{P}(\boldsymbol{X}))$, $\boldsymbol{I}(A) = \inf_A \boldsymbol{I}$.

# Time reversal and level-3 fluctuation relation

Suppose there is a natural "time-reversal operation" that can be lifted to an involution $\boldsymbol{\theta} : \mathcal{P}_s(\boldsymbol{X}) \to \mathcal{P}_s(\boldsymbol{X})$. For instance, setting $\boldsymbol{X}^t = X \times \cdots \times X$ ($t$ times), let us set

$$\theta_t : \boldsymbol{X}^t \to \boldsymbol{X}^t, \quad [v_1, \dots, v_t] \mapsto [v_t, \dots, v_1].$$

This defines an involution $\boldsymbol{\theta}_t : \mathcal{P}(\boldsymbol{X}^t) \to \mathcal{P}(\boldsymbol{X}^t)$. Now note that if $\boldsymbol{\lambda}_t \in \mathcal{P}(\boldsymbol{X}^t)$ is the projection of a measure $\boldsymbol{\lambda} \in \mathcal{P}_s(\boldsymbol{X})$ to $\boldsymbol{X}^t$, then the sequence $\{\bar{\boldsymbol{\lambda}}_t = \boldsymbol{\lambda}_t \circ \boldsymbol{\theta}_t\}$ is consistent, so that we can find a measure $\bar{\boldsymbol{\lambda}} \in \mathcal{P}_s(\boldsymbol{X})$ whose projection to $\boldsymbol{X}^t$ coincides with $\bar{\boldsymbol{\lambda}}_t$. We denote $\bar{\boldsymbol{\lambda}}$ by $\boldsymbol{\lambda} \circ \boldsymbol{\theta}$.

## Definition

We say that level-3 fluctuation relation holds if $\exists$ affine function $\mathrm{ep} : \mathcal{P}_s(\boldsymbol{X}) \to \mathbb{R}$ s.t., for a "large class" of measures $\boldsymbol{\lambda} \in \mathcal{P}_s(\boldsymbol{X})$,

$$\boldsymbol{I}(\boldsymbol{\lambda} \circ \boldsymbol{\theta}) = \boldsymbol{I}(\boldsymbol{\lambda}) + \mathrm{ep}(\boldsymbol{\lambda}). \tag{1}$$

# Entropy production

The level-3 FR (1) may be valid under rather general hypotheses. A natural problem is then the description of the functional $\mathrm{ep}(\boldsymbol{\lambda})$. In particular, one may ask the following questions:

- Is there $\sigma : \boldsymbol{X} \to \mathbb{R}$ called entropy production observable such that $\mathrm{ep}(\boldsymbol{\lambda}) = \langle \sigma, \boldsymbol{\lambda} \rangle$?
- If the answer to the first question is negative, can one find $\sigma_t : \boldsymbol{X}^t \to \mathbb{R}$ called entropy production in time $t$ such that

$$\mathrm{ep}(\boldsymbol{\lambda}) = \lim_{t\to\infty} t^{-1} \langle \sigma_t, \boldsymbol{\lambda} \rangle \, ? \tag{2}$$

- What is the relation of $\sigma$ and $\sigma_t$ with physically relevant quantities of the system in question?

A candidate for $\sigma_t$ is the logarithmic density

$$\sigma_t = \log \frac{\mathrm{d}\boldsymbol{\lambda}_t}{\mathrm{d}\bar{\boldsymbol{\lambda}}_t}, \quad \bar{\boldsymbol{\lambda}}_t = \boldsymbol{\lambda}_t \circ \boldsymbol{\theta}_t \, ; \tag{3}$$

however, it is far from being obvious that (2) holds.

# Law of large numbers and Stein exponent

Suppose it is possible to identify $\sigma_t$, the entropy production in time $t$. Then one may ask the question about the large time behaviour of $\sigma_t$. We shall say that the law of large numbers holds if

$$\lim_{t\to\infty} t^{-1}\sigma_t = \mathrm{ep}(\mu) \quad \text{in probability with respect to } \mu, \tag{4}$$

where $\mu$ is the law of the process we are observing. If this is true, then

$$\lim_{t\to\infty} t^{-1}\log\mathfrak{s}_\gamma(t) = -\,\mathrm{ep}(\mu) \quad \text{for any } \gamma\in(0,1), \tag{5}$$

where $\mathfrak{s}_\gamma$ is the Stein error exponent defined by

$$\mathfrak{s}_\gamma(t) = \inf\{\bar{\mu}_t(\Gamma) : \Gamma\subset \boldsymbol{X}^t, \mu_t(\Gamma^c)\le\gamma\}. \tag{6}$$

# Positivity of $\mathrm{ep}(\boldsymbol{\mu})$, LDP, and Hoeffding exponent

The validity of (5) raises the question as to whether $\mathrm{ep}(\boldsymbol{\mu}) > 0$, since in this case the measures $\boldsymbol{\mu}_t$ and $\bar{\boldsymbol{\mu}}_t$ separate from each other as $t \to +\infty$ and become singular in the limit $t = +\infty$.

A natural question is a quantitative description of separation of $\boldsymbol{\mu}_t$ and $\bar{\boldsymbol{\mu}}_t$. This can be done if $\{t^{-1}\sigma_t\}$ satisfies the LDP

$$t^{-1} \log \mathbb{P}\{t^{-1}\sigma_t \in \Gamma\} \sim -I(\Gamma) \quad \text{as } t \to \infty. \tag{7}$$

Namely, defining the Hoeffding exponent

$$\mathfrak{h}_\theta = \inf\Big\{ \lim_{t\to\infty} t^{-1} \log \bar{\boldsymbol{\mu}}_t(\Gamma_t) : \Gamma_t \in \boldsymbol{X}^t, \limsup_{t\to\infty} t^{-1} \log \boldsymbol{\mu}_t(\Gamma_t^c) < -\theta \Big\},$$

one can express it in terms of $I$. Moreover, if (7) holds, then the rate function has to satisfy the Gallavotti–Cohen fluctuation relation:

$$I(-r) = I(r) + r \quad \text{for all } r \in \mathbb{R}. \tag{8}$$

# Summary

Summarising the above discussion, we can single out the following questions arising in the study of entropic fluctuations:

**(a)** LDP for the occupation measures of trajectories.

**(b)** Level-3 fluctuation relation for the rate function.

**(c)** Well-posedness of the entropy production and its relation with the physical notion of transport.

**(d)** Law of large numbers for the time average of the entropy production.

**(e)** Strict positivity of the mean entropy production.

**(f)** Local and global LDP for the time average of the entropy production.

Each of these questions may be a difficult problem and most of them can be studied independently.

# Some references

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**Kurchan (1998), Lebowitz–Spohn (1999):** GC relation for some classes of stochastic dynamics.

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**Baiesi–Maes (2005):** Interpretation of the Navier–Stokes equations as a heat conduction network

**Bodineau–Lefevere (2008), Barato–Chetrite (2015), Cuneo–Jakšić–Pillet–AS (2017):** Level-3 LDP and FR

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# Navier–Stokes equations with smooth forcing

Let us consider the Navier–Stokes system on $\mathbb{T}^2$:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{9}$$

The noise is assumed to be smooth in $x$, while its dependence on time should be such that the family of solutions of (9) form a Markov process. Under some non-degeneracy hypotheses, the latter has a unique stationary measure, and our goal is to study entropic fluctuations for the corresponding stationary trajectory.

The validity of the level-3 LDP was proved for various type of random perturbations, and in the discrete-time setting the rate function is given by the Donsker–Varadhan entropy formula:

$$\boldsymbol{I}(\boldsymbol{\lambda}) = \int_{\boldsymbol{H}_-} \operatorname{Ent}\big(\lambda(\boldsymbol{u}, \cdot) \,|\, P(u_0, \cdot)\big) \boldsymbol{\lambda}_-(d\boldsymbol{u}), \tag{10}$$

where $\boldsymbol{H}_- = H^{\mathbb{Z}_-}$, and $P(u, dv)$ is the time-1 transition function.

# Difficulties: Donsker–Varadhan relation

One may try to use (10) to prove the level-3 fluctuation relation:

$$
\begin{aligned}
\boldsymbol{I}(\boldsymbol{\lambda}) &= \int_{\boldsymbol{H}_-} \int_H \log \frac{\lambda(\boldsymbol{u}, \mathrm{d}u_1)}{P(u_0, \mathrm{d}u_1)} \lambda(\boldsymbol{u}, \mathrm{d}u_1)\, \boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{u}) \\
&= \int_{\boldsymbol{H}_- \times H} \log \frac{\boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{u})\lambda(\boldsymbol{u}, \mathrm{d}u_1)}{\boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{u})P(u_0, \mathrm{d}u_1)}\, \boldsymbol{\lambda}_-^1(\mathrm{d}\boldsymbol{u}, \mathrm{d}u_1) \\
&= \mathrm{Ent}\big(\boldsymbol{\lambda}_-^1 \,|\, \boldsymbol{\lambda}_- \otimes P\big),
\end{aligned}
$$

where $\boldsymbol{\lambda}_- \otimes P$ stands for the measure defined by the formula

$$
\langle F, \boldsymbol{\lambda}_- \otimes P \rangle = \int_{\boldsymbol{H}_-} \left\{ \int_H F(\boldsymbol{u}, u_1) P(u_0, \mathrm{d}u_1) \right\} \boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{u}).
$$

Assuming that $P(u_0, \mathrm{d}u_1) = \rho(u_0, u_1)\ell(\mathrm{d}u_1)$, we get

$$
\boldsymbol{I}(\boldsymbol{\lambda} \circ \boldsymbol{\theta}) - \boldsymbol{I}(\boldsymbol{\lambda}) = \int_{\boldsymbol{H}_-} \log \frac{\rho(u_0, u_1)}{\rho(u_1, u_0)}\, \boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u}). \tag{11}
$$

Denoting by $\sigma(\boldsymbol{u})$ the integrand in (11), we obtain the level-3 fluctuation relation (1), in which $\mathrm{ep}(\boldsymbol{\lambda}) = \langle \sigma, \boldsymbol{\lambda} \rangle$.

# Difficulties: linear case

Even in the linear case, one cannot justify the above calculation. For instance, for the discrete-time system

$$u_k = S(u_{k-1}) + \eta_k, \quad k \geq 1,$$

the transition function has the form

$$P(u_0, \cdot) = \ell\big(\cdot - S(u_0)\big), \quad \ell \text{ is the law of } \eta_k.$$

Thus, the measures $P(u_0, \cdot)$ an $P(u_1, \cdot)$ are equivalent if and only if the difference $S(u_0) - S(u_1)$ is an admissible shift for $\ell$. In the case of Gaussian measures, this is equivalent to a lower bound on the covariance operator. A similar claim is true when comparing the forward and backward stationary laws.

# Lagrangian formulation

Let us consider the ODE

$$\dot{y} = u(t, y), \quad y(t) \in \mathbb{T}^2, \tag{12}$$

where $u(t, x)$ is a stationary (in the probabilistic sense) solution of the Navier–Stokes system with random forcing:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{13}$$

We assume that $\eta$ is a bounded random process, smooth in both variables and piecewise independent:

$$\eta(t, x) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1,k)}(t) \eta_k(t - k + 1, x), \tag{14}$$

where $\{\eta_k\}$ are i.i.d. random variables in $L^2([0, 1] \times \mathbb{T}^2)$. We assume in addition that the mean values in $x$ is zero.

# Decomposability hypothesis

We assume that the laws $\ell$ of $\eta_k$ is decomposable:

**(D)** *There is an orthonormal basis $\{e_j(t,x)\}$ in $L^2([0,1]\times\mathbb{T}^2)$ that consists of smooth functions such that*

$$\eta_k(t,x)=\sum_{j=1}^{\infty} b_j\xi_{jk}e_j(t,x), \tag{15}$$

*where $\{b_j\}$ is sequence of non-zero numbers going to zero sufficiently fast, $\xi_{jk}$ are independent random variables whose laws are smooth, supported by $[-1,1]$, and positive on some interval $[-\delta,\delta]$.*

This hypothesis ensures that the Navier–Stokes system (13) has a unique stationary distribution $\mu$. In the following theorem, we fix an arbitrary stationary solution $u(t,x)$ for (13).

# Large deviation principle

## Theorem

*There is a $\mathbb{T}^2$-valued random process $z_t$, whose almost every trajectory satisfies* (12)*, such that the following assertions hold.*

**Stationarity**. *The process $\{z_t, t \geq 0\}$ is stationary.*

**Convergence**. *Let $p \in \mathbb{T}^2$ be an arbitrary initial point and let $y_t(p)$ be the corresponding trajectory of* (12). *Then, for any $s \geq 1$, the law of $(y_t(p), \dots, y_{t+s}(p))$ converges exponentially fast in the total variation norm, as $t \to \infty$, to that of $(z_0, \dots, z_s)$.*

**Large deviations**. *For any $p \in \mathbb{T}^2$, the empirical measures*

$$\boldsymbol{\nu}_t^p = \frac{1}{t} \sum_{k=0}^{t-1} \delta_{\boldsymbol{y}_k(p)}, \quad \boldsymbol{y}_k(p) = (y_l(p), l \geq k),$$

*satisfy LDP with some good rate function $\boldsymbol{I} : \mathcal{P}(\mathbb{T}^{\mathbb{Z}_+}) \to [0, +\infty]$.*

# Entropy production

We now assume that

$$\eta(t,x) = a \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t,x), \quad |a| \geq 1.$$

## Theorem

*For any $t \geq 1$, the law of $(z_1, \ldots, z_t)$ has a smooth density $\rho_t(x_1, \ldots, x_t)$ with respect to the Lebesgue measure on $\mathbb{T}^{2t}$. Moreover, there is $a_0 > 0$ such that, for $|a| \geq a_0$, the density $\rho_t$ is strictly positive, and the entropy production*

$$\sigma_t(\boldsymbol{x}^t) = \log \frac{\rho_t(x_1, \ldots, x_t)}{\rho_t(x_t, \ldots, x_1)}, \quad \boldsymbol{x}^t := (x_1, \ldots, x_t), \qquad (16)$$

*satisfies the following inequality for any $t \geq 1$:*

$$-C \leq t^{-1} \sigma(\boldsymbol{x}^t) \leq C \quad \text{for all } \boldsymbol{x}^t \in \mathbb{T}^{2t}.$$

# A class of RDS

Let $H$ and $E$ be two Hilbert spaces and let $S : H \times E \to H$ be a twice continuously differentiable map. Consider the RDS

$$U_k = S(U_{k-1}, \eta_k), \quad k \geq 1, \tag{17}$$

supplemented with the initial condition

$$U_0 = U \in H. \tag{18}$$

Here $\{\eta_k\}_{k\geq 1}$ is a sequence of i.i.d. random variables in $E$ satisfying the decomposability hypothesis:

**(D)** *There is an orthonormal basis $\{e_j\}_{j\geq 1}$ in $E$ such that*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{19}$$

*where $\{b_j\}$ are non-zero numbers such that $\sum_j b_j^2 < \infty$, and $\xi_{jk}$ are independent random variables with smooth laws supported by $[-1, 1]$.*

# Hypotheses

Let us denote by $\mathcal{K}$ the support of the law of $\eta_k$ and assume that there is a compact subset $X \subset H$ such that $S(X \times \mathcal{K}) \subset X$.

**(AC)** *For any $\varepsilon > 0$ there is an integer $n \geq 1$ such that, given $U, \widehat{U} \in X$, we can find $\zeta_1, \dots, \zeta_n \in \mathcal{K}$ satisfying*

$$\|S_n(U, \zeta_1, \dots, \zeta_n) - \widehat{U}\|_H < \varepsilon. \tag{20}$$

**(ACL)** *The derivative $(D_\eta S)(U, \eta) : E \to H$ has a dense image for any $U \in X$ and $\eta \in \mathcal{K}$.*

The above hypotheses ensure that the Markov process defined by (17) in $X$ has a unique stationary measure $\mu \in \mathcal{P}(X)$.

Let us define the empirical measures

$$\boldsymbol{\nu}_t := \frac{1}{t} \sum_{k=0}^{t-1} \delta_{\boldsymbol{U}_k}, \quad \boldsymbol{U}_k = (U_l, l \geq k). \tag{21}$$

# Main result

## Theorem

*Suppose that Hypotheses* (D), (AC), *and* (ACL) *are satisfied. Then, for any* $U \in X$, *the sequence* $\{\boldsymbol{\nu}_t\}_{t\geq 1}$ *considered as a random measure on* $\boldsymbol{X} = X^{\mathbb{Z}_+}$ *satisfies the LDP with a good rate function* $\boldsymbol{I} : \mathcal{P}(\boldsymbol{X}) \to \mathbb{R}_+$ *given by the Donsker–Varadhan entropy*

$$\boldsymbol{I}(\boldsymbol{\lambda}) = \int_{X^{\mathbb{Z}_-}} \mathrm{Ent}\big(\lambda(\boldsymbol{U},\cdot)\,|\,P_1(U_0,\cdot)\big)\boldsymbol{\lambda}_-(\mathrm{d}\boldsymbol{U}), \qquad (22)$$

*where* $\boldsymbol{\lambda} \in \mathcal{P}(\boldsymbol{X})$ *is an arbitrary stationary measure,* $\boldsymbol{\lambda}_-$ *denote its projection to the negative half-line, and* $\lambda(\boldsymbol{U},\cdot)$ *is the projection to the first component of the conditional measure of* $\boldsymbol{\lambda}$ *given* $\boldsymbol{U} \in X^{\mathbb{Z}_-}$.

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