An infinite-dimensional Cartan development

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Developing fluids?

Part I

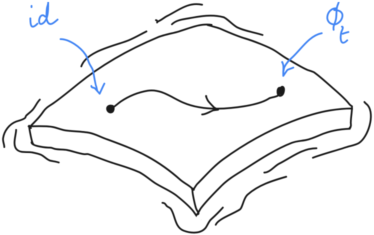
The Lagrangian approach to fluid mechanics

I. Fluid mechanics

Fix a (finite dimensional) closed Riemannian manifold M of dimension d, and call μ its measure.
Consider $\mathcal{D}$ *some* space of maps $M \to M$, and $\mathcal{D}_\mu \subset \mathcal{D}$ *some* space of volume-preserving diffeomorphisms.

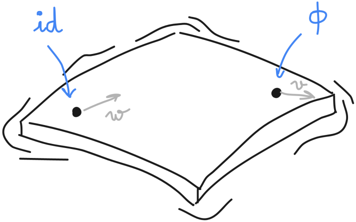
We think of points in $\mathcal{D}$ as fluid configurations, of curves in $\mathcal{D}$ as fluid motions. The objective is to define Brownian fluids (and more general stochastic fluids).

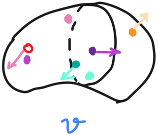
I. Fluid mechanics



- Point in $\mathcal{D} \leftrightarrow$ map
- Curve in $\mathcal{D} \leftrightarrow$ flow
- Tangent vector at id $\in \mathcal{D} \leftrightarrow$ vector field
- Tangent vector at $\phi \in \mathcal{D} \leftrightarrow$ vector field rooted at ϕ.

I. Fluid mechanics





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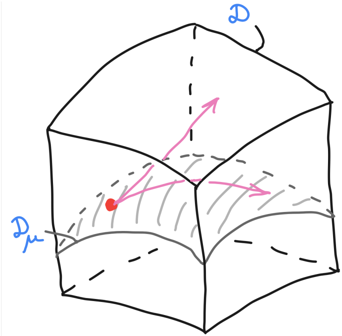
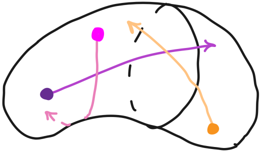
I. Fluid mechanics

For *some* Riemannian metric on $\mathcal{D}$, the geodesics follow the inviscid Burgers equation: every fluid particle goes straight ahead (we may have immediate collision).

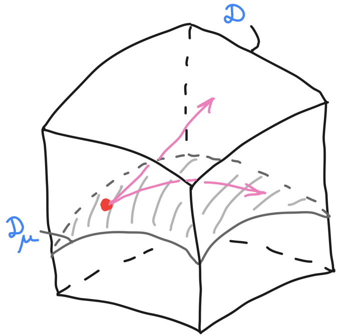


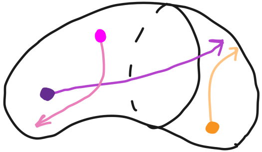


$$\overline{\nabla}_{\dot\phi}\dot\phi = 0, \qquad \nabla_{\dot\phi(x)}\big(\dot\phi(x)\big) = 0 \qquad \text{for all } x \in M.$$

I. Fluid mechanics

For the same Riemannian metric restricted to $\mathcal{D}_\mu$, the geodesics follow the incompressible inviscid Euler equations: the fluid particles cannot collide (they have an associated volume) but try to go straight ahead as much as possible.





$$\overline{\nabla}_{\dot\phi}\dot\phi = -\nabla p.$$

I. Fluid mechanics

Objective: define a common framework for classical fluids, Brownian fluids, stochastic perturbations of classical fluids, and possibly more.
Solution: the Cartan development

Part II

Stochastic processes on manifolds

II. Stochastic processes on manifolds

Brownian motion is the process with generator $\frac{1}{2}\Delta_M$.

Locally, up to order 1, Δ_M is a sum of squares: if $(X_1(x), \ldots, X_d(x))$ is an orthonormal basis at each point,

$$\Delta_M f = X_1 \cdot (X_1 \cdot f) + \cdots + X_d \cdot (X_d \cdot f) + X_0 \cdot f,$$

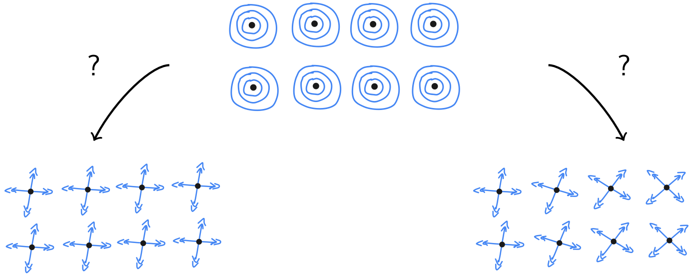
for some vector field X_0. It means the Brownian motion x satisfies

$$\mathrm{d}x_t = X_1(x_t) \circ \mathrm{d}W_t^1 + \cdots + X_d(x_t) \circ \mathrm{d}W_t^d + X_0(x_t)\mathrm{d}t.$$

II. Stochastic processes on manifolds

Somewhat unsatisfactory for a few reasons.

- Arbitrary choice for the basis; we break the symmetry.
- X_0 depends on the choice of basis.
- Patching the local SDEs requires a higher-dimensional noise.



II. Stochastic processes on manifolds

Idea of Eells, Elworthy and Malliavin: enlarge the space to the orthonormal frame bundle OM.

$u = (x, u)$ is a point x on the manifold together with some orthonormal basis $\big(u(\epsilon_i)\big)_i$ at x ($u : \mathbb{R}^d \to T_x M$ is an isometry).

Informal equation:

$$\begin{aligned} \mathrm{d}x_t &= u_t(\epsilon_1) \circ \mathrm{d}W_t^1 + \cdots + u_t(\epsilon_d) \circ \mathrm{d}W_t^d, \qquad \nabla_{\mathrm{d}x_t} u_t = 0. \\ &= u_t(\circ\, \mathrm{d}W_t) \end{aligned}$$

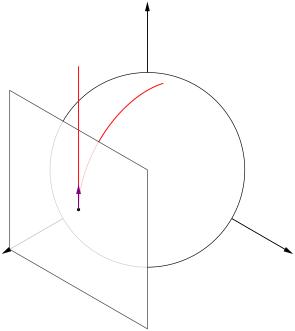
Actual equation:

$$\mathrm{d}(x_t, u_t) = \underbrace{H(x_t, u_t)}_{\text{horizontal lift}} \circ \mathrm{d}W_t.$$

In fact, we can do much more than just Brownian motion: Cartan development.

$$\mathrm{d}u_t = H(u_t)\mathrm{d}z_t$$

Geodesics:



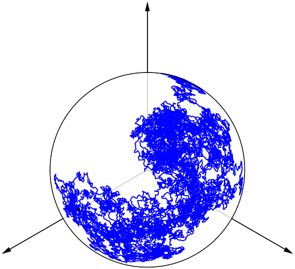
$$z_t = z_0 + \dot{z}_0 t$$

II. Stochastic processes on manifolds

In fact, we can do much more than just Brownian motion: Cartan development.

$$\mathrm{d}u_t = H(u_t)\mathrm{d}z_t$$

Brownian motion:



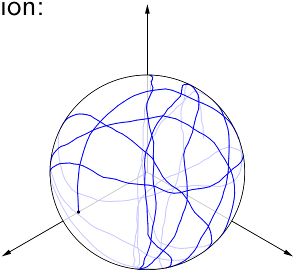
$$z_t = W_t$$

II. Stochastic processes on manifolds

In fact, we can do much more than just Brownian motion: Cartan development.

$$\mathrm{d}u_t = H(u_t)\mathrm{d}z_t$$

Kinetic Brownian motion:



$\dot{z} =$ Brownian motion on the sphere

The plan: Define infinite-dimensional versions $\mathcal{O}_{(\mu)}$ of the orthonormal frame bundle, with smooth horizontal lifts

$$\overline{H}_{(\mu)} : \mathcal{O} \times \underbrace{\mathcal{E}}_{\text{flat space}} \to T\mathcal{O}.$$

Then, for any flat motion z of your choosing, we can try to solve the smooth controlled equation

$$\mathrm{d}u_t = \overline{H}_{(\mu)}(u_t) \circ \mathrm{d}z_t.$$

Part III

The infinite-dimensional orthonormal bundle

III. Infinite-dimensional geometry

1. Topology

Recall M is a (finite-dimensional) closed Riemannian manifold of dimension d, and fix $s > d/2$.
There is a way to define sections of regularity H^s for every given bundle F over M, and they comes with a natural manifold structure. For instance:

- the H^s vector fields over M are the H^s sections of TM;
- the H^s maps $\phi : M \to M$ are the H^s sections of $M \times M$;
- the H^s "unrooted" vector fields are the H^s sections of $TM \times M$.

We write $H^s(F)$ for the manifold of H^s-regular sections of F.

1. Topology

Lemma (Omega lemma)

Pointwise operations are smooth.

Example:
For M an oriented surface, the operation R that sends

- a collection of vectors $v(x)$ (a vector field $v \in H^s(TM)$)
- and a collection of angles $\theta(x)$ (a function $\theta \in H^s(\mathbb{R} \times M)$)
- to the collection of vectors $R_{\theta(x)}(v(x))$ rotated clockwise (a vector field $R(\theta, v) \in H^s(TM)$)

is pointwise. By the Omega lemma, it is smooth.

1. Topology

Recall $s > d/2$. Define

$$\mathcal{D} := \{\phi : M \xrightarrow{H^s} M\},$$
$$\mathcal{D}_\mu := \{\phi : M \xrightarrow{H^{s+1}} M \text{ volume-preserving diffeomorphism}\}.$$

(Not obvious but true: $\mathcal{D}_\mu$ is a submanifold of $\mathcal{D}^{s+1}$.)

The tangent space at $\phi \in \mathcal{D}$ is the space of H^s vector fields rooted at ϕ, so the total tangent space is $H^s(TM)$.
The tangent space at $\phi \in \mathcal{D}_\mu$ is the space of divergence-free H^{s+1} vector fields rooted at ϕ.

2. Geometry

Metric = total kinetic energy.
For $v \in T_\phi \mathcal{D}$ (vector field rooted at ϕ),

$$\frac{1}{2}|v|^2_{L^2} = \frac{1}{2}\int_M |v(x)|^2 \mathrm{d}x.$$

Weak Riemannian structure: need not admit geodesics, a connection, etc. However, in $\mathcal{D}_{(\mu)}$, it does!
In $\mathcal{D}$, everything is easy because it works pointwise:

- $t \mapsto \phi_t$ geodesic means $t \mapsto \phi_t(x)$ geodesic for all x;
- for $t \mapsto v_t$ a vector field along $t \mapsto \phi_t$,

$$(\overline{\nabla}_{\dot\phi} v)(x) = \nabla_{\dot\phi(x)} v(x).$$

III. Infinite-dimensional orthonormal bundle

2. Geometry

In $\mathcal{D}_\mu$, everything is also nice because the orthogonal projections $P_\phi : T_\phi\mathcal{D} \to T_\phi\mathcal{D}$ on the space of divergence-free vector fields rooted at $\phi \in \mathcal{D}_\mu$ patch into a smooth $P : T\mathcal{D}_{|\mathcal{D}_\mu} \to T\mathcal{D}$.

Important difference: the geodesics are only defined locally.

Part IV

The infinite-dimensional Cartan development

IV. Infinite-dimensional Cartan development

Pairs (ϕ, u), where $\phi : T_{\mathrm{id}}\mathcal{D}_{(\mu)} \to T_\phi \mathcal{D}_{(\mu)}$ is an isometry.

- Pointwise (easy) isometries: choose an isometry $e(x) : T_x M \to T_{\phi(x)} M$ for each x, and compose at the target. We write $u = [e]$.
- Non-local (subtle) isometries: anything else. For instance, move the energy across the Fourier modes.

There is an easy way to construct the pointwise orthonormal frame bundle and the pointwise Cartan development. It turns out that they are enough to construct the Cartan development in $\mathcal{D}$, but not in $\mathcal{D}_\mu$.

IV. Infinite-dimensional Cartan development

Because of the non-local character, it is not obvious that the set of pairs (ϕ, u) should admit a manifold structure.

Trick: pull back the non-local issues to a fixed Banach space. For any isometry $u : T_{\mathrm{id}}\mathcal{D} \to T_\phi \mathcal{D}$ and collection e of isometries $e(x) : T_x M \to T_{\phi(x)} M$,

$$u = [e] \circ \underbrace{([e]^{-1} \circ u)}_{=:\, u^0}.$$

The space of triples (ϕ, e, u^0) is a product

$$H^s(F) \times \mathrm{Isom}(T_{\mathrm{id}}\mathcal{D})$$

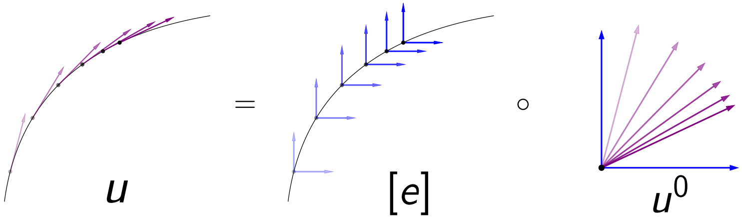
for F well-chosen, and $\mathrm{Isom}(T_{\mathrm{id}}\mathcal{D})$ inside to a fixed Banach space.

This is big enough to be used as a $\mathcal{O}_{(\mu)}$, and the horizontal lift H over $\mathcal{O}$ is easily defined.

IV. Infinite-dimensional Cartan development

What about $\mathcal{D}_\mu$?
Rough idea: $u = [e] \circ u^0$ with

- e parallel transported with no constraint,
- u^0 continuously adjusting to preserve the volume.



Actual construction technical but not clever; same as parallel transport in finite-dimensional submanifolds.

Bibliography

Thank you for your attention.

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