Heat kernels, their derivatives, and the bridge process in small time

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Acknowledgments

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Also thank you for not scheduling me to speak at 3am my time…

This is based on joint work with Ludovic Sacchelli (Lyon).

sub-Riemannian geometry

- M is a smooth, connected, complete (for now) sub-Riemannian manifold of dim. n,
- with a sub-Laplacian Δ, locally

$$\Delta = \sum_{i=1}^{k} V_i^2 + V_0$$

  where $V_0, V_1, \ldots, V_k$ are smooth and $V_1, \ldots, V_k$ are orthonormal and bracket-generating (strong Hörmander), and a smooth volume (as a reference measure). (Well, rank-varying is also possible…)
- X_t is the associated diffusion and $p_t(x, y)$ the heat kernel (for $\partial_t u_t(x) = \Delta u_t(x)$– we *try* to stick to the analysts' normalization).
- Riemannian manifolds are viewed as a special case.

Abnormals

- A sub-Riemannian manifold may admit abnormal minimizers in addition to (normal) geodesics. The diagonal is always (non-strictly) abnormal on a properly sR manifold.
- In several important classes of sub-Riemannian manifolds, such as contact and CR geometry, non-trivial abnormals do not arise.
- No such thing as abnormals in the Riemannian case.

Away from abnormals and the diagonal, the exponential map and cut and conjugate loci are largely analogous to the Riemannian case, although note that $\mathrm{Cut}(x)$ is adjacent to x.

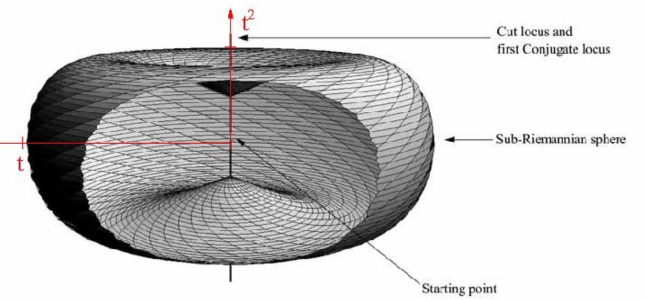
The cut locus

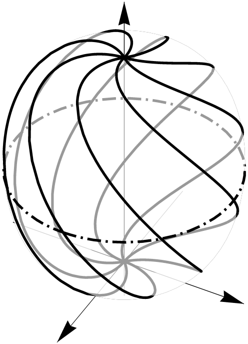
For $x \in M$, the non-abnormal cut locus $\mathrm{Cut}(x)$ is

- the set of $y \in M$ such that there is more than one minimal geodesic from x to y, or there is a minimal geodesic from x to y which is conjugate (or both);
- the closure of the set where $\mathrm{dist}(x, \cdot)$ is not differentiable;
- the points where geodesics cease to minimize distance.

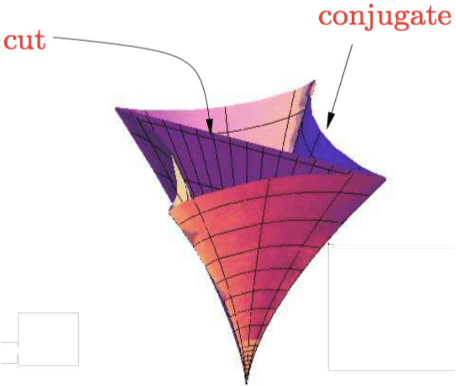
The Heisenberg group (prototype for Morse-Bott singularities)

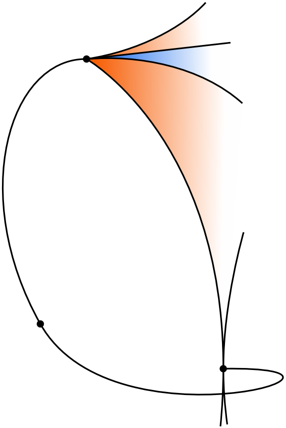
Let $X = \partial_x - (y/2)\partial_z$ and $Y = \partial_y + (x/2)\partial_z$ be orthonormal in $\mathbb{R}^3$ (here $\Delta = X^2 + Y^2$ and the volume is the Euclidean one):





Perturbed 3D contact case (prototype for A_n-singularities)





A_3-singularity: suspension of $(x, y) \mapsto (x^3 + xy, y)$

(Folk) theorems...

For vector fields Z^i, multi-index α and non-negative integer l, we have

Léandre asymptotics (also Bailleul-Norris), Varadhan for Riemannian:

- $2t \log p_t(x,y) \to -d(x,y)^2$ as $t \searrow 0$,
- $\limsup_{t \searrow 0} 2t \log \left(\left| \partial_t^l Z_y^\alpha p_t(x,y) \right| \right) \leq -d^2(x,y)$

uniformly on compacts.

Ben Arous asymptotics, Minakshisundaram and Pleijel for Riemannian:

- $\partial_t^l Z_y^\alpha p_t(x,y) =$
  $t^{-(|\alpha|+2l+d/2)} e^{-\frac{d(x,y)^2}{2t}} \left(\sum_{k=0}^{N} c_k(x,y) t^k + t^{n+1} r_{n+1}(t,x,y) \right)$

uniformly on compacts subsets of $M \times M$ avoiding the cut locus and abnormals.

Molchanov's technique

In the 70's, Molchanov discussed a method (later formalized by the Bellaiches and Hsu) to get an expansion similar to that of Minakshisundaram and Pleijel at the cut locus in the Riemannian case. It is quite flexible, requiring 3 ingredients

- a "global" coarse estimate, like Varadhan/Leandre above
- a finer estimate off of the cut locus, like Minakshisundaram-Pleijel/Ben Arous above
- the Markov property/Chapman-Kolmogorov equation

We develop this idea for the sub-Riemannian case. The idea is to glue two copies of the expansion.

Other approaches

Integral representations of hypoelliptic heat kernels for left-invariant structures on Lie groups have been studied algebraically going back to Gaveau and Hulanicki (Heisenberg group, late 70s) and Beals-Gaveau-Greiner (higher-dimension extension of this, mid-90s). Asaad-Gordina '16 gave a general treatment for nilpotent Lie groups via generalized Fourier transform.

The positively and negatively curved sub-Riemannian model spaces, de Sitter and anti-de Sitter, space also admit explicit integral representations for the heat kernel, as developed by Bonnefont, Badoin-Bonnefont, and Baudoin-Wang ('09-'12).

Recently, Inahama-Taniguchi '17 used Watanabe's distributional Malliavin calculus to give an approach to sub-Riemannian heat kernel asymptotics, and Ludewig '18 gave similar asymptotics for Riemannian vector bundles via a path-integral-type approach, both emphasizing complete expansions. (Also Kusuoka-Stroock…)

Plus plenty of related directions…

Basic objects

Take $x, y \in M$, let Γ be the (compact) set of midpoints of minimal geodesics from x to y and let Γ_ϵ be an ϵ-neighborhood. For example, if M is the standard sphere and x, y the north and south poles, Γ is the equator.

Let

$$h_{x,y}(z) = \frac{\mathrm{dist}(x,z)^2}{2} + \frac{\mathrm{dist}(z,y)^2}{2}$$

be the *hinged energy function*. Note

- $h_{x,y}(z)$ achieves its minimum (of $d^2(x,y)/4$) exactly on the set Γ.
- For $z \in \Gamma$, $\nabla^2 h_{x,y}(z)$ is non-degenerate if and only if the geodesic from x to y through z is non-conjugate.

Laplace integrals and the role of conjugacy

- One finds that

$$p_t(x,y) = \left(\frac{1}{2\pi t}\right)^n \int_{\Gamma_\epsilon} c_0(x,z)c_0(z,y)e^{-h_{x,y}(z)/t}\,dz.$$

  Moreover, you can keep as many terms from the Ben Arous expansion as you want, and also take derivatives of p_t.
- The germ/normal form of $h_{x,y}$ near its minima governs the power of $1/t$ appearing in the expansion of these integrals. The behavior of $h_{x,y}$, in turn is governed by the exponential map; a "more degenerate" Hessian corresponds to "more conjugacy."
- Thus "more conjugacy leads to a larger power of $1/t$."

(Local) real-analyticity

- Suppose that every $z \in \Gamma$ is contained in a coordinate patch such that $h_{x,y}$ is (locally) real-analytic (automatic if M is real-analytic). Then for any $z \in \Gamma$, there is a rational $\alpha(z) \in [d/2, d - (1/2)]$, a non-negative integer $\beta(z)$, so for any small ball,

$$\int_{B_z(r)} e^{-\frac{h_{x,y}(u) - h_{x,y}(z)}{4t}} \mu(du) \sim \frac{C}{t^{\alpha}} \left(\log \frac{1}{t} \right)^{\beta} .$$

- With lexigraphical order, $(\alpha(z_1), \beta(z_1)) < (\alpha(z_2), \beta(z_2))$ means that the integral around z_2 dominates the integral around z_1 as $t \searrow 0$.
- Let

$$\Gamma^m_{x,y} = \Gamma^m = \{ z \in \Gamma : (\alpha(z), \beta(z)) = \text{the max} \} .$$

  Then Γ^m is a non-empty, closed subset of Γ (corresponding to geodesics of "maximal degeneracy") (Hsu '90s).

The probability measure

- Asymptotics of p_t can be worked out from the above (Boscain-Barilari-Charlot-N. '12-'19); can extend to complete expansions and derivatives of p_t (N.-Sacchelli '20-) that's another talk.
- But we move on...
consider the one-parameter family of probability measures:

$$m_t(dz) = \frac{\mathbf{1}_{\Gamma_\epsilon}(z)}{Z_t} c_0(x,z) c_0(y,z) \exp\left(-\frac{h_{x,y}(z)}{t}\right) dz$$

$$\text{where} \quad Z_t = \int_{\Gamma_\epsilon} c_0(x,z) c_0(y,z) \exp\left(-\frac{h_{x,y}(z)}{t}\right) dz.$$

- The m_t are subsequentially compact, with all limits supported on Γ. In general, consider any limit m_0 along some sequence of times t_n.
- If locally real-analytic, m_0 is unique with support Γ^m.

Law of large numbers for the bridge

Let μ_t be the distribution, on pathspace, of the associated bridge process from x to y in time t. If there is a single minimal geodesic from x to y, Bailleul-Norris recently showed that μ_t converges to point mass on that geodesic, as $t \searrow 0$.

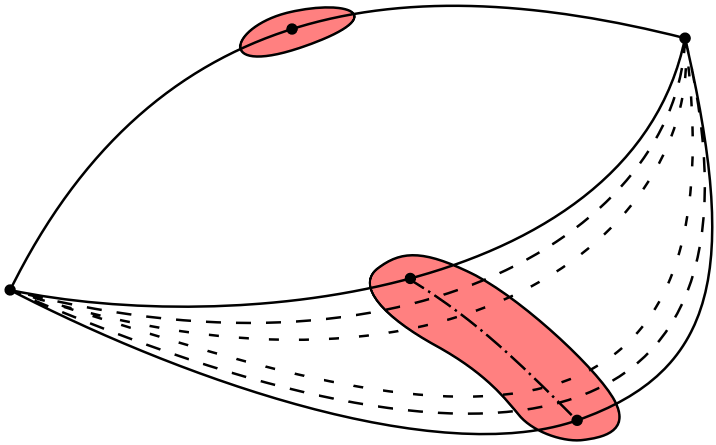
By Molchanov-style gluing, we show that

Theorem (N.-Sacchelli ’20-)

If every minimal geodesic from x to y is strongly normal, $\mu_{t_n} \to \tilde{m}_0$, where $\tilde{m}_0$ is the natural lift of m_0 to pathspace.

The Riemannian case was done by Hsu in the 90s, as a consequence of a large deviation principle. There are some large deviation results in the sR situation (Bailleul, Inahama), but none pushed through to this result.

The picture



Simple examples

For spheres or Heisenberg, m_0 is uniform on Γ, by symmetry.

If the exponential map at γ is A_m-conjugate, near the midpoint of γ,

$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^2 + \ldots + z_{n-1}^2 + z_n^{m+1}.$$

($A_1 \iff$ non-conjugate)

Theorem (N.-Sacchelli ’20-, Hsu ’90s in Riemannian case)

Assume all minimal geodesics from x to y are strictly normal, there is $\ell \in \{1,3,5,\ldots\}$ such that for every $z \in \Gamma$, γ_z is A_m-conjugate for $1 \le m \le \ell$, and for at least one $z \in \Gamma$, γ_z is A_ℓ-conjugate. Then the support of m_0 is exactly those $z \in \Gamma$ for which γ_z is A_ℓ-conjugate.

A non-analytic example

- Let $g(z_1)$ be a smooth, non-negative function with zeroes at $\pm\frac{1}{n}$ for all positive integers n and at 0, with Hessian non-degenerate at all of the $\pm\frac{1}{n}$.
- g necessarily vanishes to all orders at 0.
- Let Γ be contained in a coordinate patch such that

$$h = \frac{d^2(x,y)}{4} + g(z_1) + z_2^2 + \cdots z_n^2$$

- This can be realized on a Riemannian surface and on a sR 3D-contact structure.
- Then m_0 is a point mass at the geodesic through the origin.
- n.B. The leading term of p_t is not known in this case; Arnold et. al.'s real-analytic results don't apply.

Log-derivaives

Theorem (N.-Sacchelli '20-)

Assume alll minimal geodesics from x to y are strictly normal, $Z^1, \ldots, Z^N$ *are smooth vector fields near y. Then*

$$Z^N \cdots Z^1 \log p_t(x,y) =$$
$$\left(-\frac{1}{t}\right)^N \left\{\kappa^{m_t}\left(d(\cdot,y)Z^1 d(\cdot,y), \ldots, d(\cdot,y)Z^N d(\cdot,y)\right) + O(t)\right\},$$

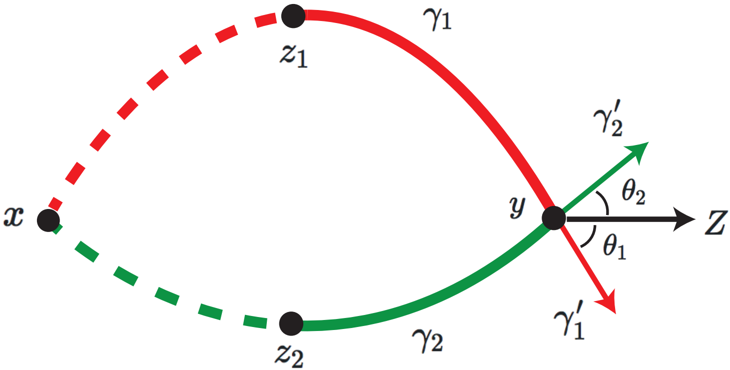
where κ^{m_t} *is the joint cumulant w.r.t.* m_t*, so*

$$\lim_{n\to\infty} t_n Z_y \log p_{t_n}(x,y) = -\frac{1}{2} d(x,y)\mathbb{E}^{m_0}\left[Z_y d(\cdot,y)\right],$$
$$\lim_{n\to\infty} t_n^2 Z'_y Z_y \log p_{t_n}(x,y) = \frac{d^2(x,y)}{4}\operatorname{Cov}^{m_0}\Big(Z_y d(\cdot,y), Z'_y d(\cdot,y)\Big),$$

and $\lim_{n\to\infty} t_n^N Z_y^N \cdots Z_y^1 \log p_{t_n}(x,y) =$

$$\left(-\frac{d(x,y)}{2}\right)^N \kappa^{m_0}\Big(Z_y^1 d(\cdot,y), \ldots, Z_y^N d(\cdot,y)\Big).$$

Riemannian example with two points in support



Take $\|Z\| = 1$ and $m_0 = p\delta_{z_1} + (1-p)\delta_{z_2}$.

Then $Z_y d(z_i, y) = \cos\theta_i$ and

$$\mathrm{Var}^{m_0}\left(Z_y d(\cdot, y)\right) = p(1-p)\left(\cos\theta_1 - \cos\theta_2\right)^2$$

Characterizing the cut locus

Theorem (N.-Sacchelli '20-)

Assume all minimal geodesics from x to y are strictly normal, $\mathcal{Z}$ a set of vector fields on a near y, C^1-bounded, such that $\mathcal{Z}|_{T_yM}$ contains a neighborhood of the origin. Then $y \notin \mathrm{Cut}(x)$ if and only if

$$\limsup_{t \searrow 0} \left[\sup_{Z \in \mathcal{Z}} t \left| Z_y Z_y \log p_t(x, y) \right| \right] < \infty$$

and $y \in \mathrm{Cut}(x)$ if and only if

$$\lim_{t \searrow 0} \left[\sup_{Z \in \mathcal{Z}} t Z_y Z_y \log p_t(x, y) \right] = \infty$$

Compare with Barilari-Rizzi '19 – non-abnormal cut locus is characterized by the square of the distance failing to be semi-convex.

... and localization

Localization methods for small-time asymptotics imply we can treat incomplete manifolds:

- N.-Sacchelli '20- localize to a compact K at points where $d(x, \partial K) + d(y, \partial K) > d(x, y)$, and thus all of the above holds if $d(x, \infty) + d(y, \infty) > d(x, y)$. This is what you can expect in general.
- Bailleul-Norris localized the heat kernel itself, but not its derivatives, to points with $\inf_{z \in \partial K}(d(x, z) + d(y, z)) > d(x, y)$ under a sector condition that limits the asymmetry. This extends the LLN for the bridge process, but not the log-derivatives, to this situation.

Related to work on "not feeling the boundary" by Hsu in the 90s.