I HYDRODYNAMICS AS inf.-dim. RIEMANNIAN GEOMETRY.

1. Motivation: KOLMOGOROV'S 1958/59 Seminar on Turbulence
   - understand turbulence in low viscosity regime,
   - identify „chaotic attractors",
   - explain experimental results (e.g. KOLM. '41);
   - „geometrization programs"
     (fluids – ARNOLD, statistics – CHENTSOV ...?)

2. ARNOLD (1966):
   - M^n – compact, Rie. mfld with or w/o bdry, „fixed fluid vessel"
     (e.g.: T^n, S^n, $\Omega \subset \mathbb{R}^n$ (domain) – $n = 2, 3$ ...)
     μ – Riem. volume form,
   - Fluid's configuration space:

     fluid particles do not split/fuse

$$D_\mu = \text{group of diffeomorphisms of } M \text{ preserving } \mu$$

fluid is incompressible

   - As in classical mechanics, fluid motions trace out in the config. space geodesic paths of the metric defined by the Kinetic energy

Here: ass. the L^2-inner product of vector fields on M.

3. EBIN-MARSDEN (1970):

Topologize the configuration space $\mathcal{D}_\mu$ with a „reasonable" topology, e.g.:

H^s - Sobolev ($s > \frac{n}{2} + 1$) or $C^{1,\alpha}$ - Hölder ($0 < \alpha < 1$)

or a more „exotic" function space

... etc.

so that the corresp. completion, e.g.

Sobolev $\mathcal{D}^s_\mu$ — becomes a topological group and a C^∞ Hilbert mfld

and the L^2-geodesic equations do not „lose derivatives" $\Longrightarrow$ O.D.E.!

4. Eulerian and Lagrangian formulations are only FORMALLY equivalent. E.g., the corresp. data-to-solution maps:

<u>Eulerian</u>:

$u_0 \to u$ is at best cts!

while

<u>Lagrangian</u>:

$u_0 \to (\gamma, \dot{\gamma})$ is C^∞-smooth!

The Cauchy Problem for the Euler eqns has a long history:

LICHTENSTEIN, GÜNTHER, WOLIBNER (1920/30's)

...

Banach/Picard iterations

<u>In part.</u>: the L^2 metric has a well-defined exponential map:

$$\exp_e : \underset{\text{open}}{\mathcal{U}} \subset T_e D^s_\mu \longrightarrow D^s_\mu$$

$$\exp_e(t u_0) := \gamma(t)$$

the unique L^2-geodesic from e in dir. u_0

<u>COR.</u> (EBIN-MARSDEN):

$\exp_e$ is a <u>local</u> diffeomorphism (by the IFT).

$\Rightarrow$ locally O.K. but <u>globally</u> we encounter <u>singularities</u> ...

$\Rightarrow$ study singularities of the L^2-$\exp_e$ in $\mathcal{D}^s_\mu$ as conjugate points of the classical RG.

- In infinite-dim. expect trouble:
  - 2 types: $\nearrow$ mono-conjugate ($d\exp_e$ is not 1-1); $\searrow$ epi-conjugate ($d\exp_e$ is not onto).
  - ∞-order of conjugacy (of either type).
  - accumulation pts along finite geodesic segments.
- Need more examples - incl. pathological. ← Later ...
- No such pathologies in 2D!

b/c.:

EBIN-M.-PRESTON (2006): ← ... BENN (2018)

- M^n - compact, Riem. poss. with bdry
- $n=2 \Rightarrow L^2$-$\exp_e$ is a nonlinear Fredholm map of index $=0$;
- $n=3 \Rightarrow$ Fredholm property fails in general.

E.g.: take $M=S^3$, $u_0=$ left-inv. Killing field

then: $\operatorname{ran} d\exp_e(\pi u_0)$ is not closed in H^s

↑ special case? ...

· SMALE (1965)
· 2D L^2-$\exp_e$ like a map between fin-dim. mflds.

... some results:

(higher-order metrics)

- M.-PRESTON (2010):

M^3 - compact, Riem. $\partial M = \emptyset$ $(r>0)$

$\mathcal{D}^s_\mu(M^3)$ equipped with a right-inv. Sobolev H^r-metric

$\Longrightarrow$ H^r-$\exp_e$ is Fredholm of index $= 0$.

- LICHTENFELZ-M.-PRESTON (2018): (presence of symmetries)

M^3 - compact or asympt. euclidean Riem. with a Killing field K,

$u_0 \in T_e\mathcal{D}^s_\mu$ - K/axi-symmetric v. field with no swirl/small swirl

$\Longrightarrow$ $d\exp_e(tu_0)$ is Fredholm of index $= 0$.

K-structures on M:

- $v \in T_e\mathcal{D}^s_\mu$ is K-symmetric if $[v, K] = 0$.

↳ $T_e\mathcal{A}^s_\mu$ is a „Lie algebra" and $\mathcal{A}^s_\mu := \left\{ \eta \in \mathcal{D}^s_\mu : \begin{array}{l}\eta \text{ commutes}\\ \text{with } K\text{-flow}\end{array} \right\}$

is a top. group and C^∞ Hilbert submfld of $\mathcal{D}^s_\mu$ (totally geodesic!)

- $v \in T_e\mathcal{A}^s_\mu$ is swirl-free if $\underbrace{\langle v, K\rangle}_{=:\text{swirl of } v} = 0$.

↳ $T_e\mathcal{A}^s_{\mu,0}$ is a „Lie algebra" iff $K^\perp := \{ v : \langle v, K\rangle = 0 \}$ is an integrable distribution in M^3;

and

$$\mathcal{A}^s_{\mu,0} := \left\{ \eta \in \mathcal{A}^s_\mu : \begin{array}{l}\eta \text{ preserves}\\ \text{leaves of } K^\perp\end{array} \right\}$$

is also a top. group and a C^∞ Hilbert submfld of $\mathcal{A}^s_\mu$ (totally geodesic!).

• Another geometric result on 2D/3D „dichotomy":

SHNIRELMAN (1985/1994): $\text{diam}^{L^2}\left(\mathcal{D}^s_\mu(M^3)\right) < +\infty$

compact, simply connected Riemannian mfld

ELIASHBERG-RATIU (1991): $\text{diam}^{L^2}\left(\mathcal{D}^s_\mu(B^{2n})\right) = +\infty$

unit 2n-ball
or any compact, exact symplectic (with bdry) mfld

• More consequences of fredholmness:

- Hydrodyn. Morse Index Theorem [M.-PRESTON, 2010];
- Calculation of normal forms for the L^2-$\exp_e$ and the classification of singularities (as cusps, folds ...etc) [LICHTENFELZ, 2018];
- L^2-$\exp_e$ is not 1-1 near its conjugate pts.

HD interpretation: any H^s neighb. of a conjugate pt in $\mathcal{D}^s_\mu(M^2)$ contains fluid configurations that can be reached from e by two distinct fluid flows in same time.