Regularity properties of some infinite-dimensional hypoelliptic diffusions

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Smooth measures

Definition

A measure μ on $\mathbb{R}^d$ is smooth *if μ is abs cts with respect to Lebesgue measure and the RN derivative is strictly positive and smooth – that is,*

$$\mu = \rho \, dm, \ \textit{for some } \rho \in C^\infty(\mathbb{R}^d, (0,\infty)).$$

Hypoellipticity

In the theory of diffusions, hypoellipticity of the generator is a sufficient (and nearly necessary) condition to ensure smoothness.

Theorem (Hörmander)

Given vector fields $X_0, X_1, \ldots, X_k : \mathbb{R}^d \to \mathbb{R}^d$*, a second order differential operator*

$$L = \sum_{i=1}^{k} X_i^2 + X_0$$

is hypoelliptic *if*

$$\operatorname{span}\{X_{i_1}(x), [X_{i_1}, X_{i_2}](x), [[X_{i_1}, X_{i_2}], X_{i_3}](x), \ldots : \\ i_\ell \in \{0, 1, \ldots, k\}\} = \mathbb{R}^d$$

for all $x \in \mathbb{R}^d$*.*

If $\{X_t\}_{t \geq 0}$ is a diffusion on $\mathbb{R}^d$ with hypoelliptic generator L, then $\mu_t = \mathrm{Law}(X_t)$ is smooth.

Kolmogorov diffusion

Let $\{B_t\}_{t\geq 0}$ be BM on $\mathbb{R}^d$. The Kolmogorov diffusion on $\mathbb{R}^d \times \mathbb{R}^d$

$$X_t := \left(B_t, \int_0^t B_s\, ds \right)$$

has generator

$$\begin{aligned} (Lf)(p,\xi) &:= \frac{1}{2}\sum_{j=1}^d \frac{\partial^2 f}{\partial p_j^2}(p,\xi) + \sum_{j=1}^d p_j \frac{\partial f}{\partial \xi_j}(p,\xi) \\ &= \frac{1}{2}(\Delta_p f)(p,\xi) + p\cdot(\nabla_\xi f)(p,\xi). \end{aligned}$$

The operator L is hypoelliptic, and thus $\mathrm{Law}(X_t)$ is smooth.

For example, for $d=1$, $d\mathrm{Law}(X_t)(p,\xi) = p_t(p,\xi)\, dp\, d\xi$ where

$$p_t(p,\xi) = \frac{\sqrt{3}}{\pi t^2}\exp\left(-\frac{2p^2}{t} + \frac{6p\xi}{t^2} - \frac{6\xi^2}{t^3} \right).$$

Smooth measures

Definition

A measure μ on $\mathbb{R}^d$ is smooth *if μ is abs cts with respect to Lebesgue measure and the RN derivative is strictly positive and smooth – that is,*

$$\mu = \rho\, dm, \text{ for some } \rho \in C^\infty(\mathbb{R}^d, (0, \infty)).$$

(*) for any multi-index α, there exists a function $g_\alpha \in C^\infty(\mathbb{R}^d) \cap L^{\infty-}(\mu)$ such that

$$\int_{\mathbb{R}^d} (-D)^\alpha f \, d\mu = \int_{\mathbb{R}^d} f g_\alpha \, d\mu, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

$$\text{smoothness} \iff (*)$$

A first step to smoothness: Quasi-invariance

Definition

A measure μ on Ω is quasi-invariant *under a transformation $T : \Omega \to \Omega$ if μ and $\mu \circ T^{-1}$ are mutually absolutely continuous.*

In particular, we're interested in quasi-invariance under transformations of the type

$$T = T_h = \text{translation (in some sense) by some } h \in \Omega_0 \subset \Omega,$$

where typically Ω_0 is some distinguished subset of Ω.

Quasi-invariance

The canonical ∞-dim example

The *Wiener space* construction is a triple

- $\mathcal{W} = \mathcal{W}(\mathbb{R}^k) = \{w : [0,1] \to \mathbb{R}^k : w \text{ is cts and } w(0) = 0\}$ equipped with the sup norm,
- $\mu = \mathrm{Law}(B_\cdot) =$ *Wiener measure* on $\mathcal{W}$, and
- $\mathcal{H} = \mathcal{H}(\mathbb{R}^k) =$ *Cameron-Martin space*, that is,

$$\mathcal{H} = \left\{ h \in \mathcal{W} : h \text{ is abs cts and } \int_0^1 |\dot{h}(t)|^2\, dt < \infty \right\}$$

equipped with the inner product

$$\langle h, k \rangle_{\mathcal{H}} := \int_0^1 \dot{h}(t) \cdot \dot{k}(t)\, dt.$$

Quasi-invariance

The canonical ∞-dim example

- $\mathcal{W}$ is a Banach space
- μ is a Gaussian measure
- The mapping $h \in \mathcal{H} \mapsto \dot{h} \in L^2([0,1], \mathbb{R}^k)$ is an isometric isomorphism and $\mathcal{H}$ is a separable Hilbert space.
- $\mathcal{H}$ is dense in $\mathcal{W}$ and $\mu(\mathcal{H}) = 0$

Canonical Wiener space

Theorem (Cameron-Martin-Maruyama)

The Wiener measure μ *is qi under translation by elts of* $\mathcal{H}$*.*
That is, for $h \in \mathcal{H}$ *and* $d\mu^h := d\mu(\cdot - h)$*,*

$$\mu^h \ll \mu \quad \text{and} \quad \mu^h \gg \mu.$$

More particularly,

$$d\mu^h(x) = J^h(x)\, d\mu(x) := e^{-|h|_H^2/2 + \text{“}\langle x,h\rangle\text{”}}\, d\mu(x).$$

Moreover, if $h \notin \mathcal{H}$, then $\mu^h \perp \mu$.

Theorem (Integration by parts)

For all $h \in \mathcal{H}$*,*

$$\int_{\mathcal{W}} (\partial_h f)(x)\, d\mu(x) = \int_{\mathcal{W}} f(x)\, \text{“}\langle x, h\rangle\text{”}\, d\mu(x).$$

Gross' abstract Wiener space

An *abstract Wiener space* is a triple (W, H, μ) where

- W is a Banach space
- μ is a Gaussian measure on W
- H is a Hilbert space densely embedded in W and (when $\dim(H) = \infty$) $\mu(H) = 0$

The Cameron-Martin-Maruyama QI Theorem and IBP hold on any abstract Wiener space.

other QI and IBP references:
Shigekawa (1984), Driver (1992), Hsu (1995,2002), Enchev-Stroock (1995), Albeverio-Daletskii-Kondratiev (1997), Kondratiev-Silva-Streit (1998), Albeverio-Kondratiev-Röckner-Tsikalenko (2000), Kuna-Silva (2004), Airault-Malliavin (2006), Driver-Gordina (2008), Hsu-Ouyang (2010),...

One approach to QI

Driver–Gordina (2008), Gordina (2017)

Let M be an inf dim manifold with measure μ and $T : M \to M$.

- Suppose M_n are submanifolds approximating M such that $T : M_n \to M_n$, and μ_n are measures on M_n approximating μ.
- Suppose that for each n, μ_n is qi under T; that is, $\exists$ $J_T^n : M_n \to (0, \infty)$ so that for any $f \in C_b(M)$

$$\int_{M_n} |f(x)| \, d(\mu_n \circ T^{-1})(x) = \int_{M_n} |f(x)| J_T^n(x) \, d\mu_n(x) \leq \|f\|_{L^p(M_n,\mu_n)} \|J_T^n\|_{L^q(M_n,\mu_n)}.$$

- Finally, suppose that for all n

$$\|J_T^n\|_{L^q(M_n,\mu_n)} \leq C_T < \infty \tag{IH}$$

One approach to QI

Driver–Gordina (2008), Gordina (2017)

Then taking the limit in the first inequality gives

$$\int_M |f(x)|\, d(\mu \circ T^{-1})(x) \leq C_T \|f\|_{L^p(M,\mu)},$$

which implies that the linear functional

$$\varphi_T(f) := \int_M f(x) d(\mu \circ T^{-1})(x)$$

is bounded on $L^p(M,\mu)$. Thus there exists $J_T \in L^q(M,\mu)$ such that

$$\varphi_T(f) = \int_M f(x) J_T(x)\, d\mu(x)$$

and $\|J_T\|_{L^q(M,\mu)} \leq C_T$.

Integrated Harnack inequalities

In the case of diffusions where $\mu = \mu_t = \mathrm{Law}(X_t)$, one often has fin dim approx X_t^n with $\mu_t^n = \mathrm{Law}(X_t^n)$ where

$$d\mu_t^n(x) = p_t^n(x)\,dx.$$

Thus, when $T = T_h =$ "translation" by h

$$J_{T_h}^n(x) = \frac{p_t^n(h,x)}{p_t^n(x)},$$

and these estimates look like

$$\int_{M_n} \left(\frac{p_t^n(h,x)}{p_t^n(x)}\right)^p p_t^n(x)\,dx \le C^p.$$

Integrated Harnack inequalities

- via lower bounds on Ricci curvature (Wang 2004, Driver–Gordina 2008) — not available in the hypoelliptic setting
- via modified Bakry-Émery + "transverse symmetry" (Baudoin–Bonnefont–Garofalo 2010, Baudoin–Garofalo 2011) $\implies$ reverse log Sobolev $\implies$ Wang-type Harnack $\iff$ (IH)

Other inf dim hypoelliptic results

- via modified Bakry-Émery: inf dim hypoelliptic Heisenberg groups (Baudoin–Gordina–M 2013)
- via other techniques:
  - stronger smoothness results for inf dim Heisenberg groups in elliptic (Dobbs–M 2013) and hypoelliptic (Driver–Eldredge–M 2016) settings
  - qi and ibp for path space measure of hypoelliptic BM on foliated compact manifolds (Baudoin–Gordina–Feng 2019)
  - qi and ibp for measures on path space of subRiemannian manifolds (Cheng–Grong–Thalmaier, 2021)
- nothing previously for diffusions under "weak" Hörmander condition

Generalized Kolmogorov diffusion

Let (W, H, μ) be an abstract Wiener space, and let $\{B_t\}_{t \geq 0}$ denote Brownian motion on W. Let V be a vector space. Fix a cts $F : W \to V$ and define the diffusion on $W \times V$

$$Y_t := \left(B_t, \int_0^t F(B_s)\, ds \right).$$

We're interested in when the law $\nu_t^{(h,k)}$ of

$$Y_t^{(h,k)} := \left(B_t + h, \int_0^t F(B_s + h)\, ds + k \right)$$

is mutually abs cts wrt $\nu_t := \nu_t^0 := \mathrm{Law}(Y_t)$.

Note that in the case $V = W$ and $F = I$, this is a natural notion of an inf dim Kolmogorov diffusion.

The fin dim approximations

For simplicity, consider first $V = \mathbb{R}$, in which case $\{Y_t\}$ has generator

$$\mathcal{L} = \Delta_p + F(p)\frac{\partial}{\partial \xi}.$$

We can approximate Y_t by

$$Y_t^d := \left(B_t^d, \int_0^t (F \circ i_d)(B_s^d)\, ds \right)$$

where $\{B_t^d\}_{t \geq 0}$ is BM on $\mathbb{R}^d$, with analogous generator L^d.

The fin dim estimates

For now, just write $L = L^d$.

For each $\alpha, \beta \geq 0$, define

$$\Gamma^{\alpha,\beta}(f,g) := \sum_{i=1}^{d} \left(\frac{\partial f}{\partial p_i} - \alpha \frac{\partial f}{\partial \xi} \right) \left(\frac{\partial g}{\partial p_i} - \alpha \frac{\partial g}{\partial \xi} \right) + \beta \left(\frac{\partial f}{\partial \xi} \right) \left(\frac{\partial g}{\partial \xi} \right).$$

and

$$\Gamma_2^{\alpha,\beta}(f) := \frac{1}{2} L \Gamma^{\alpha,\beta}(f) - \Gamma^{\alpha,\beta}(f, Lf).$$

The fin dim estimates

Assumption A There exist $m, M > 0$ such that for every $i = 1, \ldots, d$ and $p \in \mathbb{R}^d$

$$m \leq \frac{\partial F}{\partial p_i}(p) \leq M.$$

Proposition (Bakry-Émery type)

Suppose that F satisfies Assumption A. Then for every $\alpha, \beta \geq 0$ and $f \in C^\infty\left(\mathbb{R}^d \times \mathbb{R}\right)$,

$$\Gamma_2^{\alpha,\beta}(f) \geq -\frac{M-m}{4\alpha}\Gamma(f) + m\sum_{i=1}^{d}\left(\alpha\left(\frac{\partial f}{\partial \xi}\right)^2 - \frac{\partial f}{\partial \xi}\frac{\partial f}{\partial p_i}\right).$$

The fin dim estimates

Let $p_t(\cdot,\cdot)$ denote the RN derivative of $\mu_t = \mu_t^d$ wrt Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}$.

Proposition (Integrated Harnack inequality)

For any $t > 0$*,* $(p,\xi) \in \mathbb{R}^d \times \mathbb{R}$*, and* $q \in (1,\infty)$*,*

$$\left(\int_{\mathbb{R}^d \times \mathbb{R}} \left[\frac{p_t((p,\xi),(p',\xi'))}{p_t(p',\xi')} \right]^q p_t(p',\xi')\, dp'\, d\xi' \right)^{1/q} \leq A_q(p,\xi)$$

where

$$A_q(p,\xi) := \exp\left(\frac{3(1+q)M}{m^3 t^3} \left(\frac{mt}{2} \sum_{i=1}^{d} p_i + \xi \right)^2 \right) \exp\left(\frac{(1+q)M}{4mt} \|p\|^2 \right).$$

A qi result for generalized Kolmogorov diffusions

Assumption A′ Suppose $F : W \to \mathbb{R}$ is in the domain of ∇, and assume that there exist a "good" onb $\{e_j\}_{j=1}^{\infty}$ of H and $m, M > 0$ so that for all $w \in W$

$$m \leq \langle \nabla F(w), e_j \rangle \leq M.$$

A qi result for generalized Kolmogorov diffusions

Theorem

Suppose F satisfies Assumption A'. Fix $h \in H$, $k \in \mathbb{R}$. If $\forall q \in (1, \infty)$

$$A_q(h,k) := \exp\left(\frac{3(1+q)M}{m^3t^3}\left(\frac{mt}{2}\sum_{i=1}^{\infty}\langle h, e_i\rangle + k\right)^2\right) \times \exp\left(\frac{(1+q)M}{4mt}\|h\|^2\right) < \infty$$

then $\nu_t^{(h,k)}$ is mutually abs cts wrt $\nu_t := \nu_t^0$

$$\left\|\frac{d\nu_t^{(h,k)}}{d\nu_t}\right\|_{L^q(W\times\mathbb{R},\nu_t)} \leq A_q(h,k).$$

A better starting assumption

Assumption B For $F = (F_1, \dots, F_r) : \mathbb{R}^d \to \mathbb{R}^r$, there exist non-empty disjoint $I_1, \dots, I_r \subset \{1, \cdots, d\}$ and $m_1, M_1, \dots, m_r, M_r > 0$ such that for each $j = 1, \dots, r$

$$m_j \leqslant \frac{\partial F_j}{\partial p_i}(p) \leqslant M_j, \qquad \forall i \in I_j$$

and, for every $i \notin I_j$, $\frac{\partial F_j}{\partial p_i}(p) = 0$.

In this case, the generator may be written as

$$\begin{aligned} L &= \sum_{j=1}^{r} L^{I_j} + \sum_{i \notin \cup I_j} \frac{\partial^2}{\partial p_i^2} \\ &= \sum_{j=1}^{r} \left(\sum_{i \in I_j} \frac{\partial^2}{\partial p_i^2} + F_j(p) \frac{\partial}{\partial \xi_j} \right) + \sum_{i \notin \cup I_j} \frac{\partial^2}{\partial p_i^2}. \end{aligned}$$

The fin dim estimates for the better assumption

Proposition (Integrated Harnack inequality II)

Suppose $F : \mathbb{R}^d \to \mathbb{R}^r$ *satisfies Assumption B. Then for any* $t > 0$, $(p, \xi) \in \mathbb{R}^d \times \mathbb{R}^r$, *and* $q \in (1, \infty)$,

$$\left(\int_{\mathbb{R}^d \times \mathbb{R}^r} \left[\frac{p_t((p,\xi),(p',\xi'))}{p_t(p',\xi')} \right]^q p_t(p',\xi')\, dp'\, d\xi' \right)^{1/q} \leqslant \left(\prod_{j=1}^{r} A_q^j(p,\xi) \right) \exp\left(\frac{1+q}{4t} \|p\|_{I^c}^2 \right)$$

where $I^c := (\cup_{j=1}^r I_j)^c$ *and*

$$A_q^j(p,\xi) := A_q^j(p_{I_j}, \xi_j)$$

$$:= \exp\left(\frac{3(1+q)M_j}{m_j^3 t^3} \left(\frac{m_j t}{2} \sum_{i \in I_j} p_i + \xi_j \right)^2 \right) \exp\left(\frac{(1+q)M_j}{4 m_j t} \|p\|_{I_j}^2 \right).$$

A (better) qi result for generalized Kolmogorov

Assumption B′ Suppose $F = (F_1, \ldots, F_r) : W \to \mathbb{R}^r$ such that each F_j is H-differentiable, and assume that there exist a "good" onb $\{e_i\}_{i=1}^{\infty}$ of H, non-empty disjoint $I_1, \ldots, I_r \subset \mathbb{N}$, and $m_1, M_1, \ldots, m_r, M_r > 0$ such that for each $j = 1, \ldots, r$

$$m_j \leqslant \langle \nabla F_j(w), e_i \rangle \leqslant M_j, \qquad \text{for all } i \in I_j$$

and

$$\langle \nabla F_j(w), e_i \rangle = 0, \qquad \text{for all } i \notin I_j.$$

A (better) qi result for generalized Kolmogorov

Theorem (Baudoin–Gordina–M, 2021)

Suppose that Assumption B' holds for $F : W \to \mathbb{R}^r$. Fix $h \in H$ and $k \in \mathbb{R}^r$. If for each $j = 1, \ldots, r$,

$$\sum_{i \in I_j} |\langle h, e_i \rangle| < \infty,$$

then $\nu_t^{(h,k)}$ is mutually abs cts wrt $\nu_t := \nu_t^0$ and $\forall q \in (1, \infty)$

$$\left\| \frac{d\nu_t^{(h,k)}}{d\nu_t} \right\|_{L^q(W \times \mathbb{R}^r, \nu_t)} \leqslant \left(\prod_{j=1}^{r} A_q^j(h,k) \right) \exp\left(\frac{1+q}{4t} \|h\|_{I^c}^2 \right).$$

A (better) qi result for generalized Kolmogorov

Here $I^c := (\cup_{i=1}^r I_j)^c$ and

$$A_q^j(h,k) := \exp\left(\frac{3(1+q)M_j}{m_j^3 t^3}\left(\frac{m_j t}{2}\sum_{i\in I_j}\langle h, e_i\rangle + k_j\right)^2\right)\exp\left(\frac{(1+q)M_j}{4m_j t}\|h\|_{I_j}^2\right)$$

with $\{e_i\}_{i=1}^\infty$ is the onb, $I_j \subset \mathbb{N}$, and m_j and M_j are the bounds introduced in Assumption B′.

So, for example, we have qi when $F = (F_1, \ldots, F_r)$ is component-wise cylinder-functions with $F_i(B) \perp F_j(B)$ for $i \neq j$, satisfying the requisite derivative bounds.

For $F : W \to W$

Proposition

Suppose that $F : W \to W$ is cts and there exists a "good" onb $\{h_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{d} \langle F(B_t^d), h_j \rangle h_j \to F(B_t)$$

a.s. in W. Let $\{Q_d\}_{d=1}^{\infty}$ denote the sequence of projections associated to $\{h_j\}_{j=1}$ and consider

$$\tilde{Y}_d(t) := \left(B_t^d, \int_0^t Q_d F(B_s^d)\, ds \right).$$

Then

$$\lim_{d \to \infty} \max_{0 \leqslant t \leqslant T} \| Y(t) - \tilde{Y}_d(t) \|_{W \times W} = 0 \text{ a.s.}$$

For $F : W \to W$

Assumption B$''$ Suppose $F : W \to W$ is cts and there exists a "good" onb $\{h_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{d} \langle F(B_t^d), h_j \rangle h_j \to F(B_t) \quad \text{a.s. in } W.$$

Additionally, assume that $F_j := \langle F, h_j \rangle$ is H-differentiable for all j and that there exists a "good" onb $\{e_i\}_{i=1}^{\infty}$ of H, non-empty disjoint $I_j \subset \mathbb{N}$ and $m_j, M_j > 0$ such that, for each j

$$m_j \leqslant \langle \nabla F_j(w), e_i \rangle \leqslant M_j, \qquad \text{for all } i \in I_j$$

and

$$\langle \nabla F_j(w), e_i \rangle = 0, \qquad \text{for all } i \notin I_j.$$

For $F : W \to W$

Theorem (Baudoin–Gordina–M, 2021)

Suppose that Assumption B'' holds for $F : W \to W$. Fix $h, k \in H$. For $q \in (1, \infty)$ and each $j \in \mathbb{N}$, let

$$A_q^j(h,k) := \exp\left(\frac{3(1+q)M_j}{m_j^3 t^3}\left(\frac{m_j t}{2}\sum_{i\in I_j}\langle h, e_i\rangle + \langle k, h_j\rangle\right)^2\right) \times \exp\left(\frac{(1+q)M_j}{4m_j t}\|h\|_{I_j}^2\right).$$

If $\prod_{j=1}^\infty A_q^j(h,k) < \infty$, then $\nu_t^{(h,k)} \ll \nu_t$ and $\nu_t \ll \nu_t^{(h,k)}$ and

$$\left\|\frac{d\nu_t^{(h,k)}}{d\nu_t}\right\|_{L^q(W\times W,\nu_t)} \leqslant \left(\prod_{j=1}^\infty A_q^j(h,k)\right)\exp\left(\frac{1+q}{4t}\|h\|_{I^c}^2\right).$$

The “standard” inf dim Kolmogorov diffusion

In the case that $F = I$, we are back in the setting of a “standard” inf-dim Kolmogorov diffusion

$$X_t = \left(B_t, \int_0^t B_s \, ds \right).$$

This is a Gaussian process and qi follows from the Cameron-Martin-Maruyama theorem.

Alternatively, we can see qi as an application of the Cameron-Martin-Maruyama theorem on path space

$$\mathcal{W}_t := \mathcal{W}_t(W) := \{w : [0, t] \to W : w \text{ is cts and } w(0) = 0\}.$$

The “standard” inf dim Kolmogorov diffusion

Fix $h, k \in H$. CMM on $\mathcal{W}_t \implies$ for any $\gamma \in \mathcal{H}_t$, the translation $B \mapsto B + \gamma$ gives

$$\begin{aligned}\mathbb{E}[f(X_t^{(h,k)})] &= \mathbb{E}\left[f\left(B_t + h, \int_0^t (B_s + h)\,ds + k\right)\right] \\ &= \mathbb{E}\left[f\left(B_t + \gamma(t) + h, \int_0^t (B_s + \gamma(s) + h)\,ds + k\right) J_t^\gamma(B)\right],\end{aligned}$$

where

$$\begin{aligned}J_t^\gamma(w) &= \exp\left(\text{“}\langle \gamma, w\rangle_{\mathcal{H}_t}\text{”} + \|\gamma\|^2_{\mathcal{H}_t}\right) \\ &= \exp\left(\int_0^t \langle \dot\gamma(s), dw(s)\rangle - \frac{1}{2}\int_0^t \|\dot\gamma(s)\|_H^2\,ds\right).\end{aligned}$$

The “standard” inf dim Kolmogorov diffusion

So, for example, taking the path $\gamma(s) = sa + s^2 b$ with

$$a = -\frac{4}{t}h - \frac{6}{t^2}k \qquad \text{and} \qquad b = \frac{3}{t^2}h + \frac{6}{t^3}k,$$

we have

$$\mathbb{E}[f(X_t^{(h,k)})] = \mathbb{E}\left[f\left(B_t, \int_0^t B_s\,ds\right) J_t^\gamma(B)\right].$$

The “standard” inf dim Kolmogorov diffusion

We can compute exactly

$$\mathbb{E}\left[J_t^\gamma(B)^q\right] = \mathbb{E}\left[\exp\left(q\int_0^t \langle\dot\gamma(s), dB_s\rangle\right)\right]\exp\left(-\frac{q}{2}\int_0^t \|\dot\gamma(s)\|_H^2\, ds\right)$$
$$= \exp\left(\frac{q^2-q}{2}\|\gamma\|_{\mathcal{H}_t}^2\right).$$

and

$$\|\gamma\|_{\mathcal{H}_t}^2 = \frac{4}{t}\|h\|_H^2 + \frac{12}{t^2}\langle h, k\rangle_H + \frac{12}{t^3}\|k\|_H^2$$

and thus

$$\left\|\frac{d\nu_t^{h,k}}{d\nu_t}\right\|_{L^q(W\times W,\nu_t)} \leq \|J_t^\gamma(B)\|_{L^q(\mathcal{W}_t} = \mathbb{E}[(J_t^\gamma(B))^q]^{1/q}$$
$$= \exp\left(2(q-1)\left(\frac{\|h\|_H^2}{t} + \frac{3\langle h, k\rangle_H}{t^2} + \frac{3\|k\|_H^2}{t^3}\right)\right).$$

The "standard" inf dim Kolmogorov diffusion

To prove qi instead as an application of our main theorem, we can take $h_j = e_j$, and we have $I_j = \{j\}$ and $m_j = M_j = 1$ for all j, which gives the bound

$$\begin{aligned}
&\left\| \frac{d\nu_t^{(h,k)}}{d\nu_t} \right\|_{L^q(W\times W,\nu_t)} \\
&\quad \leqslant \exp\left(\frac{3(1+q)}{t^3} \sum_j \left(\frac{t}{2}\langle h, e_j\rangle + \langle k, e_j\rangle \right)^2 \right) \exp\left(\frac{1+q}{4t} \|h\|_H^2 \right) \\
&\quad = \exp\left((1+q) \left(\frac{\|h\|_H^2}{t} + \frac{3\langle h, k\rangle}{t^2} + \frac{3\|k\|_H^2}{t^3} \right) \right).
\end{aligned}$$