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ERGODICITY OF GROUP ACTIONS AND SPECTRAL GAP, APPLICATIONS TO RANDOM WALKS AND MARKOV SHIFTS

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Abstract. Let $(X, \mathcal{B}, \nu)$ be a probability space and let Γ be a countable group of ν-preserving invertible maps of X into itself. To a probability measure μ on Γ corresponds a random walk on X with Markov operator P given by $P\psi(x) = \sum_a \psi(ax)\,\mu(a)$. We consider various examples of ergodic Γ-actions and random walks and their extensions by a vector space: groups of automorphisms or affine transformations on compact nilmanifolds, random walks in random scenery on non amenable groups, translations on homogeneous spaces of simple Lie groups, random walks on motion groups. A powerful tool in this study is the spectral gap property for the operator P when it holds. We use it to obtain limit theorems, recurrence/transience property and ergodicity for random walks on non compact extensions of the corresponding dynamical systems.

Contents

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Introduction. Let $(X, \mathcal{B}, \nu)$ be a metric space endowed with its Borel σ-algebra $\mathcal{B}$ and a probability measure ν, and let Γ be a countable group of Borel invertible maps of X into itself which preserve ν.

Let μ be a probability measure on Γ such that the group generated by $A := \operatorname{supp}(\mu)$ is Γ. We consider the random walk on X defined by μ, with Markov operator P given by

$$P\psi(x) \;=\; \sum_{a \in A} \psi(ax)\, \mu(a), \;\; x \in X. \tag{1}$$

These data, i.e., the probability space (X, ν), the group Γ acting on (X, ν) and the probability measure μ on Γ, will be denoted by (X, ν, Γ, μ).

The operator P is a contraction of $L^p(X, \nu)$, $\forall p \geq 1$, and it preserves the subspace $L_0^2(X, \nu)$ of functions φ in $L^2(X, \nu)$ such that $\nu(\varphi) = 0$. P is said to be *ergodic* if the constant functions are the only P-invariant functions in $L^2(X, \nu)$.

Ergodicity of P is equivalent to ergodicity of the action of Γ on the measure space $(X, \mathcal{B}, \nu)$. Indeed, any Γ-invariant function is obviously P-invariant. Conversely, if φ in $L^2(\nu)$ is P-invariant, then, by strict convexity of $L^2(X, \nu)$, we have $\varphi(ax) = \varphi(x)$, ν-a.e. for every $a \in \operatorname{supp}(\mu)$. Therefore φ is Γ-invariant, hence ν-a.e. constant if Γ acts ergodically on (X, ν).

Our aim is to consider some examples of ergodic actions and extensions of these actions by a vector space. We will use a strong reinforcement of the ergodicity, the spectral gap property for the operator P when it holds and we will develop some of its consequences. Let us recall its definition and related notions.

Definition 0.1. We denote by Π_0 the restriction of P defined by (1) to $L_0^2(X, \nu)$ and by $r(\Pi_0) := \lim_n \|\Pi_0^n\|^{\frac{1}{n}}$ its spectral radius. If $r(\Pi_0) < 1$, we say that (X, ν, Γ, μ) satisfies the *spectral gap property* (we will use the shorthand "property (SG)").

We recall that a unitary representation ρ of a group Γ in a Hilbert space H is said to *contain weakly* the identity representation if there exists a sequence (x_n) in H with $\|x_n\| = 1$ such that, for every $\gamma \in \Gamma$, $\lim_n \|\rho(\gamma)x_n - x_n\| = 0$. See [3] for this notion.

Recall also that Γ is said to have property (T) if, when the identity representation is weakly contained in a unitary representation ρ of Γ, then it is contained in ρ.

The natural action of Γ on $L_0^2(X)$ defines a unitary representation ρ_0 of Γ in $L_0^2(X)$. Property (SG) implies that the identity representation of Γ is not weakly contained in ρ_0. The converse is true if $(\operatorname{supp}(\mu))^k$ generates Γ for every $k \geq 1$ (see below Corollary 3.12). Property (SG) depends only on the support of μ.

For a countable group Γ acting measurably on a probability measure space (X, ν) where ν is Γ-invariant, according to [11] the Γ-action on (X, ν) is said to be *strongly ergodic* if ν is the unique Γ-invariant continuous positive normalized functional on $L^\infty(X, \nu)$. Property (SG) implies strong ergodicity, hence ergodicity of the action of Γ on (X, ν).

Our framework will be essentially algebraic. As examples, we study the action of groups of automorphisms or affine transformations on tori and compact nilmanifolds, and translations on homogeneous spaces of simple Lie groups. In Section 1 we show for nilmanifolds that the ergodicity of P follows from the ergodicity of its restriction to the maximal torus factor. In Section 2, we recall property (SG) for subgroups of $\mathrm{SL}(d, \mathbb{Z})$ acting on $\mathbb{T}^d$, as well as recent results on property (SG) for nilmanifolds. In Section 3 we consider random walks on non compact extensions of dynamical systems and apply property (SG) to recurrence and ergodicity. The last section is devoted to examples.

1. **Ergodicity of groups of affine transformations on nilmanifolds.** In this section, we consider a group of affine transformations Γ on a compact nilmanifold X. In order to obtain ergodicity of Markov operators on X, as described in the introduction, we study the question of ergodicity of the action of Γ.

Let N be a connected, simply connected, nilpotent Lie group and D a lattice in N, i.e., a discrete subgroup D such that the quotient $X = N/D$ is compact. If L_1, L_2 are two subgroups of N, we denote by $[L_1, L_2]$ the closed subgroup generated by the elements $\{n_1\, n_2\, n_1^{-1}\, n_2^{-1}, n_1 \in L_1, n_2 \in L_2\}$, $L' := [L, L]$ the derived group of L, e the neutral element of N. The descending series of N is defined by

$$N \supset N^1 \supset ... \supset N^{k-1} \supset N^k \supset \{e\},$$

where $N^{\ell+1} := [N^\ell, N]$, for $\ell \geq 0$, with $N^0 = N$.

The elements $g \in N$ act on N/D by left translation: $nD \in N/D \to gnD$. We say that τ is an automorphism of the nilmanifold N/D if τ is an automorphism of the group N such that $\tau D = D$. The group of automorphisms of N/D is denoted by $\mathrm{Aut}(N/D)$. The action of $\tau \in \mathrm{Aut}(N/D)$ on N/D is $nD \to \tau(n)D$. An affine transformation γ of N/D is a map of the form:

$$nD \to \gamma(n)D = \alpha_\gamma\, \tau_\gamma(n)D, \tag{2}$$

with $\alpha_\gamma \in N$ and $\tau_\gamma \in \mathrm{Aut}(N/D)$.

Let Γ be a group of affine transformations of the nilmanifold. The measure m on N/D deduced from a Haar measure on N is Γ invariant. The group Γ acts on the factor torus $T := N/N^1.D$. When Γ is a group of automorphisms, ergodicity of the action on the torus is equivalent to the fact that every non trivial character has an infinite Γ-orbit.

When Γ is generated by a single automorphism (or more generally by an affine transformation), W. Parry has proved ([28], [29]) that the ergodicity of the action on the maximal torus factor T implies the ergodicity of the action on the nilmanifold. We will show (Theorem 1.4) the analogous statement for a group of affine transformations.

Notations: For a given group Γ of affine transformations of N/D, $\tilde{\Gamma}$ denotes the subgroup of $\mathrm{Aut}(N/D)$ generated by $\{\tau_\gamma, \gamma \in \Gamma\}$, where τ_γ is the automorphism associated with γ as in (2). We denote by N_e^ℓ the Lie algebra of N^ℓ and by $d\tau_e$ the linear map tangent at e to an automorphism τ of N.

We will use the following lemmas.

Lemma 1.1. *(cf. CoGu74) If Γ is a subgroup of $\mathrm{GL}(\mathbb{R}^d)$ such that the eigenvalues of each element of Γ has modulus 1, then there is a Γ-invariant subspace $W \neq \{0\}$ of $\mathbb{R}^d$ such that the action of Γ on W is relatively compact. If Γ is a subgroup of $\mathrm{GL}(\mathbb{Z}^d)$, the action of Γ on W is that of a finite group of rotations and reduces to the identity for γ in a subgroup Γ_0 of finite index in Γ.*

Proof. We extend the action of Γ to $\mathbb{C}^d$. Let $\tilde{W}$ be a subspace of $\mathbb{C}^d$ which is different from $\{0\}$ and invariant by Γ on which the action of Γ is irreducible. Let (e_i) be a basis of $\tilde{W}$, and let E_{ij} be the maps defined by $E_{ij}(e_k) = \delta_{kj} e_i, \forall k$. We denote by $\tilde{\tau}$ the endomorphism corresponding to the action of $\tau \in \Gamma$ on $\tilde{W}$.

There is a constant C such that the trace of each automorphism $\tilde{\tau}$, for $\tau \in \Gamma$, satisfies: $\mathrm{trace}(\tilde{\tau}) \leq C \dim(\tilde{W})$.

The action of Γ on $\tilde{W}$ being irreducible, by Burnside's theorem, for each E_{ji} there are constants b_k and elements $\tau_k \in \Gamma$ such that $E_{ji} = \sum_k b_k \tilde{\tau}_k$. The coefficients of the transformations $\tilde{\tau}$ satisfy then:

$$|a_{ij}(\tilde{\tau})| = |\mathrm{trace}(\tilde{\tau} E_{ji})| \leq \sum_k |b_k||\mathrm{trace}(\tilde{\tau}\tilde{\tau}_k)| \leq C \dim(\tilde{W}) \sum_k |b_k|.$$

Therefore $\sup_{\tau \in \Gamma} |a_{ij}(\tilde{\tau})| < \infty$, which implies the relative compactness of the action of Γ on $\tilde{W}$, as well on the Γ-invariant subspace W of $\mathbb{R}^d$ generated by $\{\Re\mathrm{e}\, v, v \in \tilde{W}\}$.

Now assume that Γ is a subgroup of $\mathrm{GL}(\mathbb{Z}^d)$. The symmetric functions of the eigenvalues of $\gamma \in \Gamma$ take values in $\mathbb{Z}$ and remain bounded when γ runs through Γ. This implies that the set of the characteristic polynomials of the elements γ is finite. If λ is an eigenvalue of γ, the set $(\lambda^n)_{n\in\mathbb{Z}}$ is finite and therefore λ is a root of the unity. The order of these roots remains bounded on Γ. This implies the last assertion. □

Lemma 1.2. *If a group Γ of affine transformations on $\mathbb{T}^d$ has an invariant square integrable non a.e. constant function f, then it has an invariant function which is a non identically constant trigonometric polynomial. If the action of Γ is ergodic, every eigenfunction is a trigonometric polynomial.*

Proof. Let $f \in L^2(\mathbb{T}^d)$ be a Γ-eigenfunction, $f \circ \gamma = \beta(\gamma) f, \forall \gamma \in \Gamma$. By invariance of the measure we have $|\beta(\gamma)| = 1$ and for every $\gamma \in \Gamma$

$$f = \sum_{p\in\mathbb{Z}^d} \hat{f}(p) e^{2\pi i <p,.>} = \overline{\beta(\gamma)} \sum_{p\in\mathbb{Z}^d} \hat{f}(p) e^{2\pi i <p,\alpha_\gamma>} e^{2\pi i <{}^t\tau_\gamma p,.>}; \qquad (3)$$

hence: $|\hat{f}(p)| = |\hat{f}({}^t\tau_\gamma p)|, \forall p \in \mathbb{Z}^d$.

Let $R := \{p \in \mathbb{Z}^d : |\hat{f}(p)| \neq 0\}$. For two automorphisms τ, τ' of the torus and $p \in \mathbb{Z}^d$ such that ${}^t\tau p \neq {}^t\tau' p$, the characters $e^{2\pi i <{}^t\tau p,.>}$ and $e^{2\pi i <{}^t\tau' p,.>}$ are orthogonal. Therefore the orbit $\{{}^t\tau_\gamma p,\ \gamma \in \Gamma\}$ of every element p of R is finite. The set R decomposes into finite disjoint subsets R_k, with each R_k permuted by the automorphisms $\tau_\gamma \in \tilde{\Gamma}$.

The subspaces W_k of L^2 generated by $e^{2\pi i <p,.>}$, for $p \in R_k$, have finite dimension, are pairwise orthogonal and are invariant by each $\gamma \in \Gamma$. The orthogonal projections of f on these subspaces give Γ-eigenfunctions with the same eigenvalue as for f. This shows the existence of a non constant eigenfunction (invariant if f is invariant) which is a trigonometric polynomial. If the group Γ acts ergodically, only one of these projections is non null. Hence f is a trigonometric polynomial. □

Lemma 1.3. *If a group of affine transformations Γ of a torus $\mathbb{T}^d$ is ergodic, then every subgroup Γ_0 of Γ with finite index is also ergodic on $\mathbb{T}^d$.*

Proof. Let Γ_0 be a subgroup of Γ with finite index. As the action of Γ is ergodic, the σ-algebra of the Γ_0-invariant subsets is an atomic finite σ-algebra whose elements are permuted by $\gamma \in \Gamma$. From Lemma 1.2, if Γ_0 is not ergodic, there exists a non constant trigonometric polynomial which is invariant by Γ_0. This polynomial should be measurable with respect to the σ-algebra of the Γ_0-invariant subsets which is atomic. The connectedness of the torus implies that it is constant; hence a contradiction. □

Ergodicity of a group of affine transformations

Theorem 1.4. *Let Γ be a group of affine transformations on N/D. If its action on the torus factor $N/N^1.D$ is ergodic, then every eigenfunction for the action of Γ on N/D factorizes into an eigenfunction on $N/N^1.D$. In particular, the action of Γ is ergodic on N/D if and only if its action on the maximal torus factor $N/N^1.D$ is ergodic.*

Proof. We follow essentially the method of W. Parry ([28]). We make an induction on the length k of the descending central series of N. The property stated in the theorem is clearly satisfied if $k = 0$.

The induction assumption is that, for every group of affine transformations of N/D, ergodicity of the action on $N/N^1.D$ implies ergodicity of the action on $N/N^k.D$ and every eigenfunction for the action on $N/N^k.D$ factorizes through $N/N^1.D$. (The quotient $(N/N^k)/(N/N^k)'$ can be identified with the quotient N/N^1.)

Remark that, for every subgroup Γ_0 with finite index in a group Γ of affine transformations on N/D, the action of Γ_0 on $N/N^1.D$ is ergodic if the action of Γ on $N/N^1.D$ is also ergodic (Lemma 1.3). The induction assumption implies then that Γ_0 acts ergodically on $N/N^k.D$ and that every eigenfunction for the action of Γ_0 on $N/N^k.D$ factorizes through $N/N^1.D$.

Let $f \in L^2(N/D)$ be a Γ-eigenfunction, i.e., such that for complex numbers $\beta(\gamma)$ of modulus 1,

$$f(\gamma(n)D) = f(\alpha_\gamma \, \tau_\gamma(n)D) = \beta(\gamma) f(nD), \forall \gamma \in \Gamma. \tag{4}$$

We are going to show that f factorizes into an eigenfunction on the quotient $N/N^k.D$ and therefore, by the induction hypothesis, into an eigenfunction on $N/N^1.D$.

The proof is given in several steps.

a) We denote by Z the center of N. We have $N^k \subset Z \cap N^{k-1}$. The torus $H := Z \cap N^{k-1} / Z \cap N^{k-1} \cap D$ acts by left translation on N/D and its action commutes with the translation by elements of N. Let Θ be the group of characters of H. The space $L^2(N/D)$ decomposes into pairwise orthogonal subspaces V_η, where η belongs to Θ and V_η stands for the subspace of functions transformed according to the character η under the action of H:

$$V_\eta = \{\varphi \in L^2(N/D) : \varphi(hnD) = \eta(h) \; \varphi(nD), \forall h \in H\}.$$

If τ is an automorphism of N/D, $h \in H \to \eta(\tau(h))$ defines a character on H denoted by $\tau\eta$. We have $\varphi \in V_\eta \Leftrightarrow \varphi \circ \gamma \in V_{\tau_\gamma \eta}$, for $\gamma \in \Gamma$.

By (4) we have the orthogonal decomposition of f into components in V_η:

$$f = \sum_{\eta \in \Theta} f_\eta = \overline{\beta(\gamma)} \sum_{\eta \in \Theta} \; f_\eta \circ \gamma, \; \forall \gamma \in \Gamma, \tag{5}$$

with $f_\eta \in V_\eta$, $f_\eta \circ \gamma \in V_{\tau_\gamma \eta}$. We will show that the components f_η, hence also f, are invariant by translation by the elements of N^k.

Let us fix $\theta \in \Theta$ such that $\|f_\theta\|_2 \neq 0$. Let $\Gamma_0 := \{\gamma \in \Gamma : \tau_\gamma \theta = \theta\}$. For two automorphisms τ, τ' of N/D, if $\tau\theta \neq \tau'\theta$, the subspaces $V_{\tau\theta}$ and $V_{\tau'\theta}$ are orthogonal. Equality of the norms $\|f_\theta \circ \gamma\|_2 = \|f_\theta\|_2, \forall \gamma \in \Gamma$, and Equation (5) imply that there are only a finite number of distinct images in the orbit $\{\tau_\gamma \theta, \gamma \in \Gamma\}$. Hence the group Γ_0 has a finite index in Γ. As remarked above, its action on $N/N^k.D$, like the action of Γ, is ergodic.

¿From now on we consider Γ_0 and the component f_θ (denoted f for simplicity).

¿From what precedes, we have an ergodic action of a group of affine transformations Γ_0, a character $\theta \in \Theta$ such that $\tau_\gamma \theta = \theta, \forall \gamma \in \Gamma_0$, and a function $x \to f(x)$ on N/D satisfying

$$f(\alpha_\gamma \tau_\gamma(x)) = \beta(\gamma) f(x),\ \ f(h.x) = \theta(h) f(x), \forall h \in Z \cap N^{k-1}.$$

We can assume that θ is non trivial on N^k. By replacing N/D by $N/H_0.D$, where H_0 is the connected component of the neutral element of $Ker\,\theta$, we can also assume that $N^k/N^k \cap D$ has dimension 1.

The previous equations imply that $|f|$ is γ-invariant for every $\gamma \in \Gamma_0$ and N^k-invariant. Therefore the function $|f|$ is a.e. equal to a constant that we can assume to be 1.

b) For $g \in N$, let $\theta(g) := \int f(gx)\overline{f(x)}\, dx$. Remark that the restriction of θ to $Z \cap N^{k-1}$ coincides with the character θ previously defined on $Z \cap N^{k-1}$. The function θ is continuous and $\theta(e) = 1$; therefore $\theta(g) \neq 0$ on a neighborhood of e. The invariance of the measure implies:

$$\theta(g^{-1}) = \overline{\theta(g)}. \tag{6}$$

Denote by G the subgroup of N^{k-1} defined by

$$G := \{g \in N^{k-1} : f(gx) = \theta(g) f(x)\} = \{g \in N^{k-1} : |\theta(g)| = 1\}, \tag{7}$$

(the equality in (7) follows from the equality in Cauchy-Schwarz inequality), and by G_0 the connected component of the neutral element in G.

For g in N^{k-1} and h in N, we have: $\theta(hgh^{-1}) = \theta(hgh^{-1}g^{-1})\theta(g)$; hence

$$|\theta(hgh^{-1})| = |\theta(g)|,\ \forall g \in N^{k-1}, h \in N. \tag{8}$$

For g in G and h in N, the relation $f(gx) = \theta(g) f(x)$ implies $\theta(gh) = \theta(g)\,\theta(h)$. Therefore we have by applying (6) to gh :

$$g \in G, h \in N \Rightarrow \theta(gh) = \theta(hg) = \theta(g)\,\theta(h). \tag{9}$$

Equation (8) implies: $|\theta(hgh^{-1})| = 1,\ \forall g \in G, h \in N$. The group G (and therefore G_0) is a closed normal subgroup of N.

¿From the equation of eigenfunction (4), we have:

$$\begin{aligned} f(\alpha_\gamma \tau_\gamma(g)\, \tau_\gamma(x))\overline{f(\alpha_\gamma \tau_\gamma(x))} &= f(\gamma(gx))\overline{f(\gamma(x))} \\ &= \beta(\gamma)\overline{\beta(\gamma)} f(gx)\overline{f(x)} = f(gx)\overline{f(x)} \end{aligned} \tag{10}$$

and therefore, by invariance of the measure:

$$\int f(\alpha_\gamma \tau_\gamma(g)\alpha_\gamma^{-1} x)\overline{f(x)}\ dx = \int f(\alpha_\gamma \tau_\gamma(g)\tau_\gamma(x))\overline{f(\alpha_\gamma \tau_\gamma(x))}\ dx = \int f(gx)\overline{f(x)}\ dx,$$

which implies: $\theta(\alpha_\gamma \tau_\gamma(g)\alpha_\gamma^{-1}) = \theta(g)$; hence, from (8) :

$$|\theta(\tau_\gamma g)| = |\theta(g)|, \forall \gamma \in \Gamma_0. \tag{11}$$

We define two subsets $\exp W_1$ and $\exp W_2$ containing respectively the stable and unstable subgroups of the automorphisms τ in $\tilde{\Gamma}_0$ acting on N^{k-1}/N^k by setting

$$\begin{aligned} W_1 &= \{v \in N_e^{k-1} : \exists \tau \in \tilde{\Gamma}_0 : \lim_{n \to +\infty} d\tau_e^n v \bmod N_e^k = 0\}, \\ W_2 &= \{v \in N_e^{k-1} : \exists \tau \in \tilde{\Gamma}_0 : \lim_{n \to -\infty} d\tau_e^n v \bmod N_e^k = 0\}. \end{aligned}$$

Let us show that $\exp W_i \subset G_0$, $i = 1, 2$. Let $g \in \exp W_1$. It belongs to a one parameter subgroup (g_t) such that, for every t, there exists a sequence (g_n) of elements of N^k such that $\lim_n \tau^n(g_t)\, g_n = e$. Using (11), (7) and (9), this implies:

$$|\theta(g_t)| = |\theta(\tau^n(g_t)| = |\theta(\tau^n(g_t)|\,|\theta(g_n)| = |\theta(\tau^n(g_t)\, g_n)| \to |\theta(e)| = 1.$$

Therefore g_t is in G, for every t, hence $g \in G_0$. The analysis is the same for W_2.

c) Now we prove: $[N, G_0] = N^k$.

As we are reduced to the case where N^k is of dimension 1, the other possibility is that $[N, G_0] = \{e\}$. Let us assume that $[N, G_0] = \{e\}$. The subgroup G_0 is then in the center of N and contained in $Z \cap N^{k-1}$.

Consider the quotient $N^{k-1}/Z \cap N^{k-1}$. As G_0 contains $\exp W_1$ and $\exp W_2$, the differentials of the automorphisms $\tau \in \tilde{\Gamma}_0$ have only eigenvalues of modulus 1 for their action on $N_e^{k-1}/Z_e \cap N_e^{k-1}$.

By Lemma 1.1, there exists then in $N_e^{k-1}/Z_e \cap N_e^{k-1}$ a subspace W_3 on which the action of the transformations which are linear tangent to the automorphisms $\in \tilde{\Gamma}_0$ is compact. As the automorphisms τ preserve a lattice, the subgroup $\tilde{\Gamma}_1$ which leaves fixed the elements of W_3 has a finite index in $\tilde{\Gamma}_0$. Let Γ_1 denote the subgroup of elements $\gamma \in \Gamma_0$ such that $\tau_\gamma \in \tilde{\Gamma}_1$.

Let g be an element of N^{k-1} in $\exp W_3$. For every $\tau_\gamma \in \tilde{\Gamma}_1$, there exists $g_0 \in Z \cap N^{k-1}$ such that $\tau_\gamma g = g_0 g$. Equation (10) reads

$$\begin{aligned} f(\alpha_\gamma\, g_0 g\, \tau_\gamma(x))\overline{f(\alpha_\gamma \tau_\gamma(x))} &= f(g_0 \alpha_\gamma g \alpha_\gamma^{-1} g^{-1}\, g\, \alpha_\gamma\, \tau_\gamma(x))\overline{f(\alpha_\gamma \tau_\gamma(x))} \\ &= \theta(g_0)\, \theta(\alpha_\gamma g \alpha_\gamma^{-1} g^{-1}) f(g\, \gamma(x))\overline{f(\gamma(x))} = f(gx)\overline{f(x)}. \end{aligned}$$

Therefore $f(g.)\ \overline{f(.)}$ is an eigenfunction for every $\gamma \in \Gamma_1$. Moreover it is invariant by $h \in N^k$. It factorizes into an eigenfunction for Γ_1 on $N/N^k.D$. Either its integral is 0 (if for $\gamma \in \Gamma_1$ the corresponding eigenvalue is $\neq 1$) or it is invariant by Γ_1. In the latter case, as ergodicity holds for the action of Γ_1 on $N/N^k.D$ by the induction hypothesis, $f(g.)\ \overline{f(.)}$ is equal to a constant with modulus 1 (since $|f| = 1$).

We have therefore $|\theta(g)| = 0$ or 1. By a continuity argument $|\theta(g)| = 1$, which implies that $g \in G$. Using as above a one parameter subgroup, we obtain that $g \in G_0$. This gives a contradiction, since $G_0 \subset Z \cap N^{k-1}$.

d) Let (h_t) be a one parameter subgroup of N. For g in G_0 we have by (9):

$$\begin{aligned} \theta(h_t)\, \theta(g) &= \theta(h_t g) = \theta(h_t g h_t^{-1} g^{-1} g h_t) \\ &= \theta(h_t g h_t^{-1} g^{-1}) \theta(g h_t) = \theta(h_t g h_t^{-1} g^{-1}) \theta(g)\, \theta(h_t). \end{aligned}$$

By continuity, $\theta(h_t)$ is different from zero in a neighborhood of $t = 0$. The previous relation implies $\theta(h_t g h_t^{-1} g^{-1}) = 1$ in a neighborhood of $t = 0$ and, G_0 being connected, is equal to 1 everywhere. As $[N, G_0] = N^k$, this shows that the character θ is identically equal to 1 on N^k and therefore the announced factorization property is satisfied. $\square$

Remark. There are compact nilmanifolds N/Γ for which the group $\mathrm{Aut}(N/\Gamma)$ is non ergodic (cf. [10]). This contrasts with the case of Heisenberg nilmanifolds, for which there is a large group of automorphisms.

An example. Now we give an example of nilmanifold with an ergodic group Γ of automorphisms such that each automorphism in Γ is non ergodic.

Construction on the torus

Examples of groups of matrices such that each of them has an eigenvalue equal to 1 can be constructed by action on the space of quadratic forms. We explicit the example in dimension 2. For similar examples see [30].

Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$, with eigenvalues λ_1, λ_2. The action of A on the vector space of symmetric 2×2 matrices given by $M \to AMA^t$ is represented by the matrix

$$q(A) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}, \tag{12}$$

whose eigenvalues are $\det A, \lambda_1^2, \lambda_2^2$. The vector $(2b, d-a, -2c)^t$ is an eigenvector for $q(A)$ with eigenvalue $\det A$ and is invariant by $q(A)$ if $\det A = 1$. When A has integer coefficients, $q(A)$ is an integral matrix and the eigenvector $(2b, d-a, -2c)^t$ belongs to $\mathbb{Z}^3$.

The restriction to $\mathrm{SL}(2,\mathbb{Z})$ of the map $A \to q(A)$ defines an isomorphism onto a discrete subgroup Λ_0 of automorphisms of $\mathrm{SL}(3,\mathbb{Z})$ whose each element is non ergodic (each element $q(A)$ leaves fixed a non trivial character of the torus $\mathbb{T}^3$), but which acts ergodically on $\mathbb{T}^3$, since the orbits of the transposed action on $\mathbb{Z}^3 \setminus \{0\}$ are infinite.

Extension to a nilmanifold

Now we extend the action of Λ_0 to a nilmanifold. Let us consider the real Heisenberg group H_{2d+1} of dimension $2d+1$, $d \geq 1$, identified with the group of matrices $(d+2)\times(d+2)$ of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & I_d & y \\ 0 & 0 & 1 \end{pmatrix},$$

where x and y are respectively line and column vectors of dimension d, z a scalar, I_d the identity matrix of dimension d. The composition in H_{2d+1} can be defined by:

$$(x,y,z).(x',y',z') = (x+x', y+y', z+z' + \langle x, y'\rangle - \langle x', y\rangle).$$

The map $(x,y,z) \to (Dx, {}^tD^{-1}y, z)$, for $D \in \mathrm{SL}(d,\mathbb{R})$, defines a group of automorphisms of H_{2d+1}. If the matrices are in $\mathrm{SL}(d,\mathbb{Z})$, these automorphisms preserve the subgroup D_{2d+1} of elements of H_{2d+1} with integral coefficients.

The group $\{q(A), A \in \mathrm{SL}(2,\mathbb{Z})\}$ defined by (12) extends to a group $\Gamma = \{\tau_A, A \in \mathrm{SL}(2,\mathbb{Z})\}$ of automorphisms of the nilmanifold $N/D = H_7/D_7$, where $\tau_A(x,y,z) = (q(A)x,\, {}^tq(A)^{-1}y,\, z)$. Since the orbits of the action of Γ on $\mathbb{Z}^3 \times \mathbb{Z}^3 \setminus \{0\}$ are infinite, the group Γ acts ergodically on the torus factor $N/N'D = \mathbb{T}^3 \times \mathbb{T}^3$, hence on N/D by Theorem 1.4. But each automorphism $(x,y) \to (q(A)x,\, {}^tq(A)^{-1}y)$ is non ergodic on $\mathbb{T}^3 \times \mathbb{T}^3$.

2. **Spectral gap property.** We will now describe some classes of examples where property (SG) is satisfied.

Tori

As a basic example where property (SG) is valid, let us consider as in [11] (See also [15]) the d-dimensional torus $X = \mathbb{T}^d$ endowed with the Lebesgue measure, and the action of $\mathrm{SL}(d,\mathbb{Z})$ on $\mathbb{T}^d$ by automorphisms. The Lebesgue measure is preserved.

Every $\gamma \in \mathrm{SL}(d,\mathbb{Z})$ acts by duality on $\mathbb{Z}^d$ by γ^t. We denote by μ^t the push-forward of a probability measure μ on $\mathrm{SL}(d,\mathbb{Z})$ by the map $\gamma \to \gamma^t$.

Proposition 2.1. *Let μ be a probability measure on $\mathrm{SL}(d,\mathbb{Z})$ such that $\mathrm{supp}(\mu^t)$ has no invariant measure on the projective space $\mathbf{P}^{d-1}$. Let P be the Markov operator on $\mathbb{T}^d$ defined by*

$$P\varphi(x) = \sum_\gamma \varphi(\gamma x)\,\mu(\gamma).$$

Then the corresponding contraction Π_0 on $L_0^2(X)$ satisfies $r(\Pi_0) < 1$.

Proof. The Plancherel formula gives an isometry $\mathcal{I}$ between $L_0^2(\mathbb{T}^d)$ and $\ell^2(\mathbb{Z}^d\setminus\{0\})$. For $\gamma \in \mathrm{SL}(d,\mathbb{Z})$ we have $\mathcal{I}\circ\gamma = \gamma^t \circ \mathcal{I}$. Hence if L denotes the convolution operator on $\ell^2(\mathbb{Z}^d \setminus \{0\})$ defined by μ^t we have $r(\Pi_0) = r(L)$.

Suppose $r(L) = r(\Pi_0) = 1$ and let $e^{i\theta}$ be a spectral value. Then two cases can occur. Either there exists a sequence $(f_n) \in \ell^2(\mathbb{Z}^d \setminus \{0\})$ with $\|f_n\|_2 = 1$ and $\lim_n \|Lf_n - e^{i\theta}f_n\|_2 = 0$, or, for some $f \in \ell^2(\mathbb{Z}^d\setminus\{0\})$ with $\|f\|_2 = 1$, $L^*f = e^{-i\theta}f$.

Since $e^{i\theta}L^*$ is a contraction on $\ell^2(\mathbb{Z}^d \setminus \{0\})$, its fixed points are also fixed points of its adjoint $e^{-i\theta}L$, hence $Lf = e^{i\theta}f$. It follows that it suffices to consider the first case. The condition $\lim_n \|Lf_n - e^{i\theta}f_n\|_2 = 0$ implies $\lim_n \|Lf_n\|_2 = \lim_n \|e^{i\theta}f_n\|_2 = 1$ and

$$\lim_n [\|Lf_n\|_2^2 + \|f_n\|_2^2 - 2\,\Re\mathrm{e}\langle Lf_n, e^{i\theta}f_n\rangle] = 0;$$

hence, using the inequality $|\langle Lf_n, e^{i\theta}f_n\rangle| \le \|Lf_n\|\,\|f_n\| \le 1$,

$$\lim_n \langle Lf_n, e^{i\theta}f_n\rangle = 1.$$

Since $\langle Lf_n, e^{i\theta}f_n\rangle = \sum_\gamma \mu(\gamma)\langle f_n\circ\gamma^t, e^{i\theta}f_n\rangle$ and $|\langle f_n\circ\gamma^t, e^{i\theta}f_n\rangle| \le \|f_n\circ\gamma^t\|_2\|f_n\|_2$, we get that, for every $\gamma \in \mathrm{supp}(\mu)$,

$$\lim_n \langle f_n\circ\gamma^t, e^{i\theta}f_n\rangle = 1.$$

Since $|\langle f_n\circ\gamma^t, e^{i\theta}f_n\rangle| \le \langle |f_n|\circ\gamma^t, |f_n|\rangle$, we have also $\lim_n\langle |f_n|\circ\gamma^t, |f_n|\rangle = 1$, hence

$$\lim_n \||f_n|\circ\gamma^t - |f_n|\|_2 = 0.$$

The inequality

$$\||f_n|^2\circ\gamma^t - |f_n|^2\|_1 \le \||f_n|\circ\gamma^t - |f_n|\|_2\ \||f_n|\circ\gamma^t + |f_n|\|_2$$

implies $\lim_n \||f_n|^2\circ\gamma^t - |f_n|^2\|_1 = 0$.

In other words, if ν_n denotes the probability measure on $\ell^2(\mathbb{Z}^d\setminus\{0\})$ with density $|f_n|^2$, we have in variational norm:

$$\lim_n \|(\gamma^t)^{-1}\nu_n - \nu_n\| = 0. \tag{13}$$

Let $\overline{\nu}_n$ be the projection of ν_n on $\mathbf{P}^{d-1}$ and $\overline{\nu}$ a weak limit of ν_n. By (13), we have $(\gamma^t)^{-1}\overline{\nu} = \overline{\nu}$, hence $\gamma^t\overline{\nu} = \overline{\nu}, \forall\gamma^t \in \mathrm{supp}(\mu^t)$, which contradicts the hypothesis. □

Remarks. 1) The hypothesis on the support of μ^t is satisfied if the group generated by $\mathrm{supp}(\mu)$ has no irreducible subgroup of finite index. This is a consequence of the following fact observed by H. Furstenberg: if a linear group has an invariant measure on $\mathbf{P}^{d-1}$, then either it is bounded or it has a reducible finite index subgroup (See [39], p. 39, for a proof).

2) The above corresponds to a special case in the characterization of property (SG) given in [5], Theorem 5, for affine maps of $\mathbb{T}^d$. In particular if $\hat{\mu}$ is a probability measure on the group $\mathrm{Aut}(\mathbb{T}^d) \ltimes \mathbb{T}^d$ and μ is its projection in $\mathrm{Aut}(\mathbb{T}^d)$, property

(SG) for $\hat{\mu}$ acting on $\mathbb{T}^d$ is valid if the group generated by $\text{supp}(\mu)$ is non virtually abelian and its action on $\mathbb{R}^d$ is $\mathbb{Q}$-irreducible.

Nilmanifolds

As in Section 1, let N be a simply connected nilpotent group, D a lattice in N, $X = N/D$ the corresponding nilmanifold and T the maximal torus factor. Let μ be a probability measure on $\text{Aut}(X) \ltimes N$, $\overline{\mu}$ its projection on $\text{Aut}(T) \ltimes T$. It is shown in [5], Thm. 1, that if the convolution operator on $L_0^2(T)$ associated with $\overline{\mu}$ satisfies property (SG), then the same is true for the operator on X associated with μ.

In view of the torus situation described above, this gives various examples of measures μ where property (SG) is valid. If N is a Heisenberg group, more precise results are available, which will be recalled in Section 4.

Simple Lie groups

Let us consider a non compact simple Lie group G and let Δ be a lattice in G, i.e. a discrete subgroup such that $X = G/\Delta$ has finite volume for the Haar measure v. Let μ be a probability measure on G. It follows from Theorem 6.10 in [11] that, if μ is not supported on a coset of a closed amenable subgroup of G, then property (SG) is valid for the action of μ on X.

Compact Lie groups

We take $X = SU(d)$, ν the Haar measure on X. It is known (See [12], [2]) that for $d \geq 3$, if $\Gamma \subset SU(d)$ is a countable dense subgroup with the coefficients of every element of Γ algebraic over $\mathbb{Q}$ and if μ generates Γ, then the natural representation of Γ in $L_0^2(X)$ does not contain weakly Id_Γ (cf. Definitions 0.1).

In particular there are dense free subgroups of $SU(d)$ as above. Also, if $X = SO(d)$ and $d \geq 5$, there are countable dense subgroups of $SO(d)$ which have property (T). For example, if Γ is the group of $d \times d$ matrices with coefficients in $\mathbb{Z}[\sqrt{2}]$ which preserve the quadratic form $q(x) = \sum_{i=1}^{d-2} x_i^2 + \sqrt{2}\,(x_{d-1}^2 + x_d^2)$, Γ has property (T) (see [27], p. 136, for similar examples).

Let A be a finite set of generators for Γ and μ a probability with $\text{supp}(\mu) = A$. Then property (SG) is valid for the convolution action of μ on X, since Γ is ergodic on (X, ν), a consequence of the density of Γ.

3. Applications of the spectral gap property.

3.1. **Extensions of group actions and random walks.** As in the introduction, let X be a metric space, Γ a countable group of invertible Borel maps of X into itself which preserve a probability measure ν on X, and μ a probability measure on Γ with finite support such that $A := \text{supp}(\mu)$ generates Γ as a group. We assume that the action of Γ on (X, ν) is ergodic. We will use both notations: $\sum_a f(ax)\,\mu(a)$ or $\int f(ax)\,d\mu(a)$.

Let us consider the product space $\Omega = A^{\mathbb{N}^*}$, with $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, the shift σ on Ω and the product measure $\mathbb{P} = \mu^{\otimes \mathbb{N}^*}$ on Ω. For $\omega \in \Omega$ we write $\omega = (a_1(\omega), a_2(\omega), ...)$. The extended shift σ_1 is defined on $Y = \Omega \times X$ by $\sigma_1(\omega, x) = (\sigma\omega, a_1(\omega)x)$. Clearly σ_1 preserves the measure $\mathbb{P}_1 = \mathbb{P} \otimes \nu$.

We consider also the bilateral shift on $\hat{\Omega} := A^{\mathbb{Z}}$ still denoted by σ. It preserves the product measure $\hat{\mathbb{P}} = \mu^{\otimes \mathbb{Z}}$.

Lemma 3.1. *The system $(Y, \mathbb{P}_1, \sigma_1)$ is ergodic.*

Proof. The dual operator of the composition by σ_1 on $L^2(\mathbb{P}_1)$ is

$$Rg(\omega, x) = \sum_{b \in A} g(b\omega, b^{-1}x)\, \mu(b).$$

On functions of the form $(f \otimes \varphi)(\omega, x) = f(\omega)\, \varphi(x)$, the iterates of R reads

$$R^n(f \otimes \varphi)(x) = \sum_{b_1,\dots,b_n \in A^n} f(b_1 ... b_n\, \omega)\, \varphi(b_1^{-1} ... b_n^{-1}\, x)\, \mu(b_1) ... \mu(b_n).$$

To prove the ergodicity of σ_1, it suffices to test the convergence of the means

$$\lim_N \frac{1}{N} \sum_{k=0}^{N-1} R^k g = \int \int g \, d\mathbb{P} d\nu,$$

when g is of the form $g(\omega, x) = f(\omega)\varphi(x)$, where φ is in $L^\infty(X)$ and f on Ω depends only on the first p coordinates, for some $p \geq 0$. Setting

$$F_{f,\varphi}(x) = \sum_{b_1,\dots,b_p \in A^p} f(b_1 ... b_p)\, \varphi(b_1^{-1} ... b_p^{-1} x)\, \mu(b_1) ... \mu(b_p),$$

we have, for $k \geq p$, $R^k(f \otimes \varphi)(\omega, x) = \check{P}^{k-p} F_{f,\varphi}(x)$, where $\check{P}$ is defined by

$$\check{P}\psi(x) = \sum_{b \in A} \psi(b^{-1}x)\, \mu(b).$$

Ergodicity of P, hence of $\check{P}$, implies the convergence of the means $\frac{1}{N} \sum_{k=0}^{N-1} R^k(f \otimes \varphi)(\omega, x)$ to the constant $\int F_{f,\varphi}\, \nu(x) = (\int f \, d\mathbb{P}(\omega)) \, (\int \varphi \, d\nu(x))$. $\square$

Displacement

Let $V = \mathbb{R}^d$ be the d-dimensional Euclidean space ($d \geq 1$) and let be given, for each $a \in A$, a bounded Borel map $x \to c_a(x) = c(a, x)$ from X to V. We will call $(c_a(x), a \in A)$ a "displacement".

The *centering condition* of the displacement is assumed, i.e.

$$\int (\sum_a c_a(x)\, \mu(a))\, d\nu(x) = 0. \tag{14}$$

For $a \in A$ and $(x, v) \in X \times V$, we write $\tilde{a}(x, v) = (ax, v + c_a(x))$. Then $\tilde{a}$ defines an invertible map from $X \times V$ into itself with $(\tilde{a})^{-1}(x, v) = (a^{-1}x, v - c_a(a^{-1}x))$. We denote by $\tilde{\Gamma}$ the group of Borel maps of $X \times V$ generated by $\tilde{A} = \{\tilde{a}, a \in A\}$. The action of $\tilde{\Gamma}$ preserves the fibering of $X \times V$ over X, and the projection of $X \times V$ on X is equivariant with respect to the action of Γ on X. We have a homomorphism $\tilde{\gamma} \to \gamma$ from $\tilde{\Gamma}$ to Γ which maps $\tilde{a}$ to a, for every $a \in A$.

In other words, using the displacement $(c_a(x), a \in A)$, we can extend the action of the group Γ on X to the action of the group $\tilde{\Gamma}$ generated by the maps $\tilde{a}$, $a \in A$, on $X \times V$. Clearly the maps $\tilde{\gamma} \in \tilde{\Gamma}$ commute with the translations on the second coordinate on $X \times V$ by elements of V and therefore are of the form $\tilde{\gamma}(x, v) = (\gamma x, v + c(\tilde{\gamma}, x))$, where $c(\tilde{\gamma}, x)$ is a map from $\tilde{\Gamma} \times X$ to V which satisfies the relation

$$c(\tilde{\gamma}\tilde{\gamma}', x) = c(\tilde{\gamma}, \tilde{\gamma}' x) + c(\tilde{\gamma}', x), \ \forall \tilde{\gamma}, \tilde{\gamma}' \in \tilde{\Gamma}.$$

For $\tilde{\gamma} = \tilde{a}_r ... \tilde{a}_1$, $r \in \mathbb{N}^*$, we have $\tilde{\gamma}(x, v) = (a_r ... a_1 x, v + c(\tilde{\gamma}, x))$, with

$$c(\tilde{\gamma}, x) = c(a_1, x) + c(a_2, a_1 x) + ... + c(a_r, a_{r-1} ... a_2 a_1 x). \tag{15}$$

The displacement satisfies the *cocycle property* (for Γ) if the value of the sum in (15) depends only on the value of the product $\gamma = a_r...a_1$ in Γ.

It should be noticed that this cocycle property in general is not satisfied, since the value of the sum in (15) depends in the general case on the "path" $(a_1, ..., a_r)$. This is the case in particular if there is a with $a, a^{-1} \in A$ and $\tilde{a}^{-1} \neq \widetilde{a^{-1}}$.

A special case is when $(c_a(x), a \in A)$ is a *coboundary*, i.e. when there exists $d(x)$ measurable such that $c_a(x) = d(ax) - d(x), \forall a \in A$. The cocycle property then trivially holds in Γ. This is also the case if $c_a(x)$ is a limit of coboundaries.

Extension of the random walk

We consider the random walk on the product space $X \times V$ defined by the probability measure μ and the maps $\tilde{a}$. Its Markov operator $\tilde{P}$ extends the Markov operator P of the random walk on X given by (1) and is defined by

$$(\tilde{P}\psi)(x, v) = \sum_{a \in A} \psi(\tilde{a}(x, v))\, \mu(a) = \sum_{a \in A} \psi(ax, v + c_a(x))\, \mu(a). \tag{16}$$

Such Markov chains have been considered in the literature under various names: random walk with internal degree of freedom if X is finite ([25]), covering Markov chain ([20]), Markov additive process ([36]), etc. Intuitively the random walker moves on V with possible jumps $c_a(x)$, $a \in A$, where x represents the memory of the random walker. Here the steps are chosen according to the probability $\mu(a)$ which depends on a only. A more general scheme would be to choose the steps $c_a(x)$ according to a weight depending on (x, a). Under spectral gap conditions on certain functional spaces, it is possible to develop a detailed study of the iteration $\tilde{P}^n$ of $\tilde{P}$ (See for example [15] when the functional spaces are Hölder spaces). In the framework of the present paper, no regularity is assumed. We supposed only that the displacement consists in bounded Borel maps.

Recall that $Y = \Omega \times X$. Writing $y = (\omega, x)$, we define the extension $\tilde{\sigma}$ of σ_1 on $Y \times V$ by $\tilde{\sigma}(y, v) = (\sigma_1 y, v + c(a_1(\omega), x))$. It preserves the measure $\mathbb{P}_1 \otimes \ell$, where ℓ denotes the Lebesgue measure on V.

The set Y (resp. $Y \times V$) can be identified with the space of trajectories of the Markov chain defined by P (resp. $\tilde{P}$). With the notation of (15) we have

$$S_n(y) = S_n(\omega, x) = \sum_{k=1}^{n} c(a_k(\omega), a_{k-1}(\omega)...a_1(\omega)x).$$

Hence, $S_n(y)$ appears as a Birkhoff sum over (Y, σ_1). The iterates of $\tilde{\sigma}$ on $Y \times V$ read:

$$\tilde{\sigma}^n(y, v) = (\sigma_1^n y, v + S_n(y)), \forall n \geq 1.$$

Also if we denote by $\tilde{\mu}$ the push-forward of μ by the map $a \to \tilde{a}$, we can express the iterate $\tilde{P}^n$ of $\tilde{P}$ as

$$\tilde{P}^n \psi(x, v) = \int \psi(\tilde{\gamma}(x, v))\ d\tilde{\mu}^n(\tilde{\gamma}),$$

where $\tilde{\mu}^n$ is the n-fold convolution product of $\tilde{\mu}$ by itself.

Here we are interested in the asymptotic properties of $\tilde{\sigma}^n$ and $(S_n(y))$ with respect to the measures $\mathbb{P}_1 \otimes \ell$ and $\mathbb{P}_1$ under the condition that P has "nice" spectral properties on X (see below). The L^2-spectral gap condition can be compared to the so-called Doeblin condition for the Markov operator P.

The natural invertible extension of $(\Omega\times X\times V, \tilde\sigma, \mathbb{P}_1\otimes\ell)$ is $(\hat\Omega\times X\times V, \tilde\sigma, \hat{\mathbb{P}}_1\otimes\ell)$, with as above $\tilde\sigma(\omega,x,v) = (\sigma\omega, a_1(\omega)x, v + c(a_1(\omega),x))$, and where σ is now acting on the bilateral space $\hat\Omega$.

We will need to analyze $\tilde P^n$ using methods of Fourier analysis. Hence we are led to introduce a family of operators P_λ on $L^2(X)$, $\lambda\in V$, defined by

$$P_\lambda\varphi(x) = \sum_{a\in A} e^{i\langle\lambda,\, c_a(x)\rangle}\,\varphi(ax)\,\mu(a). \tag{17}$$

We observe that, since $\sup_{a\in A}\|c_a\|_\infty = c < \infty$, the above formula still makes sense if $\lambda\in\mathbb{R}^d$ is replaced by $z\in\mathbb{C}^d$, and we obtain an operator valued holomorphic function $z\to P_z$ satisfying, for any $\varphi,\psi\in L^2(X)$,

$$|\langle P_z\varphi,\psi\rangle| \le e^{c\,\Re ez}\langle P|\varphi|,|\psi|\rangle \le e^{c\,\Re ez}\|\varphi\|_2\|\psi\|_2.$$

This will allow us to use perturbation theory (See [16] for an analogous situation).

The V-valued function $w(x) := \sum_{a\in A} c_a(x)\mu(a)$ is square integrable and its integral is 0 in view of the centering condition (14).

Since $r(\Pi_0) < 1$, the restriction of $P - I$ to $L^2_0(X)$ is invertible, hence we can solve the equation $(P-I)u = w$, with $u\in L^2_0(X)$. The modified displacement $c'(a,x)$ defined, for $a\in\Gamma$, by

$$c'(a,x) := c(a,x) - (u(ax) - u(x))$$

satisfies ν-a.e.

$$\sum_{a\in A} c'(a,x)\mu(a) = 0. \tag{18}$$

A basic tool for the study of $\tilde P^n$ will be the analysis of the Fourier operators P_λ, and in fact their spectral gap properties. Their family satisfies (as in [16], Lemmas 1 and 2):

Lemma 3.2. *For any $\varphi\in L^2(X)$, we have*

$$P_\lambda\varphi(x) = P\varphi(x) + i\int\langle\lambda, c_a(x)\rangle\,\varphi(ax)\,d\mu(a) - \frac12\int\langle\lambda,c_a(x)\rangle^2\,\varphi(ax)\,d\mu(a) + |\lambda|^2 o(\lambda).$$

For λ small, P_λ has a dominant eigenvalue $k(\lambda)$ given by

$$k(\lambda) = 1 - \frac12\Sigma(\lambda) + |\lambda|^2 o(\lambda),$$

where

$$\Sigma(\lambda) := \int\int\langle\lambda, c'_a(x)\rangle^2\,d\mu(a)d\nu(x).$$

In order to analyze more closely the operators P_λ, we introduce some definitions related to the aperiodicity of the displacement.

Definition 3.3. We say that the displacement $(c_a, a\in A)$ satisfies

- (NR) (non reducibility): if there does not exist $\lambda\in V$, $\lambda\neq 0$, and $d\in L^2(X,\mathbb{R}^d)$ such that ν-a.e.

$$\langle\lambda, c(a,x)\rangle = \langle\lambda, d(ax) - d(x)\rangle,\ \forall a\in A;$$

- (AP) (aperiodicity): if there does not exist $(\lambda,\theta)\in V\times\mathbb{R}$, $\lambda\neq 0$, and $d(x)$ measurable, with $|d(x)| = 1$ such that ν-a.e.

$$e^{i\langle\lambda, c(a,x)\rangle} \;=\; e^{i\theta} d(ax)/d(x), \forall a\in A.$$

We observe that the formula $\tilde{a}(x,z) = (ax, z\, e^{i(\langle \lambda,\, c(a,x)\rangle - \theta)})$ defines an action of $\tilde{\Gamma}$ on $X \times \mathbb{T}$ and that ergodicity of these actions for every $\lambda \neq 0$ implies condition (AP). Clearly (AP) implies (NR).

There are special cases (corresponding to functional of Markov chains) where the previous conditions can be easily verified.

Lemma 3.4. *Assume that $c_a(x) = c(x)$ does not depend on $a \in A$, and that the $\mathbb{R}^d$-valued function c is bounded and satisfies the centering condition $\int c(x)\, d\nu(x) = 0$. Assume that A has a symmetric subset B such that B^2 acts ergodically on (X,ν).*

1) If the measure $c(\nu)$ is not supported on a hyperplane of $\mathbb{R}^d$, then (NR) is satisfied.
2) If the measure $c(\nu)$ is not supported on a coset of a proper closed subgroup of $\mathbb{R}^d$, then (AP) *is satisfied.*

Proof. 1) If there exists $\lambda \neq 0$ and a $\mathbb{R}^d$-valued functions $d(x)$ such that ν-a.e. $\langle \lambda, c(x)\rangle = \langle \lambda, d(ax) - d(x)\rangle$, for any $a \in A$, then, for any $a, a' \in A$, $\langle \lambda, d(ax)\rangle = \langle \lambda, d(a'x)\rangle$, i.e. $\langle \lambda, d(\gamma x) = \langle \lambda, d(x)\rangle$ for any $\gamma \in AA^{-1}$. Since the subgroup generated by AA^{-1} is ergodic on (X,ν) we get that, for some $c_0 \in \mathbb{R}$, $\langle \lambda, d(x)\rangle = c_0$, hence $\langle \lambda, c(x)\rangle = 0$, ν-a.e. contrary to the non degeneracy condition on $c(\nu)$.

2) If (AP) does not hold, there exist $\lambda \neq 0$ in $\mathbb{R}^d$, $\theta_\lambda \in \mathbb{R}$ and a cocycle $\sigma_\lambda(\gamma, x)$ on $\Gamma \times X$ with values in the group of complex of modulus 1, such that for ν-a.e. x

$$\sigma_\lambda(a,x) = e^{i(\langle \lambda, c(x)\rangle - \theta_\lambda)}, \forall a \in A.$$

In particular, taking $a, a^{-1} \in B$, we have

$$1 = \sigma_\lambda(a^{-1}, ax)\, \sigma_\lambda(a,x) = e^{i(\langle \lambda, c(x) + c(ax)\rangle - 2\theta_\lambda)};$$

hence, for any $a, a' \in B$, $e^{i\langle \lambda, c(ax)\rangle} = e^{i\langle \lambda, c(a'x)\rangle}$, ν-a.e.

Since B^2 acts ergodically on (X,ν) we get, for $\lambda \neq 0$ and some c_λ of modulus 1, $e^{i(\langle \lambda, c(x)\rangle} = c_\lambda$. This means that $c(x)$ belongs to the coset of the proper closed subgroup of $\mathbb{R}^d$ defined by $e^{i(\langle \lambda, v\rangle} = c_\lambda$, which contradicts the hypothesis. □

3.2. **Ergodicity, recurrence/transience.** In this section, we study ergodicity, recurrence and transience of the extended dynamical systems considered above. First we recall briefly the notion of recurrence in the framework of dynamical systems (cf. [31], [1]).

Let (Y, λ, τ) be a dynamical system with Y a metric space, λ a probability measure on Y and τ an invertible Borel map of Y into itself which preserves λ. We suppose the system ergodic. If φ is a Borel map from Y to $\mathbb{R}^d$, the ergodic sums $S_n\varphi(y) = \sum_{k=0}^{n-1} \varphi(\tau^k y)$ define a "random walk in $\mathbb{R}^d$ over the dynamical system" (Y, λ, τ). The corresponding skew product is the dynamical system defined on $(Y \times \mathbb{R}^d, \lambda \times \ell)$ by the transformation $\tau_\varphi : (y, v) \to (\tau y, v + \varphi(y))$.

We say that $y \in Y$ is *recurrent* if, for every neighborhood U of 0 in $\mathbb{R}^d$,

$$\sum_{n \geq 0} 1_U(S_n\varphi(y)) = +\infty.$$

We say that y is *transient* if, for every neighborhood U of 0 in $\mathbb{R}^d$ this sum is finite. The cocycle $(S_n\varphi)$ is *recurrent* if a.e. point $y \in Y$ is recurrent. It is transient if a.e. point $y \in Y$ is transient.

Since the set of recurrent points is invariant and the system (Y, λ, τ) is ergodic, every cocycle $(S_n\varphi)$ is either transient or recurrent.

For the sake of completeness, let us give a simple proof of the following known equivalence:

Proposition 3.5. *The recurrence of* $(S_n\varphi)$ *is equivalent to the conservativity of the system* $(Y \times \mathbb{R}^d, \lambda \otimes \ell, \tau_\varphi)$.

Proof. By definition the dynamical system $(Y \times \mathbb{R}^d, \lambda \otimes \ell, \tau_\varphi)$ is conservative if, for every measurable set A in $Y \times \mathbb{R}^d$ with positive measure and for a.e. $(y, v) \in A$ there exists $n \geq 1$ such that $\tau_\varphi^n(y, v) = (\tau^n y, v + S_n\varphi(y)) \in A$.

We show conversely that this property holds if $(S_n\varphi)_{n\geq1}$ is recurrent in the sense of the above definition.

We can suppose that A is included in $Y \times L$, where L is a compact set in $\mathbb{R}^d$. One checks easily that the set $B := \{(y, v) \in A : \forall n \geq 1, \tau_\varphi^n(y, v) \notin A\}$ has pairwise disjoint images.

Using the recurrence of $(S_n\varphi)$, one can find for every $\varepsilon > 0$ a compact set K_ε such that, for a set of measure $\geq 1-\varepsilon$ of points y, the sums $S_n\varphi(y)$ belongs to K_ε for infinitely many n (we use the fact that for a.e. y, the set $\{S_n\varphi(y), n \geq 1\}$ has a finite accumulation point and that there is a neighborhood of this accumulation point in which $S_n\varphi(y)$ returns infinitely often).

The measure of the set $F_\varepsilon := Y \times (L + K_\varepsilon)$ is finite. Since B has pairwise disjoint images, we have

$$\begin{aligned}
&\int_Y 1_B(y, v) \sum_{n\geq1} 1_{F_\varepsilon}(\tau_\varphi^n(y, v))\ \lambda(dy)\, d\ell(v) = \sum_{n\geq1} (\lambda \times \ell)(B \cap \tau_\varphi^{-n} F_\varepsilon) \\
= \quad & \sum_{n\geq1} (\lambda \times \ell)(\tau_\varphi^n B \cap F_\varepsilon) \leq (\lambda \times \ell)(F_\varepsilon) < +\infty,
\end{aligned}$$

and therefore, for a.e. (y, v) in B, $\sum_{n\geq1} 1_{F_\varepsilon}(\tau_\varphi^n(y, v)) < \infty$.

This implies that $(\lambda \times \ell)(B) \leq \varepsilon$, hence B has measure 0. The set A satisfies the announced property. $\square$

¿From the proposition it follows that, when $(S_n\varphi)$ is recurrent, the random walk visits $\mathbb{P} \otimes \nu$-a.e. infinitely often any neighborhood of 0 in V, i.e., $\liminf_{n\to\infty} \|S_n(\omega, x)\| = 0$, $\mathbb{P} \otimes \nu$-a.e. When the random walk $(S_n\varphi)$ is transient, then $\lim_{n\to\infty} \|S_n(\omega, x)\| = \infty$, $\mathbb{P} \otimes \nu$-a.e. on $\Omega \times X$.

In the transient case, the system $(Y \times \mathbb{R}^d, \lambda \otimes \ell, \tau_\varphi)$ is dissipative, i.e. there exists a Borel subset $B \subset Y \times \mathbb{R}^d$, such that

$$Y \times \mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \tau_\varphi^n B \text{ and } (\lambda \otimes \ell)(\tau_\varphi^n B \cap B) = 0,\ \forall n \in \mathbb{Z} \setminus \{0\}.$$

Now we will study these properties of recurrence and transience in the case of the random walk $(S_n(\omega, x))$ over the dynamical system $(\hat{\Omega} \times X, \hat{\mathbb{P}}_1, \sigma_1)$ and its extension $(\hat{\Omega} \times X \times V, \tilde{\sigma}, \hat{\mathbb{P}}_1 \otimes \ell)$ defined at the beginning of this section.

Theorem 3.6. *Assume that* (X, ν, Γ, μ) *satisfies property (SG).*

1a) If the displacement $(c_a, a \in A)$ *satisfies (NR), then* $(\frac{1}{\sqrt{n}} S_n(\omega, x))_{n\geq1}$ *converges in law with respect to* $\mathbb{P} \otimes \nu$ *to the centered normal law on* V *with non degenerate covariance* Σ.

1b) If the displacement $(c_a, a \in A)$ *satisfies* (AP), *then the local limit theorem holds: for any* $\varphi \in L^2(X)$ *and* f *continuous with compact support,* $\tilde{\alpha} = \alpha\nu \otimes \delta_0$, *with* $\alpha \in L^2(X)$, $\alpha \geq 0$, $\nu(\alpha) = 1$, *we have*

$$\lim_n (2\pi n)^{d/2} (det\,\Sigma)^{\frac{1}{2}} \tilde{P}^n \tilde{\alpha}(\varphi \otimes f) = \nu(\varphi)\, \ell(f). \tag{19}$$

2a) For $d \leq 2$ (S_n) *is recurrent:* $\mathbb{P} \otimes \nu$ *a.e.* $\liminf_{n\to\infty} \|S_n(\omega, x)\| = 0$.

2b) For $d \leq 2$, *if the displacement* $(c_a, a \in A)$ *satisfies* (AP), *then* $\tilde{\sigma}$ *is ergodic with respect to* $\hat{\mathbb{P}}_1 \otimes \ell$.

2c) If $d \geq 3$, *if the displacement* $(c_a, a \in A)$ *satisfies* (AP), (S_n) *is transient:* $\mathbb{P} \otimes \nu$ *a.e. on* $\Omega \times X$ $\lim_{n\to\infty} \|S_n(\omega, x)\| = \infty$.

3) For any $d \geq 1$, *if the displacement* $(c_a, a \in A)$ *satisfies* (AP), *the equation* $\tilde{P}h = h$, $h \in L^\infty(\nu \otimes \ell)$, *has only constant solutions.*

Proof. 1) We have

$$S_n(\omega, x) = \sum_{k=1}^{n} Y_k(\omega, x) + u(a_n...a_1 x) - u(x),$$

with $Y_k(\omega, x) = c'(a_k, a_{k-1}...a_1 x)$. We observe that

$$\mathbb{E}(Y_k(\omega, x)|a_1, ..., a_{k-1}) = \int c'(a, x)\; d\mu(a) = 0,$$

since $w(x) = \int c(a, x)\, d\mu(a) = (Pu - u)(x)$.

On the other hand, for any $v \in V$, by definition of $\Sigma(v)$ and from (18) we have the martingale property and in particular $\mathbb{E}(\langle Y_k, v\rangle\, \langle Y_\ell, v\rangle) = 0$, if $k \neq \ell$, and

$$\mathbb{E}(\langle Y_k, v\rangle^2) = \int \langle Y_k(\omega, x), v\rangle^2 d\mathbb{P}(\omega)\, d\nu(x) = \int\int \langle v, c'_a(x)\rangle^2\, d\mu(a)\, d\nu(x) = \Sigma(v).$$

Clearly $Y_k = Y_1 \circ \sigma_1^k$ and σ_1 preserves the measure $\mathbb{P} \otimes \nu$. Ergodicity of σ_1 and the ergodic theorem imply $\mathbb{P} \otimes \nu$-a.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle Y_k, v\rangle^2 = \Sigma(v).$$

Hence

$$\lim_n \frac{1}{n} \mathbb{E}(\langle \sum_1^n Y_k, v\rangle^2) = \Sigma(v) = \lim_n \frac{1}{n} \mathbb{E}(\langle S_n, v\rangle^2).$$

Brown's central limit theorem ([7]) applies to (Y_k), and gives the CLT for (S_n) since the coboundary term $u(a_n...a_1 x) - u(x)$ is bounded. So we get the convergence in law of $\frac{1}{\sqrt{n}} S_n(\omega, x)$ with respect to $\mathbb{P} \otimes \nu$ to the centered normal law on V with covariance Σ. The non degeneracy of Σ follows from the formula $\Sigma(v) = \int \langle v, c'_a(x)\rangle^2\, d\mu(a)\, d\nu(x)$ and Condition (NR).

The statement 1b) (the convergence (19)) is proved in Lemma 3.8 below.

2a) Using the CLT as a recurrence criterion for the $\mathbb{R}^2$-valued $\mathbb{Z}$-cocycle, $(S_n y)$ over the measure preserving transformation σ_1 (cf. [33] or [8]), the recurrence property follows:

$$\liminf_{n\to\infty} \|S_n(\omega, x)\| = 0. \tag{20}$$

2b) We observe that the trajectories of the random walk on $X \times V$ defined by μ are given by

$$X_n(\omega, x, v) = (\sigma_1^n(\omega, x), v + S_n(\omega, x)).$$

By (20), for any relatively compact open set $U \subset V$, $\mathbb{P} \otimes \nu$-a.e. on $\Omega \times X \times U$ there exists $n(\omega, x, v) \geq 1$ with $X_n(\omega, x, v) \in \Omega \times X \times U$.

In other words, the Markov kernel $\tilde{P}$ on $X \times V$ satisfies Property R defined in [18]. Hence, using Proposition 2.6 in [18], the ergodicity of $(\hat{\Omega} \times X \times V, \tilde{\sigma}, \hat{\mathbb{P}}_1 \otimes \ell)$ will follow if we show that the equation $\tilde{P}h = h$, for $h \in L^\infty(\nu \otimes \ell)$, has only constant solutions.

Since $\tilde{P}h = h$, we have for any $n \in \mathbb{N}$, $\varphi \in L^2(X)$, $f \in L^1(V)$ with $\int f(v)\, d\ell(v) = 0$,

$$\langle (\tilde{P}^*)^n (\varphi \otimes f), h\rangle = \langle \varphi \otimes f, h\rangle.$$

Lemma 3.7 below gives $\langle \varphi \otimes f, h\rangle = 0$, hence h is invariant by translation by v and defines an element $\overline{h} \in L_\infty(X, \nu)$ with $P\overline{h} = \overline{h}$. Then we have $\sum_{a\in A} \overline{h}(ax)\mu(a) = \overline{h}(x)$, hence the invariance of $\overline{h}$ by $a \in A$. Since ν is Γ ergodic, $\overline{h}$ is constant ν-a.e. Therefore h is constant $\nu \otimes \ell$-a.e. This proves 2b).

2c) Let us show that, for any relatively compact subset U of $X \times V$, for a.e. $(x, v) \in U$ we have on U: $\sum_{n=1}^{\infty} 1_U(a_n ... a_1 x, v + S_n(\omega, x))) < +\infty$.

We have, for every non negative Borel function ψ on $X \times V$:

$$\mathbb{E}(\sum_{n=1}^{\infty} \psi(a_n ... a_1 x, v + S_n(\omega, x))) = \sum_{n=1}^{\infty} \tilde{P}^n \psi(x, v).$$

Here we will prove the convergence $\sum_{n=1}^{\infty} \langle |\tilde{P}^n \psi, \psi\rangle| < \infty$ for ψ of the form $\varphi \otimes f$. Since we can choose $\psi \geq 1_U$, this will implies

$$\mathbb{E}(\sum_{1}^{\infty} 1_U(a_n ... a_1 x, v + S_n(\omega, x) 1_U(x, v)) < \infty,$$

hence the result. This convergence follows from Lemma 3.8 below.

To finish the proof of 2), we observe that if φ is a continuous function with compact support on V and $\psi = \alpha \otimes \varphi$, where $\alpha \in L^2(X)$ and $\tilde{\alpha} = \alpha\nu \otimes \delta_0$, then

$$\langle \tilde{P}^n \psi, \psi\rangle = (\tilde{P}^n \tilde{\alpha})(\psi)\ell(\varphi).$$

In particular by (19) the sequence $(n^{d/2}\langle P^n \psi, \psi\rangle)$ is bounded. If $d > 2$ the series $\sum_{n=0}^{\infty} |\langle \tilde{P}^n \psi, \psi\rangle|$ converges. Hence the result.

3) The assertion is shown in the proof of 2b). □

Now, under the assumption (AP) as in the theorem, we prove the lemmas used in the previous proof.

Lemma 3.7. *For any $\varphi \in L^2(X)$ and any $f \in L^1(V)$ with $\int f(v)\, d\ell(v) = 0$, we have*

$$\lim_{n\to\infty} \|(\tilde{P}^*)^n (\varphi \otimes f)\|_1 = 0.$$

Proof. In the proof of Proposition 3.6 of [17], a Markov operator Q on $X \times \mathbb{R}^d$ which commutes with the $\mathbb{R}^d$-translations is considered and it is proved that $\lim_{n\to\infty} \|Q^n (u \otimes f)\|_1 = 0$ for Hölder continuous functions u in $H_\varepsilon(X)$ and f as above. The essential points of the proof are: polynomial growth of $\mathbb{R}^d$ as a group and a spectral gap property for the Q-action on functions of the form $u \otimes \lambda$ where $u \in H_\varepsilon(X)$ and λ is a character of $\mathbb{R}^d$.

Here we observe that the adjoint operator $\tilde{P}^*$ of $\tilde{P}$ on $L_0^2(X \times V, \nu \otimes \ell)$ is associated with $\check{\mu}$ the symmetric of μ which has the same properties as μ as observed above.

The action of $\tilde{P}^*$ is also well defined on the functions of the form $u \otimes \lambda$ where λ is a fixed character of V and u is in $L^2(X)$. It reduces there to the action of P_λ^*

on $L^2(X)$ hence using $b) \Rightarrow a)$ of Theorem 3.9 below we get that (AP) implies that P_λ^* has a spectral gap. Hence the lemma follows from the proof of Proposition 3.6 in [17] with $Q = \tilde{P}^*$. □

Lemma 3.8. *Let α be a probability measure on X which has a L^2-density with respect to ν. Let $\tilde{\alpha}$ be the probability measure $\alpha \otimes \delta_0$ on $X \times V$ and let $\tilde{\mu}_n := (2\pi n)^{d/2}(\det\Sigma)^{\frac{1}{2}}\tilde{P}^n\tilde{\alpha}$. Then the sequence of Radon measures $(\tilde{\mu}_n)$ on $X \times V$ satisfies $\lim_n \tilde{\mu}_n(\varphi \otimes f) = \nu(\varphi)\,\ell(f)$ for any $\varphi \in L^2(X)$ and f continuous with compact support.*

Proof. Let φ be a function in $L^2(X)$ and $f \in L^1(V)$ be such that its Fourier transform $\hat{f}(\lambda) = \int f(v)\, e^{i\langle\lambda,v\rangle}\, d\ell(v)$ has a compact support on V. Then, by the inversion formula we have:

$$f(v) = (2\pi)^{-d} \int \hat{f}(\lambda)\, e^{-i\langle\lambda,v\rangle}\, d\ell(\lambda).$$

As in [6], p. 225, we test the convergence of $\tilde{\mu}_n^\varphi(f) = \tilde{\mu}_n(\varphi \otimes f)$ using functions f as above. We apply the method of [16] for proving the local limit theorem, giving only the main steps of the proof. According to $b) \Rightarrow a)$ of Theorem 3.9 below, we observe that the operator P_λ considered above satisfies $r(P_\lambda) < 1$ for $\lambda \neq 0$, in view of Condition (AP). Furthermore, by perturbation theory, for λ small enough, P_λ has a dominant eigenvalue $k(\lambda)$ and a corresponding one dimensional projection p_λ such that:

$$\begin{aligned} P_\lambda &= k(\lambda)p_\lambda + R_\lambda, \\ R_\lambda p_\lambda &= p_\lambda R_\lambda,\ r(R_\lambda) < |k(\lambda)|, \\ k(\lambda) &= 1 - \frac{1}{2}\Sigma(\lambda) + |\lambda|^2 o(|\lambda|). \end{aligned}$$

Also p_λ, r_λ depend analytically on λ. These facts will allow us to adapt the analogous proof of [16]. We write $\tilde{P}^n\tilde{\alpha}$ as follows:

$$\begin{aligned} \tilde{P}^n\tilde{\alpha} &= \int \delta_{\gamma x} \otimes \delta_{c(\tilde{\gamma},x)}\, d\tilde{\mu}^n(\tilde{\gamma})\, d\alpha(x); \\ P^n\tilde{\alpha}(\varphi \otimes f) &= (2\pi)^{-d} \int \varphi(\gamma x)\, e^{-i\langle\lambda, c(\tilde{\gamma},x)\rangle}\, \hat{f}(\lambda)\, d\tilde{\mu}^n(\tilde{\gamma})\, d\alpha(x)\, d\ell(\lambda) \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \alpha(P_{-\lambda}^n \varphi)\, \hat{f}(\lambda)\, d\ell(\lambda), \end{aligned}$$

hence

$$\tilde{\mu}_n(\varphi \otimes f) = (2\pi)^{-d/2}(\det\Sigma)^{\frac{1}{2}}\, n^{d/2} \int_{\mathbb{R}^d} \alpha(P_{-\lambda}^n \varphi)\, \hat{f}(\lambda)\, d\ell(\lambda).$$

Since $r(P_\lambda) < 1$ for $\lambda \neq 0$, the integration can be reduced, in the limit, to a small neighborhood U of 0 in $\mathbb{R}^d$ and it suffices to consider

$$I_n := (2\pi)^{-d/2}(\det\Sigma)^{\frac{1}{2}} \int_{\sqrt{n}U} \alpha(P_{-\lambda/\sqrt{n}}^n(\varphi)\, \hat{f}(\lambda/\sqrt{n})\ d\ell(\lambda);$$

Using the spectral decomposition of P_λ we see that $\lim_{n\to\infty} I_n = \lim_{n\to\infty} J_n$ with

$$J_n = (2\pi)^{-d/2}(\det\Sigma)^{\frac{1}{2}} \int_{\sqrt{n}U} k(\lambda/\sqrt{n})^n \alpha(P_{\lambda/\sqrt{n}}(\varphi))\, \hat{f}(\lambda/\sqrt{n})\ d\ell(\lambda).$$

Since $\lim_{n\to\infty} k(\lambda/\sqrt{n})^n = e^{-\frac{1}{2}\Sigma(\lambda)}$ and $\lim_{\lambda\to 0} p_\lambda(\varphi) = \nu(\varphi)$, we get

$$\lim_{n\to\infty} J_n = (2\pi)^{-d/2}(det\,\Sigma)^{\frac{1}{2}} \int e^{-\frac{1}{2}\Sigma(\lambda)}\,\nu(\varphi)\,\hat{f}(0)\;d\ell(\lambda);$$

hence $\lim_{n\to\infty} J_n = (\nu\otimes\ell)(\varphi\otimes f)$. □

The following theorem will play an important role, since it allows to pass from approximate coboundary equation to an exact one.

Theorem 3.9. *Let Π_0 be the restriction of P to $L^2_0(X)$ and let P_λ, $\lambda\in V$, $\lambda\neq 0$, be defined on $L^2(X)$ by (17) with $c_a(x)\in L^\infty(X)$ for every $a\in\text{supp}(\mu)$. Assume that $r(\Pi_0)<1$. Then the following properties are equivalent:*
a) the spectral radius $r(P_\lambda)$ of P_λ acting on $L^2(X)$ is 1;
b) the condition (AP) *is not satisfied at λ: there exists a real θ and a measurable function α such that*

$$e^{i(\langle\lambda,\,c_a(x)\rangle-\theta)} = e^{i(\alpha(ax)-\alpha(x))},\;\; \nu - a.e.$$

c) there exists $\theta\in\mathbb{R}$ such that $e^{i(\langle\lambda,\,c_a(x)\rangle-\theta)}$ extends as a cocycle $\sigma_\lambda(\gamma,x)$ on $\Gamma\times X$, with values in the group of complex numbers of modulus 1, and the representation ρ_λ of Γ on $L^2(X)$ contains Id_Γ, where

$$(\rho_\lambda(\gamma)\varphi)(x) = \sigma_\lambda(\gamma^{-1},x)\,\varphi(\gamma^{-1}x). \tag{21}$$

Proof. a) $\Rightarrow$ b)

We begin as in the proof of Proposition 2.1. Assume $r(P_\lambda)=1$ and let $e^{i\theta}$, ($\theta\in[0,2\pi[$) be a spectral value of P_λ. Then, either the subspace $\text{Im}(e^{i\theta}-P_\lambda)$ is not dense in $L^2(X)$ or there exists $\varphi_n\in L^2(X)$, with $\|\varphi_n\|=1$, such that

$$\lim_n \|P_\lambda\varphi_n - e^{i\theta}\varphi_n\|_2 = 0.$$

In the first case, there exists $\varphi\in L^2(X)$ with $e^{i\theta}P^*_\lambda\varphi=\varphi$. Since $e^{-i\theta}P_\lambda$ is a contraction of $L^2(X)$, this implies $P_\lambda\varphi = e^{i\theta}\varphi$. Hence it suffices to consider the second case. We have:

$$\begin{aligned} 0\le \|P_\lambda\varphi_n - e^{i\theta}\varphi_n\|_2^2 &= \|P_\lambda\varphi_n\|_2^2 + \|\varphi_n\|_2^2 - 2\,\Re\text{e}\langle P_\lambda\varphi_n, e^{i\theta}\varphi_n\rangle \\ &\le 2 - 2\,\Re\text{e}\langle P_\lambda\varphi_n, e^{i\theta}\varphi_n\rangle. \end{aligned}$$

Then the condition $\lim_n \|P_\lambda\varphi_n - e^{i\theta}\varphi_n\|_2 = 0$ is equivalent to:

$$\lim_n \Re\text{e}\langle P_\lambda\varphi_n, e^{i\theta}\varphi_n\rangle = 1,$$

i.e. to

$$\lim_n \langle P_\lambda\varphi_n, e^{i\theta}\varphi_n\rangle = 1.$$

We have also, since $\|\varphi_n\|_2=1$,

$$0\le |\langle P_\lambda\varphi_n, e^{i\theta}\varphi_n\rangle| \le \langle P|\varphi_n|, |\varphi_n|\rangle \;\le 1.$$

It follows $\lim_n\langle P|\varphi_n|,|\varphi_n|\rangle = 1$, i.e. $\lim_n \|P|\varphi_n| - |\varphi_n|\|_2 = 0$.

We can write $|\varphi_n| = \langle|\varphi_n|,1\rangle + \psi_n$, where $\psi_n := (|\varphi_n| - \langle|\varphi_n|,1\rangle 1\in L^2_0(X)$ and therefore $\lim_n \|(P-I)\psi_n\|_2 = 0$ and $\langle|\varphi_n|,1\rangle \le \|\varphi_n\|_2\le 1$.

Since $r(\Pi_0)<1$, the restriction of $\Pi_0 - I$ to $L^2_0(X)$ is invertible. Hence the condition $\lim_n\|(P-I)\psi_n\|_2=0$ implies $\lim_n\|\psi_n\|_2=0$.

If $c\in\mathbb{R}_+$ is a limit of a subsequence $\langle|\varphi_{n_i}|,1\rangle$ of $\langle|\varphi_n|,1\rangle$, we get $\lim_i \||\varphi_{n_i}| - c\|_2 = 0$.

Since $\|\varphi_n\|_2 = 1$, we have $c=1$, hence the convergence $\lim_n \||\varphi_n| - 1\|_2 = 0$.

On the other hand, the condition $\lim_n \langle P_\lambda \varphi_n, e^{i\theta}\varphi_n\rangle = 1$ can be written as

$$\lim_n \int \langle e^{i\langle\lambda, c_a\rangle}\varphi_n \circ a, e^{i\theta}\varphi_n\rangle \, d\mu(a) = 1,$$

where, for each $a \in \mathrm{supp}(\mu)$, $|\langle e^{i\langle\lambda, c_a\rangle}\varphi_n \circ a, e^{i\theta}\varphi_n\rangle| \leq 1$.

It follows, for any $a \in \mathrm{supp}(\mu)$,

$$\lim_n \|e^{i\langle\lambda, c_a\rangle}\varphi_n \circ a - e^{i\theta}\varphi_n\|_2 = 0.$$

We can write $\varphi_n(x) = |\varphi_n(x)|e^{i\alpha_n(x)}$, with $\alpha_n(x) \in [0, 2\pi[$. Hence:

$$\begin{aligned}&e^{i\langle\lambda, c_a\rangle}\varphi_n \circ a - e^{i\theta}\varphi_n \\ &= e^{i\langle\lambda, c_a\rangle + i\alpha_n\circ a}(|\varphi_n \circ a| - 1) - e^{i\theta}(|\varphi_n| - 1)e^{i\alpha_n} + e^{i\langle\lambda, c_a\rangle + i\alpha_n\circ a} - e^{i(\alpha_n+\theta)}.\end{aligned}$$

Hence

$$\lim_n \|e^{i(-\alpha_n + \alpha_n\circ a + \langle\lambda, c_a\rangle - \theta)} - 1\|_2 = 0,$$

therefore, for a subsequence (n_k)

$$e^{i(\langle\lambda, c_a(x)\rangle - \theta)} = \lim_k e^{i(\alpha_{n_k}(ax) - \alpha_{n_k}(x))}, \ \nu - a.e.$$

Clearly $\lim_k e^{i(\alpha_{n_k}(ax) - \alpha_{n_k}(x))}$ is the restriction to $A \times X$ of a cocycle $\sigma_\lambda(\gamma, x)$ of $\Gamma \times X$.

On the other hand property (SG) implies strong ergodicity of the action of Γ on X; hence proposition 2.3 of [32] gives the existence of a measurable function α on X such that $\sigma_\lambda(a, x) = e^{i(\langle\lambda, c_a(x)\rangle - \theta)} = e^{i(\alpha(ax) - \alpha(x))}$, ν-a.e.

b) $\Rightarrow$ c)

With $\varphi(x) = e^{-i\alpha(x)}$, we have by condition b) $\sigma_\lambda(a, x) = \varphi(x)/\varphi(ax)$ which extends to $\Gamma \times X$ as a cocycle. By the definition of $\rho_\lambda(a^{-1})$ (cf. (21)), we have

$$(\rho_\lambda(a^{-1})\,\varphi)(x) = \varphi(x).$$

Since A generates Γ as a group, this means that the representation ρ_λ contains Id_Γ.

c) $\Rightarrow$ a)

Let $\check{\mu}$ be the push-forward of μ by the map $\gamma \to \gamma^{-1}$. We observe that, by the definition of ρ_λ, $\rho_\lambda(\check{\mu}) = e^{-i\theta}P_\lambda$.

Therefore $e^{i\theta}$ is an eigenvalue of P_λ, so that $r(P_\lambda) = 1$. $\square$

Remarks. 1) In general, it is non trivial to calculate the set of $\lambda \in \mathbb{R}^d$ such that $r(P_\lambda) = 1$. However, Corollary 3.10 is useful for this question.

2) Also we observe that condition c) in the proposition implies that the action of Γ on $X \times \mathbb{T}$ given by $\gamma(x, t) = (\gamma x, t\sigma_\lambda(\gamma, x))$ is not ergodic.

Corollary 3.10. *Assume property (SG). The set $R_\mu = \{\lambda \in \mathbb{R}^d : r(P_\lambda) = 1\}$ is a closed subgroup of $\mathbb{R}^d$. It is discrete if (NR) is valid.*

Proof. Assume λ_1, λ_2 satisfy $r(P_{\lambda_1}) = r(P_{\lambda_2}) = 1$. Then condition b) of the Proposition gives the existence of $\theta_1, \theta_2 \in \mathbb{R}$, φ^1, φ^2 with $|\varphi^1| = |\varphi^2| = 1$, such that, for $j = 1, 2$ and ν-a.e.

$$e^{i(\langle\lambda_j, c_a(x)\rangle - \theta_j)} = \varphi^j(ax)/\varphi^j(x).$$

It follows
$$e^{i(\langle\lambda_1-\lambda_2,c_a(x)\rangle-\theta_1+\theta_2)}=\frac{\varphi^1(ax)\overline{\varphi^2}(ax)}{\varphi^1(x)\overline{\varphi^2(x)}},$$
i.e. condition b) is satisfied with $\lambda_1-\lambda_2$, $\theta_1-\theta_2$, $\varphi^1(x)\overline{\varphi^2(x)}$. Hence R_μ is a group.

The definition of P_λ, $P_{\lambda'}$ gives the following inequality:
$$|(P_\lambda-P_{\lambda'})\varphi(x)|\le\sup_{x,a}|e^{i\langle\lambda-\lambda',\,c_a(x)\rangle}-1|\,P|\varphi|(x),$$
hence, since $|c_a(x)|$ is bounded by a constant c,
$$\|(P_\lambda-P_{\lambda'})\varphi\|_2\le\sup_{a\in A}\|c_a\|_\infty\,|\lambda-\lambda'|\,\|\varphi\|_2\le c|\lambda-\lambda'|\|\varphi\|_2.$$

Therefore, we have $\|P_\lambda-P_{\lambda'}\|\le c|\lambda-\lambda'|$, which implies that, if λ is fixed with $r(P_\lambda)<1$, we have also $r(P_{\lambda'})<1$ for λ' sufficiently close to λ. Hence R_μ is closed.

Assume now Condition (NR) is satisfied. Observe that $\|P_\lambda-P_0\|\le c|\lambda|$. Since $r(\Pi_0)<1$, the spectrum of $P=P_0$ consists of $\{1\}$ and a compact subset of the open unit disk. Hence P has a dominant isolated eigenvalue, which is a simple eigenvalue. By perturbation theory, this property remains valid in a neighborhood of 0.

Using Lemma 3.2 and the fact (noted in Theorem 3.6 part 1) that the covariance matrix Σ is non degenerate, the dominant spectral value $k(\lambda)$ satisfies: $r(P_\lambda)=|k(\lambda)|<1$, for λ small and $\neq 0$. Hence $R_\mu\cap W=\{0\}$ for some neighborhood W of 0, i.e. R_μ is a discrete subgroup of $\mathbb{R}^d$. □

Remark. If $c_a(x)$ takes values in $\mathbb{Z}^d$, the previous results have an analogue if we replace the space $X\times\mathbb{R}^d$ by $X\times\mathbb{Z}^d$. The character $\lambda\in V$ should be replaced by a character of $\mathbb{Z}^d$, i.e. $\lambda\in\mathbb{T}^d$, and the Lebesgue measure ℓ by the counting measure on $\mathbb{Z}^d$. This will be used in 4.3 below.

The following corollary makes explicit the result in Theorem 3.6 for a functional $c(x)$ of a Markov chain.

Corollary 3.11. *Assume that (X,ν,Γ,μ) satisfies property (SG), that $c_a(x)=c(x)$ does not depend on $a\in A$, and that the $\mathbb{R}^d$-valued function c is bounded and satisfies $\int c(x)\,d\nu(x)=0$. Moreover, assume that $A\subset\Gamma$ has a symmetric subset B such that B^2 acts ergodically on (X,ν). Then we have:*
1) if $d\le 2$, we have $\mathbb{P}\times\nu$-a.e. $\liminf_{n\to\infty}\|S_n(\omega,x)\|=0$;
2) if the measure $c(\nu)$ is not supported on a coset of a proper closed subgroup of $\mathbb{R}^d$,
- for $d\le 2$, $\tilde\sigma$ is ergodic with respect to $\mu\times\nu\times\ell$,
- for $d\ge 3$, the local limit theorem is valid for $(S_n(\omega,x))$ and $\lim_{n\to\infty}\|S_n(\omega,x)\|=+\infty$, $\mathbb{P}\times\nu$-a.e.

Proof. The result follows from Lemma 3.4 and Theorem 3.6. □

The arguments in the proof of the proposition give also the following corollary, which is a direct consequence of the main result of [19] (see also Theorem 6.3 in [11]).

Corollary 3.12. *Assume* $\operatorname{supp}(\mu)$ *is finite, generates Γ and the representation ρ_0 of Γ in $L^2_0(X)$ does not contain weakly* Id_Γ. *Let Γ^* be the group of characters of Γ*

and Γ_X^* *be the subset of elements of* Γ^* *contained in the natural representation of* Γ *in* $L^2(X,\nu)$. *Then* Γ_X^* *is a finite subgroup of* Γ^*. *The measure* μ *satisfies property (SG) if and only if* $\operatorname{supp}(\mu)$ *is not contained in a coset of the subgroup* $\ker\chi$ *for some* $\chi \in \Gamma_X^*$, $\chi \neq 1$. *In particular, if* $(\operatorname{supp}(\mu))^k$ *generates* Γ *for any* $k > 0$, *then* (SG) *is satisfied.*

Proof. Let ρ be the natural representation of Γ in $L^2(X,\nu)$. For a given $\chi \in \Gamma_X^*$, there exists $\varphi \in L^2(X,\nu)$ with $\varphi(\gamma x) = \chi(\gamma)\varphi(x)$, $\|\varphi\|_2 = 1$. The ergodicity of the action of Γ on X implies that φ is uniquely defined up to a scalar, with $|\varphi(x)| = 1$, ν-a.e.

Also it is clear that Γ_X^* is a subgroup of Γ^*. To obtain that Γ_X^* is closed in Γ^*, we note that if $\chi \in \Gamma^*$ satisfies for some sequence (φ_n) with $|\varphi_n(x)| = 1$, $\chi(\gamma) = \lim_n \varphi_n(\gamma x)/\varphi_n(x)$, then $\chi \in \Gamma_X^*$. As in the proof of the proposition this follows from Proposition 2.3 of [32], since the subgroup of $\mathbb{T}$-valued coboundaries of (Γ, X, ν) is closed in the group of cocycles endowed with the topology of convergence in measure.

In order to show that each element of Γ_X^* has finite order, we observe that, using [19], (Γ, X, ν) has no non atomic $\mathbb{Z}$-factor up to orbit equivalence. Hence, for every $\chi \in \Gamma_X^*$ and some $n \in \mathbb{N}^*$ one has $\chi^n = 1$. Since Γ is finitely generated, Γ_X^* is a closed subgroup of a torus. Therefore Γ_X^* is finite.

If $\operatorname{supp}(\mu)$ is contained in the coset $\{\gamma \in \Gamma : \chi(\gamma) = c\}$ and $\varphi(\gamma x) = \chi(\gamma)\,\varphi(x)$ with $\varphi \in L^2(X)$, one has:

$$P\varphi(x) = \sum_{a \in \operatorname{supp}(\mu)} \varphi(ax)\,\mu(a) = c\,\varphi(x),$$

hence μ does not satisfy (SG).

Conversely, if μ does not satisfy (SG), then for some c of modulus 1 and a sequence (φ_n) in $L^2(X)$ with $\|\varphi_n\| = 1$, we have $\lim_n \|P\varphi_n - c\varphi_n\|_2 = 0$. As in the proof of the proposition, we can use the condition that ρ_0 does not contain weakly Id_Γ to get that $P - I$ is invertible on $L_0^2(X)$ and obtain that $\lim_n \||\varphi_n| - 1\|_2 = 0$. Then, writing $\varphi_n(x) = |\varphi_n(x)|\,e^{i\alpha_n(x)}$, we get that for a subsequence (n_k) of integers, $\lim_k e^{i(\alpha_{n_k}(x) - \alpha_{n_k}(\gamma x))} = c$.

Then there exists $\chi \in \Gamma^*$ with $\chi(a) = c$ for every $a \in A$, and for every $\gamma \in \Gamma$, $\chi(\gamma) = \lim_k e^{i(\alpha_{n_k}(x) - \alpha_{n_k}(\gamma x))}$. From above $\chi \in \Gamma_X^*$. The condition $\chi(a) = c$ for every $a \in A$ implies $\operatorname{supp}(\mu) \subset \{\gamma \in \Gamma : \chi(\gamma) = c\}$. Hence the result.

For the last assertion, we observe that, if $\chi \in \Gamma_X^* \setminus \{1\}$ satisfies $\chi(a) = c$ for some $c \in \mathbb{T}$ and every $a \in \operatorname{supp}(\mu)$, then, for some $k \in \mathbb{N}^*$, $\chi^k(a) = c^k = 1$. Then any $\gamma \in (\operatorname{supp}(\mu))^k$ satisfies $\chi(\gamma) = 1$. Since $(\operatorname{supp}(\mu))^k$ generates Γ, we get $\chi = 1$, which contradicts the hypothesis on χ. $\square$

4. **Examples.**

4.1. **Random walk in random scenery.**

As an example corresponding to Corollary 3.11, we consider a group Γ, a probability measure μ on Γ such that $A := \operatorname{supp}(\mu)$ is symmetric and $(\operatorname{supp}(\mu))^2$ generates Γ as a group. We denote by $(\Sigma_n(\omega), \omega \in A^{\mathbb{Z}})$ the left random walk on Γ defined by μ and we consider the visits of $\Sigma_n(\omega)$ to a random scenery on Γ.

Such a random scenery is defined by a finite set $C \subset \mathbb{R}^d$, a probability measure η on C with $\operatorname{supp}(\eta) = C$ and $\sum_{v \in C} \eta(v)\,v = 0$. To each $\gamma \in \Gamma$, one associates a random variable x_γ with values in C. The variables x_γ are assumed to be i.i.d. with law η.

The scenery defines a point $x = (x_\gamma)_{\gamma\in\Gamma}$ of the Bernoulli scheme $X = C^\Gamma$ endowed with the product measure $\mu^{\otimes\Gamma}$ and Γ acts on C^Γ by left translations: if $x = (x_\gamma, \gamma \in \Gamma)$, then $ax = (x_{a\gamma}, \gamma \in \Gamma)$. If we define $f(x) = x_e \in C$, the cumulated scenery is given by $S_n(\omega, x) = \sum_{k=1}^{n} f(a_k(\omega)...a_1(\omega)x)$. One can give the following interpretation: the random walker collects the quantity x_γ when visiting the site γ and his "cumulated gain" at time n along the path defined by ω is $S_n(\omega, x)$.

The probability measure $\nu = \eta^{\otimes\Gamma}$ is Γ-invariant, ergodic, and

$$\int f(x)d\eta^{\otimes\Gamma}(x) = \sum_{v\in C} \eta(v)v = 0.$$

The transformation σ_1 on $A^{\mathbb{Z}} \times C^\Gamma = \hat{Y}$ is given by $\sigma_1(\omega, x) = (\theta\omega, a_1(\omega)x)$. Since A is symmetric, σ_1 can be seen as a "$T - T^{-1}$" transformation (cf. [21]).

We consider the Markov operator P on X associated with μ and its restriction Π_0 to $L^2_0(X)$. It is well known (see [3], Ex E45, p. 394) that the action of Γ on $L^2_0(X)$ decomposes as a direct sum of tensor products of the regular representation in $\ell^2(\Gamma)$. A typical summand is $\otimes_1^k \ell^2(\Gamma)$ and if $r_0(\mu)$ is the spectral radius of the convolution operator by μ in $\ell^2(\Gamma)$, we have $r(\Pi_0) = \sup_{k\geq 1}(r_0(\mu))^k = r_0(\mu)$.

Assume that Γ is non amenable. Then we have $r_0(\mu) < 1$ (see [23]), hence property (SG) is satisfied. If we assume that $C \subset \mathbb{R}^d$ is not supported on a coset of a closed subgroup of $\mathbb{R}^d$, the hypothesis and the conclusions 2 and 3 of Corollary 3.11 are valid. Hence, with the above notations, it follows:

Proposition 4.1. *Let Γ be a non amenable group, μ a probability measure on Γ such that $\mathrm{supp}(\mu)$ is symmetric and $(\mathrm{supp}(\mu))^2$ generates Γ, $\Sigma_n(\omega)$ the corresponding random walk on Γ. We assume that Γ is endowed with an $\mathbb{R}^d$-valued random scenery with law η, that $C = \mathrm{supp}(\eta)$ is finite with $\sum_{v\in C} v\eta(v) = 0$, and $\mathrm{supp}(\eta)$ is not contained in a coset of a closed subgroup of $\mathbb{R}^d$. We denote by $S_n(\omega, x)$ the accumulated scenery along the random walk and by $\tilde{\sigma}$ the transformation on $\Omega \times C^\Gamma \times \mathbb{R}^d$ defined with $f(x) = x_e$ by*

$$\tilde{\sigma}(\omega, x, t) = (\sigma\omega, a_1(\omega)x, t + f(x)).$$

Then the convergence of $\frac{1}{\sqrt{n}} S_n(\omega, x)$ to a non degenerate normal law is valid. If $d \leq 2$, $\tilde{\sigma}$ is ergodic and (S_n) is recurrent with respect to $\mu \times \nu \times \ell$. If $d \geq 3$, $\mu^{\otimes\mathbb{Z}} \times \nu$-a.e., $\lim_n \|S_n(\omega, x)\| = +\infty$.

Remark 1. The above result should be compared with the case Γ amenable. For $\Gamma = \mathbb{Z}$, $\frac{1}{n^{3/2}} S_n(\omega, x)$ converges in law towards a non Gaussian law ([24], [26]).

Here, due to the strong transience properties of Γ, $S_n(\omega, x)$ behaves qualitatively like a sum of i.i.d. random variables. Let us consider $\Gamma = \mathbb{Z}^m$, for m large.

Using independence of the random variables $(x_\gamma, \gamma \in \Gamma)$, we see that

$$\|\sum_{k=0}^{\infty} P^k f\|_2^2 = \|f\|_2^2 \sum_{\gamma\in\Gamma} (\pi(\gamma))^2,$$

where $\pi = \sum_{k=0}^{\infty} \mu^k$ is the potential of μ on $\mathbb{Z}^m$. If $m \geq 5$, it is known that $\sum_{\gamma\in\Gamma}(\pi(\gamma))^2 < \infty$ (see for example [36]), hence $\varphi = \sum_{k=0}^{\infty} P^k f$ is finite $\eta^{\otimes\Gamma}$-a.e. and defines an element of $L^2_0(X)$ which satisfies $(I - P)\varphi = f$. This implies the convergence in law of $\frac{1}{\sqrt{n}} S_n(\omega, x)$ to a non degenerate normal law ([13]).

4.2. **Random walks on extensions of tori.** Now we present a special case where Condition (AP) can be checked.

Here the $2d$-dimensional torus $\mathbb{T}^{2d}$ is identified with $[-\frac{1}{2}, \frac{1}{2}[^{2d}$ and $\{x\}$ denotes the point of $[-\frac{1}{2}, \frac{1}{2}[^{2d}$ corresponding to $x \in \mathbb{T}^{2d}$.

Proposition 4.2. *Let μ be a probability measure on $Sp(2d, \mathbb{Z})$ acting by automorphisms on $\mathbb{T}^{2d}$ and let Γ be the subgroup generated by* $\operatorname{supp}(\mu)$. *Assume that Γ acts $\mathbb{Q}$-irreducibly on $\mathbb{R}^{2d}$ and Γ is not virtually abelian. Let ν be the Lebesgue measure on $\mathbb{T}^{2d}$. Then, with the notations of Section 3, we consider the transformation $\tilde{\sigma}$ on $\Omega \times \mathbb{T}^{2d} \times \mathbb{R}^{2d}$ defined by*

$$\tilde{\sigma}(\omega, x, v) = (\sigma\omega, a_1 x, v + \{x\}).$$

Let $S_n(\omega, x) = \sum_{k=1}^n \{a_k ... a_1 x\}$. Then, if $d = 1$, $\tilde{\sigma}$ is ergodic and (S_n) is recurrent with respect to $\mu^{\otimes \mathbb{Z}} \times \nu \times \ell$. If $d \geq 2$, $\mu^{\otimes \mathbb{Z}} \times \nu$-a.e. $\lim_n \|S_n(\omega, x)\| = +\infty$.

Since $x \to \{x\}$ is bounded and $\int \{x\} \, d\nu(x) = 0$, the proposition is a direct consequence of Theorem 3.9, Theorem 3.6 and the following lemma.

Lemma 4.3. *Let μ be a probability measure on $Sp(2d, \mathbb{Z})$ and let Γ be the subgroup generated by* $\operatorname{supp}(\mu)$. *For $\lambda \in \mathbb{R}^{2d}$ let P_λ be the operator on $L^2(\mathbb{T}^{2d})$ defined by*

$$P_\lambda \varphi(x) = \sum_a e^{i\langle \lambda, \{x\}\rangle} \varphi(ax) \, \mu(a).$$

Then, if Γ acts $\mathbb{Q}$-irreducibly on $\mathbb{R}^{2d}$ and Γ is not virtually abelian, we have $r(P_\lambda) < 1$, for $\lambda \in \mathbb{R}^{2d} \setminus \{0\}$. In particular (SG) and (AP) are valid.

Proof. We will use as an auxiliary tool the Heisenberg group H_{2d+1} and its automorphism group $Sp(2d, \mathbb{R}) \ltimes \mathbb{R}^{2d}$. The group H_{2d+1} has a one dimensional center C isomorphic to $\mathbb{R}$ and a lattice Δ such that $\Delta \cap C$ is isomorphic to $\mathbb{Z}$, and $\Delta / \Delta \cap C$ is isomorphic to $\mathbb{Z}^{2d}$.

Let $\hat{X}$ be the corresponding manifold H_{2d+1}/Δ. Up to a set of 0 measure, we can represent $\hat{X}$ as $\mathbb{T}^{2d} \times \mathbb{T}^1$ and $\hat{x} \in \tilde{X}$ as $\hat{x} = (x, z)$, with $x \in \mathbb{T}^{2d}$, $z \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

The action of an element g of $Sp(2d) \ltimes \mathbb{R}^{2d}$ on H_{2d+1} can be represented as a matrix $g = \begin{pmatrix} (a) & 0 \\ u & 1 \end{pmatrix}$ acting on $\mathbb{R}^{2d} \times \mathbb{R}$, where $a \in Sp(2d, \mathbb{R})$, $u \in \mathbb{R}^{2d}$. If g preserves Δ, we have $a \in Sp(2d, \mathbb{Z})$, $u \in \mathbb{Z}^{2d}$ and the action of g on $\hat{X}$ is given by $g(x, z) = (ax, z + [u, x])$, with $a \in Sp(2d, \mathbb{Z})$, $x = (x_1, x_2) \in \mathbb{T}^{2d}$, $u = (u_1, u_2), u' = (-u_2, u_1) \in \mathbb{Z}^{2d}$ and

$$[u, x] = \{\langle u_1, x_2\rangle - \langle u_2, x_1\rangle\} = \{\langle u', x\rangle\} \in \mathbb{T}^{2d}.$$

For a fixed $u \in \mathbb{Z}^{2d}$, we associate to $a \in Sp(2d, \mathbb{Z})$ the element $g = \hat{a}$ of $Sp(2d, \mathbb{Z}) \ltimes \mathbb{Z}^{2d}$ with components $a \in Sp(2d, \mathbb{Z})$ and $u \in \mathbb{Z}^{2d}$.

We denote by $\hat{\mu}$ the push-forward of μ by the map $a \to \hat{a}$. We denote by $\hat{\Gamma}$ the group generated by $\operatorname{supp}(\hat{\mu})$ and we consider the convolution action of $\hat{\mu}$ on $L^2(\hat{X})$. On functions ψ of the form $\psi(x, z) = \varphi(x) \, e^{2i\pi z}$, with $\varphi \in L^2(\mathbb{T}^{2d})$, the action of $\hat{\mu}$ is given by

$$\hat{P}_u \varphi(x) = \sum_a \varphi(ax) \, e^{2\pi i [u, x]} \, \mu(a) = \sum_a \varphi(ax) \, e^{2\pi i \langle u', x\rangle} \, \mu(a) = P_{2\pi u'} \, \varphi(x).$$

We denote by $\|\mu^n\|_2$ (resp. $\|\overline{\mu}^n\|_2$) the norm of the convolution operator by μ^n on $\ell^2(\Gamma)$ (resp. by $\overline{\mu}^n$ on $\ell^2(\mathbb{Z}^{2d} \setminus \{0\})$). We write

$$\begin{aligned} r_0(\mu) &= \lim_n \|\mu^n\|_2^{1/n},\ r_0(\overline{\mu}) = \lim_n \|\overline{\mu}^n\|_2^{1/n}, \\ r_n &= \sup(\|\mu^n\|_2, \|\overline{\mu}^n\|_2^{1/2d+2}). \end{aligned}$$

Now we observe that, for $\lambda, \lambda' \in \mathbb{R}^{2d}$, we have

$$|(P_\lambda - P_{\lambda'})\,\varphi| \le \frac{1}{2}|\lambda - \lambda'|\,P|\varphi|,$$

and $\|P_\lambda - P_{\lambda'}\| \le \frac{1}{2}|\lambda - \lambda'|$ since P is a contraction.

On the other hand, if $\lambda' = 2\pi u'$, $u' \in \mathbb{Z}^d$, we have for any $n \in \mathbb{N}$, using [4] (Theorem 3), $\|P^n_{\lambda'}\| \le r_n$.

We observe also that the qualitative result $r(P_{\lambda'}) < 1$ follows from the implication $b) \Rightarrow a)$ of Theorem 3.9 and the remark following it, since Γ acts ergodically on $\mathbb{T}^{2d}$, and therefore $\tilde{\Gamma}$ acts ergodically on $\tilde{X}$ (cf. section 1).

The hypothesis on Γ implies its non amenability (See [5], Corollary 6), hence (see [23]) the spectral radius $r_0(\mu)$ of the convolution operator on $\ell^2(\Gamma)$ defined by μ satisfies $r_0(\mu) < 1$. Also from [5], Corollary 6, we have $r_0(\overline{\mu}) = r(\Pi_0) < 1$.

We bound $\|P^n_\lambda\|$ as follows: we have $P^n_{\lambda'} - P^n_\lambda = \sum_{k=0}^{n-1} P^k_{\lambda'}(P_{\lambda'} - P_\lambda)P_\lambda^{n-k-1}$. Since P_λ is a contraction on $L^2(\mathbb{T}^{2d})$, we have:

$$\|P^n_{\lambda'} - P^n_\lambda\| \le \sum_{k=0}^{n-1} \|P^k_{\lambda'}\|\,\|P_\lambda - P_{\lambda'}\| \le \|P_\lambda - P_{2\pi u'}\| \sum_{k=0}^{n-1} r_k.$$

Hence, $\|P^n_\lambda\| \le c\|\lambda - 2\pi u'\| + r_n$, with $c = \frac{1}{2}\sum_{k=1}^{\infty} r_k$, which is finite since $r_0(\mu) < 1$, $r_0(\overline{\mu}) < 1$.

Let $\lambda \neq 0$. Since $\lim_n r_n = 0$, in order to show that $r(P_\lambda) < 1$, i.e., $\|P^n_\lambda\| < 1$ for some $n > 0$, it suffices to find $u' \in \mathbb{Z}^{2d}$ such that $c\|\lambda - 2\pi u'\| < 1$. This is possible at least for a multiple of λ: one can find $k \in \mathbb{N}$, $k \neq 0$, and $u' \in \mathbb{Z}^{2d}$ such that $\|k\lambda - 2\pi u'\| < c^{-1}$.

Now, if $r(P_\lambda) = 1$, one has also from Corollary 3.10 that, for any $k \in \mathbb{Z}$, $r(P_{k\lambda}) = 1$. ¿From above this is impossible; hence $r(P_\lambda) < 1$. □

4.3. **Random walks on coverings.** Let G be a Lie group, H a closed subgroup such that G/H has a G-invariant measure m. If μ is a probability measure on G, we consider the random walk on G/H defined by μ, and the corresponding skew product $\tilde{\sigma}$ on $G^{\mathbb{Z}} \times G/H$ endowed with the measure $\mu^{\otimes \mathbb{Z}} \times m$. Then one can ask for the ergodicity of such a skew product and its stochastic properties. If H is a normal subgroup of another group $L \subset G$ such that G/L is compact, G/H is fibred over G/L and one can use harmonic analysis on G/L and H/L.

A special case of Proposition 4.4 below corresponds to the abelian coverings of compact Riemann surfaces of genus $g \ge 2$. In this case, H is a subgroup Δ' of a cocompact lattice Δ in $\mathrm{SL}(2, \mathbb{R})$ and G/Δ' can be seen as the unit tangent bundle of the covering.

Proposition 4.4. *Let G be a simple non compact real Lie group of real rank 1, μ a symmetric probability measure with finite support $A \subset G$ such that the closed subgroup G_μ generated by A is non amenable. Let Δ be a co-compact lattice in G, Δ' a normal subgroup such that $\Delta/\Delta' = \mathbb{Z}^d$, m the Haar measure on G/Δ'.*

Let $\tilde{\sigma}$ be the extended shift on $\Omega \times G/\Delta'$ defined by $\tilde{\sigma}(\omega, y) = (\sigma\omega, a_1(\omega)y)$ and write $\Sigma_n(\omega) = a_n(\omega) \ldots a_1(\omega) \in G$.

If $d \leq 2$, $\tilde{\sigma}$ is ergodic with respect to $\mu^{\otimes\mathbb{Z}} \times m$. If $d \geq 3$, we have $\mu^{\otimes\mathbb{Z}} \times m$-a.e. $\lim_n \Sigma_n(\omega)y = +\infty$.

Proof. Since Δ' is normal in Δ, the group $\Lambda = \Delta/\Delta' \simeq \mathbb{Z}^d$ acts by right translations on G/Δ' and this action of Λ commutes with the left action of G.

The G-space G/Δ' can be written as $X \times \Lambda$ where $X \subset G/\Delta'$ is a Borel relatively compact fundamental domain of Λ in G/Δ'. We will denote by $\overline{y}$ the projection of $y \in G/\Delta'$ on X identified with G/Δ, by $(g, x) \to g.x$ the natural action of $g \in G$ on an element x of the fundamental domain X, and by $\overline{m}$ the Haar measure on G/Δ,.

Let $z(y)$ be the Λ-valued Borel function on G/Δ' defined by $y = \overline{y}z(y)$. Then the G-action on $X \times \Lambda$ can be written as $g(x, t) = (g.x, t + z(gx))$ where the group $\Lambda = \mathbb{Z}^d$ is written additively.

For $g \in G$ and $x \in X$, writing $Z(g, x) := z(gx)$, we obtain a cocycle:

$$Z(g_2g_1, x) = Z(g_2, g_1.x) + Z(g_1, x).$$

Since G is simple and G_μ is non amenable, we know ([11], Theorem 6.11) that the convolution operator Π_0 on $X = G/\Delta$ defined by μ has a spectral radius $r(\Pi_0) < 1$ on $L_0^2(X)$. On the other hand, if for any $a \in \text{supp}(\mu)$, $x \in X$, we write $c_a(x) = z(ax) = Z(a, x)$ and $\tilde{a}(x, t) = (a.x, t + c_a(x))$, we are in the situation of Section 3. Here the group Γ is the subgroup of G generated by the support of μ.

In order to verify this, we observe that, since X is relatively compact and $\text{supp}(\mu)$ is finite, the functions $c_a(x)$ are uniformly bounded. Furthermore, the cocycle relation for $Z(g, x)$ gives for any $g \in G$, $x \in X$: $Z(g^{-1}, x) + Z(g, g^{-1}.x) = 0$; hence, using the invariance of the measure, we have $\int (Z(g^{-1}, x) + Z(g, x))\, d\overline{m}(x) = 0$. Since μ is symmetric, we have the centering condition: $\int c_a(x)\, d\overline{m}(x)\ d\mu(a) = 0$.

For any character $\lambda \in \Lambda^*$, any $\varphi \in L^2(X)$, the formula $\rho_\lambda(g)\varphi(x) = e^{i\langle\lambda, z(g^{-1}x)\rangle}$ $\varphi(g^{-1}.x)$ defines a unitary one-dimensional representation of G, hence of the group generated by $\text{supp}(\mu)$, since $Z(g, x)$ satisfies the cocycle relation.

Hence, using Theorem 3.9 and Theorem 3.6, the proof will be finished if we show that $r(\rho_\lambda(\mu)) < 1$, for $\lambda \neq 0$.

Since G_μ is non amenable and G is simple, the result will follow from Theorem C, part 2 of [34], if we can show that ρ_λ does not contain weakly the representation Id_G. By definition, ρ_λ is the induced representation to G of the representation λ_Δ of Δ defined by the character λ. Clearly, if $\lambda \neq 0$, λ_Δ does not contain weakly Id_Δ. Since G/Δ has a finite G-invariant measure, it follows from Proposition 1.11b, p. 113 of [27] that ρ_λ does not contain weakly Id_G. ◻

4.4. **Random walks on motion groups.** Let G be the motion group $SU(d) \ltimes \mathbb{C}^d$, $d \geq 2$. Write $X = SU(d)$, ν for the Haar measure on X, $V = \mathbb{C}^d$. We identify a vector in V with the corresponding translation in G and we write $G = X\,V$. Let $\Gamma \subset SU(d)$ be a dense subgroup with property (SG) and A a finite generating set of Γ. As mentioned is Section 2, such groups exists if $d \geq 2$. To each $a \in A$ we associate $\tilde{a} \in G$ with $\tilde{a} = a\,\tau_a$, where $\tau_a \in V$. We consider a probability measure μ on A with $\text{supp}(\mu) = A$ and we denote by $\tilde{\mu}$ its push-forward on $\tilde{A} := \{\tilde{a}, a \in A\}$.

In contrast to the above examples the main role here will be played by $\tilde{\Gamma}$, the subgroup of G generated by $\tilde{A}$. Let us consider the convolutions $\tilde{\mu}^n, n \in \mathbb{N}$, on G and the natural affine action of G on V. We will use the following lemma.

Lemma 4.5. *Let H be a closed subgroup of $G = SU(d) \ltimes \mathbb{C}^d$, $d \geq 2$, such that $H \cap \mathbb{C}^d = \{0\}$ and its projection on $SU(d)$ is dense. Then H is conjugate to $SU(d)$.*

Proof. Let π be the projection of G onto $SU(d)$. Observe that $\pi(H)$ is a Lie subgroup of $SU(d)$ isomorphic to H. Also $\pi(H)$ contains a finitely generated countable subgroup Δ which is dense in $\pi(H)$, hence in $SU(d)$. Then Δ is non amenable since otherwise, using [35], Δ would have a polycyclic subgroup Δ_0 with finite index. Then the closure of Δ_0 would be solvable and equal to $SU(d)$, which is impossible since $d \geq 2$.

Let H_0 be the connected component of identity in H and observe that $\pi(H_0)$ is normal in $\pi(H)$. It follows that the Lie algebra of $\pi(H_0)$ is invariant under the adjoint action of $\pi(H)$, hence invariant under the action of its closure $SU(d)$. Then, using the exponential map, we see that $\pi(H_0)$ is a normal Lie subgroup of $SU(d)$.

Since $SU(d)$ is a simple Lie group, we get $\pi(H_0) = \{e\}$ or $\pi(H_0) = SU(d)$. In the first case, H would be a discrete subgroup of G, hence amenable like G. This imply that $\pi^{-1}(\Delta) \subset H$ would be amenable. Hence Δ itself would be amenable which is a contradiction. Hence $\pi(H) = SU(d)$ and π is an isomorphism of H onto $SU(d)$. In particular H is compact and its affine action on V has a fixed point $\tau \in V$. Hence $\tau^{-1} H \tau = SU(d)$. ☐

For $d = 1$ and μ a probability measure on $\tilde{A} \subset SU(1) \ltimes \mathbb{C}$, equidistribution properties of the natural random walk on $\mathbb{C}$ are known (see for instance [22], [14], [37]). For $d > 1$, the corresponding problem was posed by G. A. Margulis. Here we have the following result:

Theorem 4.6. *Assume that $\Gamma \subset SU(d)$ is such that the natural representation of Γ in $L_0^2(SU(d))$ does not contain weakly Id_Γ and the affine action of $\tilde{A}$ on V has no fixed point. Then there exists $c > 0$ such that for any continuous function f with compact support on G, $\lim_n \tilde{\mu}^n(f)\, n^d = c(\nu \otimes \ell)(f)$. In particular, for any f, f' continuous non negative functions on G with compact support, we have:*

$$\lim_n \frac{\tilde{\mu}^n(f)}{\tilde{\mu}^n(f')} = \frac{\int f(g)\, dg}{\int f'(g)\, dg}.$$

Furthermore the convolution equation $\tilde{\mu} * f = f$ on G, with $f \in L^\infty(\nu \otimes \ell)$, has only constant solutions.

Proof. We will use the results of Section 3; the link with Section 3 is as follows. The maps $\tilde{a}$ on $X \times V$ are defined here as left multiplication on $G = XV$ by $a\tau_a$:

$$\tilde{a}(x, v) = \tilde{a}(xv) = (ax, v + x^{-1}(\tau_a)),$$

where $x^{-1}(\tau_a)$ is the vector obtained from τ_a by the linear action of x.

Hence the action of A on X is by left multiplication on the group $SU(d)$ and $c_a(x) = x^{-1}(\tau_a)$. The centering condition $\int c_a(x)\, d\mu(a)\, d\nu(x) = 0$ is valid here, since it reduces to $\int x^{-1}(\tau_a)\, d\nu(x) = 0$, which is a consequence of the fact that this integral is the barycenter of the sphere $SU(d)\tau_a$ of center 0 and radius $\|\tau_a\|$, hence is 0.

Then the action of $\tilde{\Gamma} \subset G$ on $X \times V$ is by left multiplication on $G = XV$. This action is part of the action of G on itself by left translation.

Let us fix some notations. For $x \in X$, $v \in V$, with the above notations, $x(v)$ corresponds to the element xvx^{-1} of G. We observe that, if $g = x_g \tau_g$ and $h = x_h \tau_h$, then $x_{gh} = x_g x_h$, $\tau_{gh} = x_h^{-1}(\tau_g) + \tau_h$.

Therefore, with the action of G on X is given by $(g,x) \to x_g x$, the function $(g,x) \to x^{-1}(\tau_g)$ is a V-valued cocycle on $G \times X$, i.e. $x^{-1}(\tau_{gh}) = (x_h x)^{-1}(\tau_g) + x^{-1}(\tau_h)$. It follows that for $\tilde\gamma \in \tilde\Gamma$, $c(\tilde\gamma, x)$ as defined in Section 3 is equal to $x^{-1}(\tau_{\tilde\gamma})$ and is the restriction to $\tilde\Gamma \times X$ of the cocycle on $G \times X$ given by $c(g,x) = x^{-1}(\tau_g)$.

We show now that the closure H of $\tilde\Gamma$ is equal to G. We observe that $H \cap V$ is a normal subgroup of H and the action by conjugation of G on V reduces to the linear action of G.

Since Γ is dense in $SU(d)$ and $W = H \cap V$ is Γ-invariant, W is a closed $SU(d)$-invariant subgroup of V. Hence $W = \{0\}$ or V.

Suppose we are in the first case. Then the projection $H \to SU(d)$ is injective. In particular $\tilde\Gamma$ is isomorphic to Γ. In connection with Section 3, we may observe that $c(\tilde\gamma, x) = x^{-1}(\tau_{\tilde\gamma})$ defines also a cocycle on $\Gamma \times X$ since $\tau_{\tilde\gamma}$ depends only on γ; hence $c(\tilde\gamma, x) = c(\gamma, x)$.

The existence of a fixed point for the affine action of H on V, as shown in the previous lemma, contradicts the hypothesis on $\tilde A$, hence $W = V$. Since the projection of H on $SU(d)$ is dense, we get $H = G$.

Now we apply Theorem 3.6, part 1b). For this we have to verify (AP). If (AP) is not valid, there exists $(\lambda, \theta) \in V \times \mathbb{R}$, $\lambda \neq 0$, and $d(x)$ with $|d(x)| = 1$, such that

$$e^{i\langle \lambda, c(a,x)\rangle} = e^{i\theta} d(xa)/d(x), \ \forall a \in A.$$

As observed above, $c(a,x)$ extends to G as the cocycle $c(g,x)$ which is equal to $x^{-1}(\tau_g)$ on $(g,x) = (x_g \tau_g, x)$. Then we have $e^{i\theta} = e^{i\langle \lambda, c(a,x)\rangle} d(x)/d(a.x)$ and the right hand side is the restriction to $\tilde A \times X$ of the cocycle on $G \times X$

$$c_\lambda(g,x) = e^{i\langle \lambda, c(g,x)\rangle} d(x)/d(x_g x).$$

This cocycle takes values $e^{i\theta}$ on $\tilde A$, hence its values are also independent of x on the group $\tilde\Gamma$. Since $\tilde\Gamma$ is dense in G and c_λ is measurable on $G \times X$, using the L^2 continuity of the translation, it is also independent of x on G, hence it defines a character on G.

Since G has no non trivial character we get $e^{i\theta} = 1$. Then we have, for any $\tilde\gamma \in \tilde\Gamma$ with $\tilde\gamma = \gamma \tau_{\tilde\gamma}$ and a.e. $x \in X$, $e^{i\langle \lambda, x^{-1}(\tau_{\tilde\gamma})\rangle} = d(\gamma x)/d(x)$. This means that the function on G defined by $\psi(xv) = e^{-i\langle \lambda, v\rangle} d(x)$ is invariant by left translation by any element $\tilde\gamma \in \tilde\Gamma$. Since $\tilde\Gamma$ is dense in G, hence ergodic on G, ψ is constant, i.e. $\lambda = 0$, $d = 1$. It follows that (AP) is valid. Hence the result.

Since (AP) is valid, the last assertion is a consequence of 3) in Theorem 3.6. ◻

There exists various possibilities for the geometry of the subgroup $\tilde\Gamma$ inside G, as the following proposition shows.

Proposition 4.7. *With the above notations, assume that the finite set $A \subset SU(d)$ generates a dense subgroup Γ and the affine action of $\hat A$ on V has no fixed point.*

1) If Γ has property (T), then $\tilde\Gamma \cap V$ is dense in V.

2) If Γ is a free group, then $\tilde\Gamma \cap V = \{0\}$, so $\tilde\Gamma$ is a dense subgroup of G isomorphic to Γ.

Proof. 1) We show using arguments as in the proof of Theorem 4.6 that $\tilde\Gamma$ is dense in G. We observe that $\tilde\Gamma \cap V$ is a normal subgroup of $\tilde\Gamma$ and the action by conjugation of G on V reduces to the linear action of $SU(d)$.

Since Γ is dense in $SU(d)$ and $\tilde\Gamma \cap V$ is Γ-invariant, its closure W is a closed $SU(d)$-invariant subgroup of V. Hence $W = \{0\}$ or V.

Suppose $W = \{0\}$. Then the projection $\tilde{\Gamma} \to \Gamma$ is injective, hence $\tilde{\Gamma}$ is isomorphic to Γ and has property (T). We have also $c(\tilde{\gamma}, x) = x^{-1}(\tau_{\tilde{\gamma}}) = x^{-1}(\tau_\gamma) = c(\gamma, x)$. Then the cocycle $c(\tilde{\gamma}, x)$ from $\tilde{\Gamma} \times X$ to the vector group V is trivial (See Zimmer, p. 162), hence $c(\tilde{\gamma}, x) = \varphi(\gamma x) - \varphi(x)$ for some $\varphi \in L^2(X)$.

Also, from above, $\tau_\gamma = x(c(\tilde{\gamma}, x))$ does not depend on x. Then, for every $\gamma \in \Gamma$,

$$\tau_\gamma = x(\varphi(\gamma x) - \varphi(x)) = \int x(\varphi(\gamma x) - \varphi(x))\, d\nu(x) = \gamma^{-1}(w) - w,$$

with $w := \int x(\varphi(x))\, d\nu(x) = \int \gamma x(\varphi(\gamma x))\, d\nu(x)$.

It follows, for the affine action of $\tilde{\gamma}$ on V: $\tilde{\gamma} w = \gamma(w + \tau_\gamma) = \gamma(\gamma^{-1}(w)) = w$. This contradicts the hypothesis on $\tilde{A}$, hence $\tilde{\Gamma} \cap V = \{0\}$ is not valid. Therefore $\tilde{\Gamma} \cap V$ is dense in V. The fact that $\tilde{\Gamma}$ is dense in G follows, but was already proved in Theorem 4.6.

2) We denote by π the natural projection of G onto $SU(d)$, and we observe that $\pi(\tilde{a}) = a$ for any $a \in A$; hence $\pi(\tilde{\Gamma}) = \Gamma$. Since Γ is free it follows that the restriction of π to $\tilde{\Gamma}$ is an isomorphism of $\tilde{\Gamma}$ onto Γ.

In particular, since $\pi(V) = \{0\}$ we have $\tilde{\Gamma} \cap V = \{0\}$ and $\tilde{\Gamma}$ is free. The density of $\tilde{\Gamma}$ in G has been shown in the proof of Theorem 4.6. ☐

5. **Questions.** 1) In the situation of random walks in random scenery (example 4.1), with $\Gamma = \mathbb{Z}^m$, $m \geq 3$, is the local limit theorem for $S_n(\omega, x) \in V$ valid?

2) In the situation of motion groups (example 4.4), for $d \geq 2$, if $\tilde{\Gamma} \subset SU(d) \ltimes \mathbb{C}^d$ and $\Gamma \subset SU(d)$ is dense, is the local limit theorem for $S_n(\omega, x) \in V$ still valid?

What can be said about the equidistribution of the orbits of $\tilde{\Gamma}$ on V?

What are the bounded solutions of the equation $\tilde{\mu} * f = f$, $f \in L^\infty(\nu \otimes \ell)$, on G.

3) In the above considerations the maps $a \in A$ are chosen with probability $\mu(a)$ which does not depend on x and the product space is endowed with the product measure $\mathbb{P} = \mu^{\otimes \mathbb{N}^*}$. One can extends this framework by choosing the maps $a \in A$ according to a weight $\mu(x, a)$ depending on $x \in X$ and consider the corresponding Markovian model. One can also replace the shift invariant measure $\mathbb{P}$ by a Gibbs measure.

A question is then the validity of the results obtained above in these more general situations.

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