# COCYCLES OVER INTERVAL EXCHANGE TRANSFORMATIONS AND MULTIVALUED HAMILTONIAN FLOWS

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Abstract. We consider interval exchange transformations of periodic type and construct different classes of recurrent ergodic cocycles of dimension $\geq 1$ over this special class of IETs. Then using Poincaré sections we apply this construction to obtain recurrence and ergodicity for some smooth flows on non-compact manifolds which are extensions of multivalued Hamiltonian flows on compact surfaces.

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## 1. Introduction

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel probability space and $G$ be a locally compact abelian group with identity element denoted by 0. We will consider essentially the case $G = \mathbb{R}^\ell$, for $\ell \geq 1$.

Each measurable function $\varphi : X \to G$ determines a *cocycle* $\varphi^{(\,\cdot\,)} : \mathbb{Z} \times X \to G$ for $T$ by the formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x), & \text{if} \quad n > 0 \\ 0, & \text{if} \quad n = 0, \\ -(\varphi(T^n x) + \varphi(T^{n+1}x) + \ldots + \varphi(T^{-1}x)), & \text{if} \quad n < 0. \end{cases}$$

We consider the associated *skew product*

$$\begin{aligned} T_\varphi : (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times m_G) \quad &\to \quad (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times m_G), \\ T_\varphi(x, g) \quad &= \quad (Tx, g + \varphi(x)), \end{aligned} \tag{1.1}$$

where $\mathcal{B}_G$ denotes the $\sigma$-algebra of Borel subsets and $m_G$ the Haar measure of $G$.

The cocycle $(\varphi^{(\,\cdot\,)})$ can be viewed as a *"stationary" walk* in $G$ over the dynamical system $(X, \mu, T)$. We say that it is *recurrent* if $(\varphi^{(n)}(x))$ returns for a.e. $x$ infinitely often in any neighborhood of the identity element. The transformation $T_\varphi$ is then conservative for the invariant $\sigma$-finite measure $\mu \times m_G$. If moreover the system $(X \times G, \mu \times m_G, T_\varphi)$ is *ergodic*, we say that the cocycle $\varphi^{(\,\cdot\,)}$ is ergodic. For simplicity, the expression "cocycle $\varphi$" refers to the cocycle $(\varphi^{(\,\cdot\,)})$ generated by $\varphi$ over the dynamical system $(X, \mathcal{B}, \mu, T)$.

A problem is the construction of recurrent ergodic cocycles defined over a given dynamical system by regular functions $\varphi$ with values in $\mathbb{R}^\ell$. There is an important literature on skew products over an irrational rotation on the circle, and several classes of ergodic cocycles with values in $\mathbb{R}$ or $\mathbb{R}^\ell$ are known in that case (see [23], [25] and [26] for some classes of ergodic piecewise absolutely continuous non-continuous $\mathbb{R}$-cocycles, [16] for examples of ergodic cocycles with values in a nilpotent group, [7] for ergodic cocycles in $\mathbb{Z}^2$ associated to special directional rectangular billiard flows in the plane).

Skew products appear in a natural way in the study of the billiard flow in the plane with $\mathbb{Z}^2$ periodically distributed obstacles. For instance when the obstacles are rectangles, they can be modeled as skew products over interval exchange transformations (abbreviated as IETs). Recurrence and ergodicity of these models are mainly open questions. Nevertheless a first step is the construction of recurrent ergodic cocycles over some classes of IETs (see also a recent paper by P. Hubert and B. Weiss [17] for cocycles associated to non-compact translation surfaces).

For the rotations on the circle, a special class consists in the rotations with bounded partial quotients. For IETs, it is natural to consider the so-called interval exchange transformations of periodic type. The aim of this paper is to construct different classes of recurrent ergodic cocycles over IETs in this special class.

This is done in Sections 3, 4, and 5. In Section 2 we recall basic facts about IETs of periodic type, as well as from the ergodic theory of cocycles. In the appendix proofs of the needed results on the growth of cocycles of bounded variation (abbreviated as BV cocycles) are given, mainly adapted from [24].

In Sections 6 and 7 we present smooth models for recurrent and ergodic systems based on the previous sections. We deal with a class of smooth flows on non-compact manifolds which are extensions of multivalued Hamiltonian flows on compact surfaces of higher genus. These flows have Poincaré sections for which the first recurrence map is isomorphic to a skew product of an IET and a BV cocycle. This allows us to prove a sufficient condition for recurrence and ergodicity (see Section 6) whenever the IET is of periodic type. In Section 7 we show how to construct explicit non-compact ergodic extensions of some Hamiltonian flows.

## 2. Preliminaries

### 2.1. **Interval exchange transformations.**

In this subsection, we recall standard facts on IET's, with the presentation and notations from [32] and [33]. Let $\mathcal{A}$ be a $d$-element alphabet and let $\pi=(\pi_0,\pi_1)$ be a pair of bijections $\pi_\varepsilon:\mathcal{A}\to\{1,\ldots,d\}$ for $\varepsilon=0,1$. Denote by $\mathcal{S}^0_{\mathcal{A}}$ the subset of irreducible pairs, i.e. such that $\pi_1\circ\pi_0^{-1}\{1,\ldots,k\}\neq\{1,\ldots,k\}$ for $1\leq k<d$. We will denote by $\pi_d^{sym}$ any pair $(\pi_0,\pi_1)$ such that $\pi_1\circ\pi_0^{-1}(j)=d+1-j$ for $1\leq j\leq d$.

Let us consider $\lambda=(\lambda_\alpha)_{\alpha\in\mathcal{A}}\in\mathbb{R}_+^{\mathcal{A}}$, where $\mathbb{R}_+=(0,+\infty)$. Set

$$|\lambda|=\sum_{\alpha\in\mathcal{A}}\lambda_\alpha,\quad I=[0,|\lambda|)$$

and

$$I_\alpha=[l_\alpha,r_\alpha),\text{ where } l_\alpha=\sum_{\pi_0(\beta)<\pi_0(\alpha)}\lambda_\beta,\quad r_\alpha=\sum_{\pi_0(\beta)\leq\pi_0(\alpha)}\lambda_\beta.$$

Then $|I_\alpha|=\lambda_\alpha$. Denote by $\Omega_\pi$ the matrix $[\Omega_{\alpha\,\beta}]_{\alpha,\beta\in\mathcal{A}}$ given by

$$\Omega_{\alpha\,\beta}=\begin{cases}+1 & \text{if } \pi_1(\alpha)>\pi_1(\beta)\text{ and }\pi_0(\alpha)<\pi_0(\beta),\\ -1 & \text{if } \pi_1(\alpha)<\pi_1(\beta)\text{ and }\pi_0(\alpha)>\pi_0(\beta),\\ 0 & \text{in all other cases.}\end{cases}$$

Given $(\pi,\lambda)\in\mathcal{S}^0_{\mathcal{A}}\times\mathbb{R}_+^{\mathcal{A}}$, let $T_{(\pi,\lambda)}:[0,|\lambda|)\to[0,|\lambda|)$ stand for the *interval exchange transformation* (IET) on $d$ intervals $I_\alpha$, $\alpha\in\mathcal{A}$, which are rearranged according to the permutation $\pi_1^{-1}\circ\pi_0$, i.e. $T_{(\pi,\lambda)}x=x+w_\alpha$ for $x\in I_\alpha$, where $w=\Omega_\pi\lambda$.

Note that for every $\alpha\in\mathcal{A}$ with $\pi_0(\alpha)\neq 1$ there exists $\beta\in\mathcal{A}$ such that $\pi_0(\beta)\neq d$ and $l_\alpha=r_\beta$. It follows that

$$\{l_\alpha:\alpha\in\mathcal{A},\ \pi_0(\alpha)\neq 1\}=\{r_\alpha:\alpha\in\mathcal{A},\ \pi_0(\alpha)\neq d\}. \tag{2.1}$$

By $\widehat{T}_{(\pi,\lambda)}:(0,|I|]\to(0,|I|]$ denote the exchange of the intervals $\widehat{I}_\alpha=(l_\alpha,r_\alpha]$, $\alpha\in\mathcal{A}$, i.e. $T_{(\pi,\lambda)}x=x+w_\alpha$ for $x\in\widehat{I}_\alpha$. Note that for every $\alpha\in\mathcal{A}$ with $\pi_1(\alpha)\neq 1$

there exists $\beta \in \mathcal{A}$ such that $\pi_1(\beta) \neq d$ and $T_{(\pi,\lambda)} l_\alpha = \widehat{T}_{(\pi,\lambda)} r_\beta$. It follows that

$$\{T_{(\pi,\lambda)} l_\alpha : \alpha \in \mathcal{A},\ \pi_1(\alpha) \neq 1\} = \{\widehat{T}_{(\pi,\lambda)} r_\alpha : \alpha \in \mathcal{A},\ \pi_1(\alpha) \neq d\}. \tag{2.2}$$

A pair $(\pi, \lambda)$ satisfies the *Keane condition* if $T^m_{(\pi,\lambda)} l_\alpha \neq l_\beta$ for all $m \geq 1$ and for all $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$.

Let $T = T_{(\pi,\lambda)}$, $(\pi, \lambda) \in \mathcal{S}^0_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_+$, be an IET satisfying Keane's condition. Then $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$. Let

$$\tilde{I} = \left[0, \max\left(l_{\pi_0^{-1}(d)}, l_{\pi_1^{-1}(d)}\right)\right)$$

and denote by $\mathcal{R}(T) = \tilde{T} : \tilde{I} \to \tilde{I}$ the first return map of $T$ to the interval $\tilde{I}$. Set

$$\varepsilon(\pi, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}, \\ 1 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}. \end{cases}$$

Let us consider a pair $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1) \in \mathcal{S}^0_{\mathcal{A}}$, where

$$\begin{aligned} \tilde{\pi}_\varepsilon(\alpha) &= \pi_\varepsilon(\alpha) \text{ for all } \alpha \in \mathcal{A} \text{ and} \\ \tilde{\pi}_{1-\varepsilon}(\alpha) &= \begin{cases} \pi_{1-\varepsilon}(\alpha) & \text{if} \quad \pi_{1-\varepsilon}(\alpha) \leq \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d), \\ \pi_{1-\varepsilon}(\alpha) + 1 & \text{if} \quad \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}\pi_\varepsilon^{-1}(d) + 1 & \text{if} \quad \pi_{1-\varepsilon}(\alpha) = d. \end{cases} \end{aligned}$$

As it was shown by Rauzy in [27], $\tilde{T}$ is also an IET on $d$-intervals

$$\tilde{T} = T_{(\tilde{\pi},\tilde{\lambda})} \text{ with } \tilde{\lambda} = \Theta^{-1}(\pi, \lambda)\lambda,$$

where

$$\Theta(T) = \Theta(\pi, \lambda) = I + E_{\pi_\varepsilon^{-1}(d)\,\pi_{1-\varepsilon}^{-1}(d)} \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}}).$$

Moreover,

$$\Theta^t(\pi, \lambda)\Omega_\pi \Theta(\pi, \lambda) = \Omega_{\tilde{\pi}}. \tag{2.3}$$

It follows that $\ker \Omega_\pi = \Theta(\pi, \lambda) \ker \Omega_{\tilde{\pi}}$. We have also $\Omega^t_\pi = -\Omega_\pi$. Thus taking $H_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}}) = \ker \Omega_\pi^\perp$, we get $H_{\tilde{\pi}} = \Theta^t(\pi, \lambda) H_\pi$. Moreover, $\dim H_\pi = 2g$ and $\dim \ker \Omega_\pi = \kappa - 1$, where $\kappa$ is the number of singularities and $g$ is the genus of the translation surface associated to $\pi$. For more details we refer the reader to [33].

The IET $\tilde{T}$ fulfills the Keane condition as well. Therefore we can iterate the renormalization procedure and generate a sequence of IETs $(T^{(n)})_{n\geq 0}$, where $T^{(n)} = \mathcal{R}^n(T)$ for $n \geq 0$. Denote by $\pi^{(n)} = (\pi_0^{(n)}, \pi_1^{(n)}) \in \mathcal{S}^0_{\mathcal{A}}$ the pair and by $\lambda^{(n)} = (\lambda^{(n)}_\alpha)_{\alpha \in \mathcal{A}}$ the vector which determines $T^{(n)}$. Then $T^{(n)}$ is the first return map of $T$ to the interval $I^{(n)} = [0, |\lambda^{(n)}|)$ and

$$\lambda = \Theta^{(n)}(T)\lambda^{(n)} \text{ with } \Theta^{(n)}(T) = \Theta(T) \cdot \Theta(T^{(1)}) \cdot \ldots \cdot \Theta(T^{(n-1)}).$$

## 2.2. **IETs of periodic type.**

*Definition* (see [29]). An IET $T$ is of *periodic type* if there exists $p > 0$ (called a *period of* $T$) such that $\Theta(T^{(n+p)}) = \Theta(T^{(n)})$ for every $n \geq 0$ and $\Theta^{(p)}(T)$ (called a *periodic matrix of* $T$ and denoted by $A$ in all that follows) has strictly positive entries.

*Remark* 2.1. Suppose that $T = T_{(\pi,\lambda)}$ is of periodic type. It follows that

$$\lambda = \Theta^{(pn)}(T)\lambda^{(pn)} = \Theta^{(p)}(T)^n\lambda^{(pn)} \in \Theta^{(p)}(T)^n\mathbb{R}^{\mathcal{A}},$$

and hence $\lambda$ belongs to $\bigcap_{n\geq 0}\Theta^{(p)}(T)^n\mathbb{R}^{\mathcal{A}}$ which is a one-dimensional convex cone (see [30]). Therefore $\lambda$ is a positive right Perron-Frobenius eigenvector of the matrix $\Theta^{(p)}(T)$. Since the set $\mathcal{S}^0_{\mathcal{A}}$ is finite, multiplying the period $p$ if necessary, we can assume that $\pi^{(p)} = \pi$. It follows that $(\pi^{(p)}, \lambda^{(p)}/|\lambda^{(p)}|) = (\pi, \lambda/|\lambda|)$ and $\rho := |\lambda|/|\lambda^{(p)}|$ is the Perron-Frobenius eigenvector of the matrix $\Theta^{(p)}(T)$. Recall that similar arguments to those above show that every IET of periodic type is uniquely ergodic.

A procedure giving an explicit construction of IETs of periodic type was introduced in [29]. The construction is based on choosing closed paths on the graph giving the Rauzy classes. Every IET of periodic type can be obtained this way.

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type and $p$ be a period such that $\pi^{(p)} = \pi$. Let $A = \Theta^{(p)}(T)$. By (2.3),

$$A^t\Omega_\pi A = \Omega_\pi \text{ and hence } \ker\Omega_\pi = A\ker\Omega_\pi \text{ and } H_\pi = A^tH_\pi.$$

Multiplying the period $p$ if necessary, we can assume that $A|_{\ker\Omega_\pi} = Id$ (see Appendix C for details). Denote by $Sp(A)$ the collection of complex eigenvalues of $A$, including multiplicities. Let us consider the collection of Lyapunov exponents $\log|\rho|$, $\rho \in Sp(A)$. It consists of the numbers

$$\theta_1 > \theta_2 \geq \theta_3 \geq \ldots \geq \theta_g \geq 0 = \ldots = 0 \geq -\theta_g \geq \ldots \geq -\theta_3 \geq -\theta_2 > -\theta_1,$$

where $2g = \dim H_\pi$ and 0 occurs with the multiplicity $\kappa - 1 = \dim\ker\Omega_\pi$ (see e.g. [35] and [36]). Moreover, $\rho_1 := \exp\theta_1$ is the Perron-Frobenius eigenvalue of $A$. We will use sometimes the symbol $\theta_i(T)$ instead of $\theta_i$ to emphasize that it is associated to $T$.

*Definition.* An IET of periodic type $T_{(\pi,\lambda)}$ has *non-degenerated spectrum* if $\theta_g > 0$.

## 2.3. **Growth of BV cocycles.**

The recurrence of a cocycle $\varphi$ with values in $\mathbb{R}^\ell$ is related to the growth of $\varphi^{(n)}$ when $n$ tends to $\infty$.

For an irrational rotation $T : x \to x+\alpha \bmod 1$ (this can be viewed as an exchange of 2 intervals), when $\varphi$ has a bounded variation, the growth of $\varphi^{(n)}$ is controlled by the Denjoy-Koksma inequality: if $\varphi$ is a zero mean function on $X = \mathbb{R}/\mathbb{Z}$ with bounded variation $\operatorname{Var}\varphi$, and $(q_n)$ the denominators (of the convergents) given by the continued fraction expansion of $\alpha$, then the following inequality holds:

$$|\sum_{j=0}^{q_n-1}\varphi(x+j\alpha)| \leq \operatorname{Var}\varphi, \forall x \in X. \tag{2.4}$$

This inequality implies obviously recurrence of the cocycle $\varphi^{(\,\cdot\,)}$ and if $\alpha$ has bounded partial quotients (we say for brevity *bpq*) $\sum_{j=0}^{n-1}\varphi(x+j\alpha) = O(\log n)$ uniformly in $x \in X$.

It is much more difficult to get a precise upper bound for the growth of a cocycle over an IET. The following theorem (proved in Appendix A) gives for an IET of periodic type a control on the growth of a BV cocycle in terms of the Lyapunov exponents of the matrix $A$.

**Theorem 2.2.** *Suppose that $T_{(\pi,\lambda)} : I \to I$ is an interval exchange transformation of periodic type, $0 \le \theta_2 < \theta_1$ are the two largest Lyapunov exponents, and $M$ is the maximal size of Jordan blocks in the Jordan decomposition of its periodic matrix $A$. Then there exists $C > 0$ such that*

$$\|\varphi^{(n)}\|_{\sup} \le C \cdot \log^{M+1} n \cdot n^{\theta_2/\theta_1} \cdot \operatorname{Var} \varphi$$

*for every function $\varphi : I \to \mathbb{R}$ of bounded variation with zero mean and for each natural $n$.* □

For our purpose, this inequality is useful when $\theta_2(T)/\theta_1(T)$ is small. In Appendix B we will give examples with arbitrary small values of this ratio.

2.4. **Recurrence, essential values, and ergodicity of cocycles.**
In this subsection we recall some general facts about cocycles. For relevant background material concerning skew products and infinite measure-preserving dynamical systems, we refer the reader to [28] and [1].

Denote by $\overline{G}$ the one point compactification of the group $G$. An element $g \in \overline{G}$ is said to be an *essential value* of $\varphi$, if for every open neighbourhood $V_g$ of $g$ in $\overline{G}$ and any set $B \in \mathcal{B}$, $\mu(B) > 0$, there exists $n \in \mathbb{Z}$ such that

$$\mu(B \cap T^{-n}B \cap \{x \in X : \varphi^{(n)}(x) \in V_g\}) > 0. \tag{2.5}$$

The set of essential values of $\varphi$ will be denoted by $\overline{E}(\varphi)$. The set of finite essential values $E(\varphi) := G \cap \overline{E}(\varphi)$ is a closed subgroup of $G$. We recall below some properties of $\overline{E}(\varphi)$ (see [28]).

Two cocycles $\varphi, \psi : X \to G$ are called *cohomologous* for $T$ if there exists a measurable function $g : X \to G$ such that $\varphi = \psi + g - g \circ T$. The corresponding skew products $T_\varphi$ and $T_\psi$ are then measure-theoretically isomorphic. A cocycle $\varphi : X \to G$ is a *coboundary* if it is cohomologous to the zero cocycle.

If $\varphi$ and $\psi$ are cohomologous then $\overline{E}(\varphi) = \overline{E}(\psi)$. Moreover, $\varphi$ is a coboundary if and only if $\overline{E}(\varphi) = \{0\}$.

A cocycle $\varphi : X \to G$ is recurrent (as defined in the introduction) if and only if, for each open neighborhood $V_0$ of 0, (2.5) holds for some $n \neq 0$. This is equivalent to the conservativity of the skew product $T_\varphi$ (cf. [28]). Let $\varphi : X \to \mathbb{R}^\ell$ be an integrable function. If it is recurrent, then $\int_X \varphi \, d\mu = 0$; moreover, for $\ell = 1$ this condition is sufficient for recurrence when $T$ is ergodic.

The group $E(\varphi)$ coincides with the group of *periods* of $T_\varphi$-invariant functions i.e. the set of all $g_0 \in G$ such that, if $f : X \times G \to \mathbb{R}$ is a $T_\varphi$-invariant measurable function, then $f(x, g + g_0) = f(x, g)$ $\mu \times m_G$-a.e. In particular, $T_\varphi$ is ergodic if and only if $E(\varphi) = G$.

A simple sufficient condition of recurrence is the following:

**Proposition 2.3** (see Corollary 1.2 in [5])**.** *If $\varphi : X \to \mathbb{R}^\ell$ is a square integrable cocycle for an automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ such that $\|\varphi^{(n)}\|_{L^2(\mu)} = o(n^{1/\ell})$, then it is recurrent.* □

In view of Theorem 2.2, as a consequence we have the following.

**Corollary 2.4.** *If $T : I \to I$ is an IET of periodic type such that $\theta_2(T)/\theta_1(T) < 1/\ell$ for an integer $\ell \geq 1$, then every cocycle $\varphi : I \to \mathbb{R}^\ell$ over $T$ of bounded variation with zero mean is recurrent. If, for $j = 1, \dots, \ell$, $T_j : I^{(j)} \to I^{(j)}$ are interval exchange transformations of periodic type such that $\theta_2(T_j)/\theta_1(T_j) < 1/\ell$, then every "product" cocycle $\varphi = (\varphi_1, \dots, \varphi_\ell) : I^{(1)} \times \dots \times I^{(\ell)} \to \mathbb{R}^\ell$ of bounded variation with zero mean over $T_1 \times ... \times T_\ell$ is recurrent.* □

We continue these preliminaries by some useful observations for proving the ergodicity of cocycles. Let $(X, d)$ be a compact metric space. Let $\mathcal{B}$ stand for the $\sigma$–algebra of all Borel sets and let $\mu$ be a probability Borel measure on $X$. By $\chi_B$ we will denote the indicator function of a set $B$. Suppose that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic measure–preserving automorphism and there exist an increasing sequence of natural numbers $(q_n)$ and a sequence of Borel sets $(C_n)$ such that

$$\mu(C_n) \to \alpha > 0, \quad \mu(C_n \triangle T^{-1}C_n) \to 0 \quad \text{and} \quad \sup_{x \in C_n} d(x, T^{q_n}x) \to 0.$$

Assume that $G \subset \mathbb{R}^\ell$ for some $\ell \geq 1$. Let $\varphi : X \to G$ be a Borel integrable cocycle for $T$ with zero mean. Suppose that the sequence $(\int_{C_n} |\varphi^{(q_n)}(x)| d\mu(x))_{n \geq 1}$ is bounded. As the distributions

$$(\mu(C_n)^{-1}(\varphi^{(q_n)}|_{C_n})_*(\mu|_{C_n}), n \in \mathbb{N})$$

are uniformly tight, by passing to a further subsequence if necessary we can assume that they converge weakly to a probability Borel measure $P$ on $G$.

**Lemma 2.5.** *The topological support of the measure $P$ is included in the group $E(\varphi)$ of essential values of the cocycle $\varphi$.*

*Proof.* Suppose that $g \in \operatorname{supp}(P)$. Let $V_g$ be an open neighborhood of $g$. Let $\psi : G \to [0, 1]$ be a continuous function such that $\psi(g) = 1$ and $\psi(h) = 0$ for $h \in G \setminus V_g$. Thus $\int_G \psi(g)\, dP(g) > 0$. By Lemma 5 in [13], for every $B \in \mathcal{B}$ with $\mu(B) > 0$ we have

$$\begin{aligned} \mu(B \cap T^{-q_n}B \cap (\varphi^{(q_n)} \in V_g)) &\geq \int_{C_n} \psi\left(\varphi^{(q_n)}(x)\right) \chi_B(x)\chi_B(T^{q_n}x)\, d\mu(x) \\ &\to \alpha \int_X \int_G \psi(g)\chi_B(x)\, dP(g)\, d\mu(x) = \alpha\mu(B) \int_G \psi(g)\, dP(g) > 0, \end{aligned}$$

and hence $g \in E(\varphi)$. □

**Corollary 2.6** (see also [6])**.** *If $\varphi^{(q_n)}(x) = g_n$ for all $x \in C_n$ and $g_n \to g$, then $g \in E(\varphi)$.*

**Proposition 2.7** (see Proposition 3.8 in [28])**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism and let $\varphi : X \to G$ be a measurable cocycle for $T$. If $K \subset G$ is a compact set such that $K \cap E(\varphi) = \emptyset$, then there exists $B \in \mathcal{B}$ such that $\mu(B) > 0$ and*

$$\mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in K)) = 0 \text{ for every } n \in \mathbb{Z}.$$

**Lemma 2.8.** *Let $K \subset G$ be a compact set. If for every $B \in \mathcal{B}$ with $\mu(B) > 0$ and every neighborhood $V_0 \subset G$ of zero there exists $n \in \mathbb{Z}$ such that*

$$\mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in K + V_0)) > 0,$$

*then $K \cap E(\varphi) \neq \emptyset$. In particular, when $K = \{g, -g\}$, where $g$ is an element of $G$, then $g \in E(\varphi)$.*

*Proof.* Suppose that $K \cap E(\varphi) = \emptyset$. Since $K$ is compact and $E(\varphi)$ is closed, there exists a neighborhood $V_0$ of zero such that $\overline{V_0}$ is compact and $(K + \overline{V_0}) \cap E(\varphi) = \emptyset$. As $K + \overline{V_0}$ is also compact, by Proposition 2.7, there exists $B \in \mathcal{B}$ such that $\mu(B) > 0$ and

$$\mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in (K + \overline{V_0}))) = 0 \text{ for every } n \in \mathbb{Z},$$

contrary to assumption. The last claim is clear. □

Consider the quotient cocycle $\varphi^* : X \to G/E(\varphi)$ given by $\varphi^*(x) = \varphi(x) + E(\varphi)$. Then $E(\varphi^*) = \{0\}$. The cocycle $\varphi$ is called *regular* if $\overline{E}(\varphi^*) = \{0\}$ and *non–regular* if $\overline{E}(\varphi^*) = \{0, \infty\}$. Recall that if $\varphi$ is regular then it is cohomologous to a cocycle $\psi : X \to E(\varphi)$ such that $E(\psi) = E(\varphi)$.

**Lemma 2.9.** *If $H$ is a closed subgroup of $E(\varphi)$ such that the quotient cocycle $\varphi_H : X \to G/H$, $\varphi_H(x) = \varphi(x) + H$ is ergodic, then $\varphi : X \to G$ is ergodic as well.*

*Proof.* Let $f(x, g)$ be a measurable $T_\varphi$-invariant function. Then, since $H \subset E(\varphi)$, $f$ is $H$-invariant. Since $\varphi_H$ is ergodic, $f$ is constant. □

## 3. Ergodicity of piecewise linear cocycles

*Notations.* We denote by $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ the space of functions $\varphi : I^{(k)} \to \mathbb{R}$ such that the restriction $\varphi : I_\alpha^{(k)} \to \mathbb{R}$ is of bounded variation for every $\alpha \in \mathcal{A}$, and by $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ the subspace of functions in $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ with zero mean. We adopt the notation from [24]. The space $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ is equipped with the norm $\|\varphi\|_{\mathrm{BV}} = \|\varphi\|_{\sup} + \mathrm{Var}\,\varphi$, where

$$\mathrm{Var}\,\varphi = \sum_{\alpha\in\mathcal{A}} \mathrm{Var}\,\varphi|_{I_\alpha^{(k)}}.$$

For $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $x \in I$, $\varphi_+(x)$ and $\varphi_-(x)$ denote the right-handed and left-handed limit of $\varphi$ at $x$ respectively. We denote by $\mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the space of functions $\varphi : I \to \mathbb{R}$ which are absolutely continuous on each $I_\alpha$, $\alpha \in \mathcal{A}$ and such that $\varphi' \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. For $\varphi \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ let

$$s(\varphi) = \int_I \varphi'(x)\,dx = \sum_{\alpha\in\mathcal{A}} (\varphi_-(r_\alpha) - \varphi_+(l_\alpha)).$$

We denote by $\mathrm{BV}_*^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the subspace of functions $\varphi \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ for which $s(\varphi) = 0$, and by $\mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the set of piecewise linear (with constant slope) functions $\varphi : I \to \mathbb{R}$ such that $\varphi(x) = sx + c_\alpha$ for $x \in I_\alpha$.

**Proposition 3.1** (see [24])**.** *If $T : I \to I$ satisfies a Roth type condition, then each cocycle $\varphi \in \mathrm{BV}_*^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ for $T$ is cohomologous to a cocycle which is constant* *on each interval $I_\alpha$, $\alpha \in \mathcal{A}$. Moreover, the set of IETs satisfying this Roth type condition has full measure and contains all IETs of periodic type.*

As a consequence of Proposition 3.1 we have the following.

**Lemma 3.2.** *If $T : I \to I$ is of periodic type, then each cocycle $\varphi \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is cohomologous to a cocycle $\varphi_{pl} \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $s(\varphi_{pl}) = s(\varphi)$.*

### 3.1. **Piecewise linear cocycles.**

Now we will focus on the case where the slope of a piecewise linear cocycle is non-zero and show ergodicity. We will need an information on the distribution of discontinuities of $\varphi^{(n)}$.

Let $T : I \to I$ be an arbitrary IET satisfying Keane's condition. Denote by $\mu$ the Lebesgue measure on $I$. Each finite subset $D \subset I$ determines a partition $\mathcal{P}(D)$ of $I$ into left-closed and right-open intervals. Denote by $\min \mathcal{P}(D)$ and $\max \mathcal{P}(D)$ the length of the shortest and the longest interval of the partition $\mathcal{P}(D)$ respectively. For every $n \geq 0$ let $\mathcal{P}_n(T)$ stand for the partition given by the subset $\{T^{-k} l_\alpha : \alpha \in \mathcal{A}, 0 \leq k < n\}$. Then $T^n$ is a translation on each interval of the partition $\mathcal{P}_n(T)$. The following result shows that the discontinuities for iterations of IETs of periodic type are well distributed.

**Proposition 3.3** (see [22]). *For every IET $T$ of periodic type there exists $c \geq 1$ such that for every $n \geq 1$ we have*

$$\frac{1}{cn} \leq \min \mathcal{P}_n(T) \leq \max \mathcal{P}_n(T) \leq \frac{c}{n} \tag{3.1}$$

We begin by a preliminary result which will be proved later in a general version (see Theorem 3.5 and 3.9 for $\ell = 1$).

**Theorem 3.4.** *Let $T : I \to I$ be an IET of periodic type. If $\varphi \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi) \neq 0$, then the skew product $T_\varphi$ is ergodic.*

Now we consider cocycles taking values in $\mathbb{R}^\ell$, $\ell \geq 1$. Suppose that $\varphi : I \to \mathbb{R}^\ell$ is a piecewise linear cocycle with zero mean such that the slope $s(\varphi) \in \mathbb{R}^\ell$ is non-zero. Then, by an appropriate choice of coordinates, we obtain $s(\varphi_1) \neq 0$ and $s(\varphi_2) = 0$, where $\varphi = (\varphi_1, \varphi_2)$ and $\varphi_1 : I \to \mathbb{R}$, $\varphi_2 : I \to \mathbb{R}^{\ell-1}$. Thus $\varphi_2$ is piecewise constant and, roughly speaking, the ergodicity of $\varphi_2$ implies the ergodicity of $\varphi$. The ergodicity of piecewise constant cocycles will be studied in Sections 4 and 5.

**Theorem 3.5.** *Suppose that $T : I \to I$ is an IET of periodic type such that $\theta_2(T)/\theta_1(T) < 1/\ell$. Let $\varphi_1 \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R})$, $\varphi_2 \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$ be piecewise linear cocycles with zero mean such that $s(\varphi_1) \neq 0$ and $s(\varphi_2) = 0$. If the cocycle $\varphi_2 : I \to \mathbb{R}^{\ell-1}$ is ergodic, then the cocycle $\varphi = (\varphi_1, \varphi_2) : I \to \mathbb{R}^\ell$ is ergodic as well.*

*Proof.* Without loss of generality we can assume that $s(\varphi_1) = 1$. It suffices to show that for every $0 < a < \frac{1}{4c}$, the pair $(a, 0)$ belongs to $E(\varphi_1, \varphi_2)$. Indeed this implies that $\mathbb{R} \times \{0\} \subset E(\varphi_1, \varphi_2)$, and since the cocycle $\varphi_2$ is ergodic, by Lemma 2.9, it follows that $(\varphi_1, \varphi_2) : I \to \mathbb{R}^\ell$ is ergodic as well.

Fix $0 < a < \frac{1}{4c}$, where $c$ is given by Proposition 3.3. By a density point argument, for every measurable $B \subset I$ with $\mu(B) > 0$ and every $\varepsilon \in (0, \frac{a}{2})$, there are $B' \subset B$

with $\mu(B')>0$ and $n_0\geq 1$ such that for $n\geq n_0$,

$$\mu\left(\left(x-\frac{c}{n},x+\frac{c}{n}\right)\setminus B\right)<\frac{\varepsilon}{n}\text{ for every }x\in B'. \tag{3.2}$$

Since $\theta_2(T)/\theta_1(T)<1/\ell$, by Corollary 2.4, $(\varphi_1,\varphi_2)$ is recurrent, and hence there exists $n\geq n_0$ such that

$$\mu(B'\cap T^{-n}B'\cap(|\varphi_1^{(n)}|<\varepsilon)\cap(\|\varphi_2^{(n)}\|<\varepsilon))>0.$$

Let $x_0\in I$ be such that $x_0,T^nx_0\in B'$, $|\varphi_1^{(n)}(x_0)|<\varepsilon$ and $\|\varphi_2^{(n)}(x_0)\|<\varepsilon$. Denote by $J(x_0)\subset I$ the interval of the partition $\mathcal{P}_n(T)$ which contains $x_0$. Then $\varphi_1^{(n)}$ is a linear function on $J(x_0)$ with slope $n$. Since $2\varepsilon<a<1/(2c)-2\varepsilon$ and $|J(x_0)|>1/(cn)$ (by (3.1)), there exists $y_0$ such that $(y_0-\varepsilon/n,y_0+\varepsilon/n)\subset J(x_0)$ and

$$|\varphi_1^{(n)}(y)|\in a+(-\varepsilon,\varepsilon)\text{ for all }y\in(y_0-\varepsilon/n,y_0+\varepsilon/n).$$

Since $\varphi_2^{(n)}$ is constant on $J(x_0)$, we have

$$\|\varphi_2^{(n)}(x)\|<\varepsilon\text{ for all }x\in(y_0-\varepsilon/n,y_0+\varepsilon/n).$$

Therefore

$$\begin{aligned}&\mu\left(B\cap T^{-n}B\cap(\varphi_1^{(n)}\in\{-a,a\}+(-\varepsilon,\varepsilon))\cap(\varphi_2^{(n)}\in(-\varepsilon,\varepsilon)^{\ell-1})\right)\\&\quad\geq\mu\left((y_0-\varepsilon/n,y_0+\varepsilon/n)\cap B\cap T^{-n}B\right).\end{aligned}\tag{3.3}$$

By (3.1) we have $|J(x_0)|<c/n$, and hence $J(x_0)\subset(x_0-c/n,x_0+c/n)$. Moreover, $T^nJ(x_0)$ is an interval such that $|T^nJ(x_0)|=|J(x_0)|<c/n$, so that

$$T^nJ(x_0)\subset\left(T^nx_0-\frac{c}{n},T^nx_0+\frac{c}{n}\right).$$

Since $x_0,T^nx_0\in B'$, by (3.2), $\mu(J(x_0)\setminus B)<\varepsilon/n$ and $\mu(T^nJ(x_0)\setminus B)<\varepsilon/n$. Therefore, $\mu(J(x_0)\setminus(B\cap T^{-n}B))<2\varepsilon/n$, and hence

$$\mu\left((y_0-\varepsilon/n,y_0+\varepsilon/n)\setminus(B\cap T^{-n}B)\right)<2\varepsilon/n.$$

Thus

$$\mu\left((y_0-\varepsilon/n,y_0+\varepsilon/n)\cap B\cap T^{-n}B\right)>0.$$

In view of (3.3), it follows that

$$\mu\left(B\cap T^{-n}B\cap(\varphi_1^{(n)}\in\{-a,a\}+(-\varepsilon,\varepsilon))\cap(\varphi_2^{(n)}\in(-\varepsilon,\varepsilon)^{\ell-1})\right)>0.$$

By Lemma 2.8, we conclude that $(a,0)\in E(\varphi_1,\varphi_2)$, which completes the proof. □

3.2. **Product cocycles.**
The method used in Theorem 3.4 allows us to prove the ergodicity for Cartesian products of certain skew products. As an example, first we apply this method for cocycles taking values in $\mathbb{Z}$ over irrational rotations on the circle. This will give a class of ergodic $\mathbb{Z}^2$-cocycles driven by 2-dimensional rotations

Let $T(x,y)=(x+\alpha_1,x+\alpha_2)$ be a 2-dimensional rotation and $\varphi$ be a zero mean function on $\mathbb{T}^2$ of the form $\varphi(x,y)=(\varphi_1(x),\varphi_2(y))$ with $\varphi_1$ and $\varphi_2$ BV functions. If $\alpha_1$ and $\alpha_2$ have bounded partial quotients, then (2.4) implies $\|\varphi^{(n)}\|_{\sup}=O(\log n)$, and therefore, by Proposition 2.3, the cocycle $\varphi$ is recurrent.

Consider the function $\varphi(x,y)=(2\cdot\chi_{[0,\frac12)}(x)-1,2\cdot\chi_{[0,\frac12)}(y)-1)$ or more generally assume that $\varphi_i$, $i=1,2$, are step functions in one variable with values in $\mathbb{Z}$. For

$i = 1, 2$, we denote by $D_i \subset \mathbb{T}$ the finite set of discontinuities of $\varphi_i$ and by $J_i \subset \mathbb{Z}$ the corresponding set of jumps of the functions $\varphi_i$. It defines a recurrent $\mathbb{Z}^2$-cocycle driven by a 2-dimensional rotation. A question is then the ergodicity (with respect to the measure $\mu \times m$ the product of the uniform measure on $\mathbb{T}^2$ by the counting measure on $\mathbb{Z}^2$) of the skew-product

$$T_\varphi : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{T}^2 \times \mathbb{Z}^2, \quad T_\varphi(x, y, \bar{n}) = (x + \alpha_1, y + \alpha_2, \bar{n} + \varphi(x, y)).$$

**Theorem 3.6.** *Let $\alpha_1$ and $\alpha_2$ be two rationally independent irrational bpq numbers, and let $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ be a function on the torus with step functions components $\varphi_i : \mathbb{T} \to \mathbb{Z}$, $i = 1, 2$, such that $D_1, D_2 \subset \mathbb{Q}$ and the sets of the jumps $J_1 \times \{0\}$, $\{0\} \times J_2$ generate $\mathbb{Z}^2$. Then the system $(\mathbb{T}^2 \times \mathbb{Z}^2, \mu \times m, T_\varphi)$ is ergodic.*

*Proof.* We have seen that the cocycle $\varphi^{(n)}$ is recurrent. We prove that the group of its finite essential values is $\mathbb{Z}^2$.

Let $n$ be a fixed integer and let $(\gamma^i_{n,k})_{k=1,\ldots,d_i n}$ be the ordered set of the $d_i n$ discontinuities of $\varphi_i^{(n)}$ in $[0, 1)$ (where $d_i := \#D_i$). In the sequence of denominators of $\alpha_i$, let $q^i_{r_i(n)}$ be such that $q^i_{r_i(n)} \leq n < q^i_{r_i(n)+1}$. We write simply $q^i_{r_i}$ for $q^i_{r_i(n)}$. As $\alpha_i$ is bpq, the ratio $q^i_{r_i+1}/q^i_{r_i}$ is bounded by a constant independent from $n$.

Since $\alpha_i$ is bpq and the discontinuity points of $\varphi_i$ are rational, the distances between consecutive discontinuities of $\varphi^{(n)}$ are of the same order: there are two positive constants $c_1, c_2$ such that

$$\frac{c_1}{n} \leq \gamma^i_{n,k+1} - \gamma^i_{n,k} \leq \frac{c_2}{n}, \ k = 1, \ldots, d_i n, \ i = 1, 2. \tag{3.4}$$

Recall that, for each $t \in D_i$ and each $0 \leq \ell < q_{r_i}$, there is (mod 1) a point $t - k\alpha_i$, $0 \leq k < q_{r_i}$ in each interval $[t + \ell/q^i_{r_i}, t + (\ell + 1)/q^i_{r_i}]$. Therefore, in each interval of length greater than $2/q^i_r$ and for each $t \in D_i$, there is at least one discontinuity of $\varphi_i^{(n)}$ of the form $t - k\alpha_i$, $0 \leq k < n$.

It implies that if we move a point $x$ on the unit interval by a displacement greater than $2/q^i_{r_i}$, we cross discontinuities of $\varphi_i^{(n)}$ corresponding to each different discontinuity $t \in D_i$ of $\varphi_i$.

For $x \in \mathbb{T}$, consider the interval $[\gamma^i_{n,k}, \gamma^i_{n,k+1})$ which contains $x$ and denote it by $I^i_n(x)$. The intervals $[\gamma^i_{n,k+\ell}, \gamma^i_{n,k+\ell+1})$, where $k + \ell$ is taken mod $d_1 n$, are denoted by $I^i_{n,\ell}(x)$. This gives two collections of rectangles

$$R^n_{k,\ell}(x, y) := I^1_{n,k}(x) \times I^2_{n,\ell}(y) \text{ and } \tilde{R}^n_{k,\ell}(x, y) := T^n R^n_{k,\ell}(T^{-n}(x, y))$$

for each $(x, y) \in \mathbb{T}^2$. By (3.4), we have

$$\mu(R^n_{k,\ell}(x, y)) \quad \text{and} \quad \mu(\tilde{R}^n_{k,\ell}(x, y)) \in \left[\frac{c_1^2}{n^2}, \frac{c_2^2}{n^2}\right]. \tag{3.5}$$

Let $M$ be a natural number such that $c_1 M > 1$. Then, by (3.4), the length of $\bigcup_{k=-M}^{M} I^i_{n,k}(x)$ is greater than $2/q^i_{r_i}$, $i = 1, 2$. Let $\delta > 0$ be such that $\delta c^2(2M+1)^2 < 1/2$ with $c = c_2^2/c_1^2$. Set

$$\begin{aligned} R^n_M(x, y) &:= \bigcup_{k=-M}^{k=M} \bigcup_{\ell=-M}^{\ell=M} R^n_{k,\ell}(x, y), \\ \tilde{R}^n_M(x, y) &:= T^n R^n_M(T^{-n}(x, y)) = \bigcup_{k=-M}^{k=M} \bigcup_{\ell=-M}^{\ell=M} \tilde{R}^n_{k,\ell}(x, y). \end{aligned}$$

In view of (3.4),

$$\frac{\text{length}(R^n_M(x,y))}{\text{width}(R^n_M(x,y))} \text{ and } \frac{\text{length}(\tilde{R}^n_M(x,y))}{\text{width}(\tilde{R}^n_M(x,y))} \in \left[\frac{c_1}{c_2}, \frac{c_2}{c_1}\right]. \tag{3.6}$$

The cocycle $\varphi^{(n)}$ has a constant value on each rectangle $R^n_{k,\ell}(x,y)$ and the difference between its value on $R^n_{k+1,\ell}(x,y)$ and $R^n_{k,\ell}(x,y)$ (resp. $R^n_{k,\ell+1}(x,y)$ and $R^n_{k,\ell}(x,y)$) belongs to $J_1 \times \{0\}$ (resp. $\{0\} \times J_2$). Denote by $\kappa^n_{k,\ell}(x,y)$ the value of $\varphi^{(n)}$ on $R^n_{k,\ell}(x,y)$. Since the length of $R^n_M(x,y)$ is greater than $2/q^1_{r_1}$ and the width of $R^n_M(x,y)$ is greater than $2/q^2_{r_2}$ we have

$$\begin{aligned} &\{\kappa^n_{k+1,\ell}(x,y) - \kappa^n_{k,\ell}(x,y) : -M \le k < M\} = J_1 \times \{0\}, \\ &\{\kappa^n_{k,\ell+1}(x,y) - \kappa^n_{k,\ell}(x,y) : -M \le l < M\} = \{0\} \times J_2. \end{aligned} \tag{3.7}$$

Let

$$K := \overbrace{(J_1 \cup \{0\} + \ldots + J_1 \cup \{0\})}^{M} \times \overbrace{(J_2 \cup \{0\} + \ldots + J_2 \cup \{0\})}^{M}.$$

Let $K_1$ be the subset of elements of $K$ which are not essential values of $\varphi$, and suppose $K_1 \neq \emptyset$. By Proposition 2.7, there exists $B \subset \mathbb{T}^2$ such that $\mu(B) > 0$ and

$$\mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in K_1)) = 0 \text{ for every } n \in \mathbb{Z}. \tag{3.8}$$

Since the areas of $R^n_M(x,y)$, $\tilde{R}^n_M(x,y)$ tend to 0 as $n \to \infty$ and the rectangles satisfy (3.6), by a density point argument, there is a Borel subset $B'$ of $B$ of positive measure and there is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ and $(x,y) \in B'$:

$$\mu(B \cap R^n_M(x,y)) \ge (1-\delta)\mu(R^n_M(x,y)),\ \mu(B \cap \tilde{R}^n_M(x,y)) \ge (1-\delta)\mu(\tilde{R}^n_M(x,y)).$$

By (3.5), the areas of the small rectangles being comparable, and hence

$$\mu(R^n_M(x,y)) \le (2M+1)^2 c^2 \mu(R^n_{k,\ell}(x,y)) \text{ for all } k,\ell \in [-M,M].$$

Therefore, by the choice of $\delta$, for each $(x,y) \in B'$ we have

$$\begin{aligned} \mu(B \cap R^n_{k,\ell}(x,y)) &\ge \mu(R^n_{k,\ell}(x,y)) - \mu(B^c \cap R^n_{k,\ell}(x,y)) \\ &\ge \mu(R^n_{k,\ell}(x,y)) - \mu(B^c \cap R^n_M(x,y)) \ge \mu(R^n_{k,\ell}(x,y)) - \delta\mu(R^n_M(x,y)) \\ &\ge \mu(R^n_{k,\ell}(x,y)) - \delta(2M+1)^2 c^2 \mu(R^n_{k,\ell}(x,y)) > \frac{1}{2}\mu(R^n_{k,\ell}(x,y)). \end{aligned}$$

In the same way, if $T^n(x,y) \in B'$, then $\mu(B \cap \tilde{R}^n_{k,\ell}(T^n(x,y))) > \frac{1}{2}\mu(\tilde{R}^n_{k,\ell}(T^n(x,y)))$. Since $\tilde{R}^n_{k,\ell}(T^n(x,y)) = T^n R^n_{k,\ell}(x,y)$, we have

$$\mu(T^{-n}B \cap R^n_{k,\ell}(x,y)) > \frac{1}{2}\mu(R^n_{k,\ell}(x,y)).$$

The preceding inequalities imply

$$\mu(B \cap T^{-n}B \cap R^n_{k,\ell}(x,y)) > 0,\ \forall k,\ell \in [-M,M]. \tag{3.9}$$

By the recurrence property, there is $n > n_0$ such that

$$\mu(B' \cap T^{-n}B' \cap \{\varphi^{(n)}(\,\cdot\,) = (0,0)\}) > 0.$$

If $(x,y) \in B' \cap T^{-n}B' \cap \{\varphi^{(n)}(\,\cdot\,) = (0,0)\}$, then $\varphi^{(n)}$ is equal to $(0,0)$ on $R^n_{0,0}(x,y)$. Moreover, on each rectangle $R^n_{k,\ell}(x,y)$, $k,\ell \in [-M,M]$, the cocycle $\varphi^{(n)}$ is constant and is equal to $\kappa_{k,\ell}(x,y) \in K$. In view of (3.9), it follows that

$$\mu(B \cap T^{-n}B \cap \{\varphi^{(n)}(\,\cdot\,) = \kappa_{k,\ell}(x,y)\}) > 0,\ \forall k,\ell \in [-M,M].$$

Therefore, by (3.8) and the definition of $K_1$, $\kappa_{k,l}(x,y) \notin K_1$, and so it belongs to $E(\varphi)$ for all $k,\ell \in [-M,M]$. In view of (3.7), it follows that $J_1 \times \{0\}, \{0\} \times J_2 \subset E(\varphi)$, and hence $E(\varphi) = \mathbb{Z}^2$. □

*Remark* 3.7. The ergodicity of $T_\varphi$ can be proven also for the more general case where $\alpha_i$ is bpq and $(D_i - D_i) \setminus \{0\} \subset (\mathbb{Q} + \mathbb{Q}\alpha_i) \setminus (\mathbb{Z} + \mathbb{Z}\alpha_i)$ for $i = 1,2$. To extend the result of Theorem 3.6, we use that the discontinuities of the cocycle are "well distributed" (the condition (3.4)) which is a consequence of Lemma 2.3 in [15].

Now by a similar method we show the ergodicity of Cartesian products of skew products that appeared in Theorem 3.4. We need an elementary algebraic result:

*Remark* 3.8. Let $R$ be a real $m \times k$–matrix. Then the subgroup $R(\mathbb{Z}^k)$ is dense in $\mathbb{R}^m$ if and only if

$$\forall a \in \mathbb{R}^m, \ R^t(a) \in \mathbb{Z}^k \Rightarrow a = 0. \tag{3.10}$$

For instance, if $R = [r_{ij}]$ is a $m \times (m+1)$–matrix such that $r_{ij} = \pm\delta_{ij}$ for $1 \leq i,j \leq m$ and $1, r_{1\,m+1}, \ldots, r_{m\,m+1}$ are independent over $\mathbb{Q}$, then (3.10) holds.

**Theorem 3.9.** *Let $T_j : I^{(j)} \to I^{(j)}$ be an interval exchange transformation of periodic type such that $\theta_2(T_j)/\theta_1(T_j) < 1/\ell$ for $j = 1,\ldots,\ell$. Suppose that the Cartesian product $T_1 \times \ldots \times T_\ell$ is ergodic. If $\varphi_j \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}_|} I^{(j)}_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi_j) \neq 0$ for $j = 1,\ldots,\ell$, then the Cartesian product $(T_1)_{\varphi_1} \times \ldots \times (T_\ell)_{\varphi_\ell}$ is ergodic.*

*Proof.* Since $T_1,\ldots,T_\ell$ have periodic type, by Lemma 3.3 there exists $c > 0$ such that

$$\frac{1}{cn} \leq \min \mathcal{P}_n(T_j) \leq \max \mathcal{P}_n(T_j) \leq \frac{c}{n} \text{ for all } j = 1,\ldots,\ell \text{ and } n > 0. \tag{3.11}$$

Let $\bar{I} = I^{(1)} \times \ldots \times I^{(l)}$, $\bar{T} = T_1 \times \ldots \times T_\ell$ and let $\bar{\varphi} : \bar{I} \to \mathbb{R}^\ell$ be given by

$$\bar{\varphi}(x_1,\ldots,x_\ell) = (\varphi_1(x_1),\ldots,\varphi_\ell(x_\ell)).$$

Then $(T_1)_{\varphi_1} \times \ldots \times (T_\ell)_{\varphi_\ell} = \bar{T}_{\bar{\varphi}}$. Denote by $\bar{\mu}$ the Lebesgue measure on $\bar{I}$. Without loss of generality we can assume that $s(\varphi_j) = \pm 1$ for $j = 1,\ldots,\ell$. By Corollary 2.4, the cocycle $\bar{\varphi}$ for $\bar{T}$ is recurrent.

To prove the result, it suffices to show that, for every $r = (r_1,\ldots,r_\ell) \in [0,\frac{1}{4c})^\ell$, the set $E(\bar{\varphi})$ has nontrivial intersection with

$$\{s \bullet r := (s_1 r_1,\ldots,s_\ell r_\ell) : s = (s_1,\ldots,s_\ell) \in \{-1,1\}^\ell\}.$$

Indeed, for a fixed rational $0 < r < \frac{1}{4c}$, let us consider a collection of vectors $r^{(i)} = (r_{1i},\ldots,r_{\ell i}) \in [0,1/(4c))^\ell$, $1 \leq i \leq \ell+1$ such that $r_{ij} = r\delta_{ij}$ for all $1 \leq i,j \leq \ell$ and $1, r_{1\,\ell+1},\ldots,r_{\ell\,\ell+1}$ are independent over $\mathbb{Q}$. By Remark 3.8, for any choice $s^{(i)} \in \{-1,1\}^\ell$, $1 \leq i \leq \ell+1$ the subgroup generated by vectors $s^{(i)} \bullet r^{(i)}$, $1 \leq i \leq \ell+1$ is dense in $\mathbb{R}^\ell$. Since $E(\bar{\varphi}) \subset \mathbb{R}^\ell$ is a closed subgroup and for every $1 \leq i \leq \ell+1$ there exists $s^{(i)} \in \{-1,1\}^\ell$ such that $s^{(i)} \bullet r^{(i)} \in E(\varphi)$, it follows that $E(\bar{\varphi}) = \mathbb{R}^\ell$, and hence $\bar{T}_{\bar{\varphi}}$ is ergodic.

Fix $r = (r_1,\ldots,r_\ell) \in [0,\frac{1}{4c})^\ell$. We have to show that for every measurable set $B \subset \bar{I}$ with $\bar{\mu}(B) > 0$ and $0 < \varepsilon < 1/c$ there exists $n > 0$ such that the set of all $\bar{x} = (x_1,\ldots,x_\ell) \in B$ such that

$$(T_1^n x_1,\ldots,T_\ell^n x_\ell) \in B, \ \varphi_j^{(n)}(x_j) \in \{-r_j, r_j\} + (-\varepsilon,\varepsilon) \text{ for } 1 \leq j \leq \ell$$

has positive $\bar{\mu}$ measure. By a density point argument, there exists $B' \subset B$ and $n_0 \geq 1$ such that $\bar{\mu}(B') > 0$ and for every $(x_1, \ldots, x_\ell) \in B'$ and $n \geq n_0$ we have

$$\bar{\mu}(\prod_{j=1}^{\ell} \Big(x_j - \frac{c}{n}, x_j + \frac{c}{n}\Big) \setminus B) < \frac{\varepsilon}{4(2n)^\ell}. \tag{3.12}$$

Since $\bar{\varphi}$ (as a cocycle for $\bar{T}$) is recurrent, there exists $n \geq n_0$ such that

$$\bar{\mu}\left(B' \cap \bar{T}^{-n}B' \cap (\bar{\varphi}^{(n)} \in (-\varepsilon/2, \varepsilon/2)^\ell)\right) > 0.$$

Next choose $x^0 = (x_1^0, \ldots, x_\ell^0) \in B'$ so that $(T_1^n x_1^0, \ldots, T_\ell^n x_\ell^0) \in B'$, $|\varphi_j^{(n)}(x_j^0)| < \varepsilon/2$ for $1 \leq j \leq \ell$. For each $1 \leq j \leq \ell$ denote by $J_{j,n}(x_j^0) \subset I_j$ the interval of the partition $\mathcal{P}_n(T_j)$ such that $x_j^0 \in J_{j,n}(x_j^0)$. By assumption, $\varphi_j^{(n)}$ is continuous on every interval of $\mathcal{P}^n(T_j)$. Therefore, for every $1 \leq j \leq \ell$, the function $\varphi_j^{(n)}$ is continuous on $J_{j,n}(x_j^0)$, and hence $\varphi_j^{(n)}(x) = \pm nx + d_{n,j}$ for $x \in J_{j,n}(x_j^0)$. In view of (3.11), $\frac{1}{cn} < |J_{j,n}(x_j^0)| < \frac{c}{n}$, and hence $J_{j,n}(x_j^0) \subset (x_j^0 - c/n, x_j^0 + c/n)$ for every $1 \leq j \leq \ell$. Moreover, $T_j^n J_{j,n}(x_j^0)$ is an interval such that $|T_j^n J_{j,n}(x_j^0)| = |J_{j,n}(x_j^0)| < c/n$, so

$$T_j^n J_{j,n}(x_j^0) \subset \Big(T_j^n x_j^0 - \frac{c}{n}, T_j^n x_j^0 + \frac{c}{n}\Big). \tag{3.13}$$

Since $|\varphi_j^{(n)}(x_j^0)| < \varepsilon/2$, $\varphi_j^{(n)}$ is linear on $J_{j,n}(x_j^0)$ with slope $\pm n$ and $0 \leq r_j < \frac{1}{4c} < \frac{1}{2c} - \frac{\varepsilon}{4}$, we can find $(y_j^0 - \varepsilon/(4n), y_j^0 + \varepsilon/(4n)) \subset J_{j,n}(x_j^0)$ such that

$$|\varphi_j^{(n)}(x)| \in r_j + (-\varepsilon, \varepsilon) \text{ for all } x \in (y_j^0 - \varepsilon/(4n), y_j^0 + \varepsilon/(4n)). \tag{3.14}$$

Let $y^0 = (y_1^0, \ldots, y_\ell^0) \in \prod_{j=1}^{\ell} J_{j,n}(x_j^0)$. Since

$$\prod_{j=1}^{\ell}\Big(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\Big) \subset \prod_{j=1}^{\ell} J_{j,n}(x_j^0) \subset \prod_{j=1}^{\ell}\Big(x_j^0 - \frac{c}{n}, x_j^0 + \frac{c}{n}\Big),$$

$x^0 \in B'$ and $n \geq n_0$, by (3.12), we have

$$\bar{\mu}\left(\prod_{j=1}^{\ell}\Big(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\Big) \setminus B\right) < \frac{\varepsilon}{4(2n)^\ell}.$$

Moreover, by (3.13),

$$\bar{T}^n \prod_{j=1}^{\ell}\Big(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\Big) \subset \prod_{j=1}^{\ell} T_j^n J_{j,n}(x_j^0) \subset \prod_{j=1}^{\ell}\Big(T_j^n x_j^0 - \frac{c}{n}, T_j^n x_j^0 + \frac{c}{n}\Big).$$

Since $(T_1^n x_1^0, \ldots, T_\ell^n x_\ell^0) \in B'$ and $n \geq n_0$, by (3.12), it follows that

$$\bar{\mu}(\prod_{j=1}^{\ell}(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}) \setminus \bar{T}^{-n}B) = \bar{\mu}(\bar{T}^n \prod_{j=1}^{\ell}(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}) \setminus B) < \frac{\varepsilon}{4(2n)^\ell}.$$

Hence

$$\bar{\mu}(\prod_{j=1}^{\ell}\Big(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}) \cap (B \cap \bar{T}^{-n}B)\Big) > \frac{\varepsilon}{2(2n)^\ell} > 0.$$

By (3.14),

$$\bar{\varphi}^{(n)}(x) \in \prod_{j=1}^{\ell}(\{-r_j, r_j\} + (-\varepsilon, \varepsilon)) \text{ if } x \in \prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right).$$

Thus

$$\bar{\mu}(B \cap \bar{T}^{-n}B \cap (\bar{\varphi}^{(n)} \in \prod_{j=1}^{\ell}(\{-r_j, r_j\} + (-\varepsilon, \varepsilon)))) > \frac{\varepsilon}{2(2n)^{\ell}} > 0.$$

By Lemma 2.8, it follows that $\left(\prod_{j=1}^{\ell}\{-r_j, r_j\}\right) \cap E(\bar{\varphi}) \neq \emptyset$. This completes the proof. □

## 4. Ergodicity of certain step cocycles

In this section we apply Corollary 2.6 to prove the ergodicity of step cocycles over IETs of periodic type.

### 4.1. **Step cocycles.**

Let $T : I \to I$ be an arbitrary IET satisfying Keane's condition. Suppose that $(n_k)_{k\geq 0}$ is an increasing sequence of natural numbers such $n_0 = 0$ and the matrix

$$Z(k+1) = \Theta(T^{(n_k)}) \cdot \Theta(T^{(n_k+1)}) \cdot \ldots \cdot \Theta(T^{(n_{k+1}-1)})$$

has positive entries for each $k \geq 0$. In what follows, we denote by $(\pi^{(k)}, \lambda^{(k)})$ the pair defining $T^{(n_k)}$. By abuse of notation, we continue to write $T^{(k)}$ for $T^{(n_k)}$. With this notation,

$$\lambda^{(k)} = Z(k+1)\lambda^{(k+1)}.$$

We adopt the notation from [24]. For each $k < l$ let

$$Q(k, l) = Z(k+1) \cdot Z(k+2) \cdot \ldots \cdot Z(l).$$

Then

$$\lambda^{(k)} = Q(k, l)\lambda^{(l)}.$$

We will write $Q(l)$ for $Q(0, l)$. By definition, $T^{(l)} : I^{(l)} \to I^{(l)}$ is the first return map of $T^{(k)} : I^{(k)} \to I^{(k)}$ to the interval $I^{(k)} \subset I^{(l)}$. Moreover, $Q_{\alpha\beta}(k, l)$ is the time spent by any point of $I_{\beta}^{(l)}$ in $I_{\alpha}^{(k)}$ until it returns to $I^{(l)}$. It follows that

$$Q_{\beta}(k, l) = \sum_{\alpha\in\mathcal{A}} Q_{\alpha\beta}(k, l)$$

is the first return time of points of $I_{\beta}^{(l)}$ to $I^{(l)}$.

Suppose that $T = T_{(\pi,\lambda)}$ is of periodic type and $p$ is a period such that $\pi^{(p)} = \pi$. Let $A = \Theta^{(p)}(T)$. Considering the sequence $(n_k)_{k\geq 0}$, $n_k = pk$ we get $Z(l) = A$ and $Q(k, l) = A^{l-k}$ for all $0 \leq k \leq l$.

The norm of a vector is defined as the largest absolute value of the coefficients. We set $\|B\| = \max_{\beta\in\mathcal{A}} \sum_{\alpha\in\mathcal{A}} |B_{\alpha\beta}|$ for $B = [B_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}}$. Following [31], for every matrix $B = [B_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}}$ with positive entries, we set

$$\nu(B) = \max_{\alpha,\beta,\gamma\in\mathcal{A}} \frac{B_{\alpha\beta}}{B_{\alpha\gamma}}.$$

Then

$$\sum_{\alpha\in\mathcal{A}} B_{\alpha\beta} \le \nu(B) \sum_{\alpha\in\mathcal{A}} B_{\alpha\gamma} \text{ for all } \beta,\gamma\in\mathcal{A} \text{ and } \nu(CB)\le\nu(B), \tag{4.1}$$

for any nonnegative nonsingular matrix $C$. It follows that $\nu(B^m) \le \nu(B)$, and hence

$$\|B^m\| = \max_{\beta\in\mathcal{A}} \sum_{\alpha\in\mathcal{A}} B^m_{\alpha\beta} \le \nu(B) \min_{\beta\in\mathcal{A}} \sum_{\alpha\in\mathcal{A}} B^m_{\alpha\beta}. \tag{4.2}$$

Denote by $\Gamma^{(k)}$ the space of functions $\varphi : I^{(k)} \to \mathbb{R}$ constant on each interval $I^{(k)}_\alpha$, $\alpha\in\mathcal{A}$ and denote by $\Gamma^{(k)}_0$ the subspace of functions with zero mean. Every function $\varphi = \sum_{\alpha\in\mathcal{A}} h_\alpha \chi_{I^{(k)}_\alpha}$ in $\Gamma^{(k)}$ can be identified with the vector $h = (h_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. Moreover,

$$\varphi^{(Q(k,l)_\alpha)}(x) = (Q(k,l)^t h)_\alpha \text{ for every } x\in I^{(l)}_\alpha,\ \alpha\in\mathcal{A}. \tag{4.3}$$

The induced IET $T^{(n)} : I^{(n)} \to I^{(n)}$ determines a partition of $I$ into disjoint towers $H^{(n)}_\alpha$, $\alpha\in\mathcal{A}$, where

$$H^{(n)}_\alpha = \{T^k I^{(n)}_\alpha : 0\le k < h^{(n)}_\alpha := Q_\alpha(n)\}.$$

Denote by $h^{(n)}_{\max}$ and $h^{(n)}_{\min}$ the height of the highest and the lowest tower respectively.

Assume that $I^{(n+1)} \subset I^{(n)}_{\alpha_1}$, where $\pi^{(n)}_0(\alpha_1) = 1$. For every $\alpha\in\mathcal{A}$ denote by $C^{(n)}_\alpha$ the tower $\{T^i I^{(n+1)}_\alpha : 0 \le i < h^{(n)}_{\alpha_1}\}$.

**Lemma 4.1.** *For every $\alpha\in\mathcal{A}$ we have*

$$\mu(C^{(n)}_\alpha \triangle T C^{(n)}_\alpha) \to 0 \text{ and } \sup_{x\in C^{(n)}_\alpha} |T^{h^{(n+1)}_\alpha} x - x| \to 0 \text{ as } n\to+\infty. \tag{4.4}$$

*If $\varphi = \sum_{\alpha\in\mathcal{A}} v_\alpha \chi_{I^{(0)}_\alpha}$ for some $v = (v_\alpha)_{\alpha\in\mathcal{A}} \in \Gamma^{(0)}_0$, then*

$$\varphi^{(h^{(n+1)}_\alpha)}(x) = (Q(n+1)^t v)_\alpha \text{ for all } x\in C^{(n)}_\alpha. \tag{4.5}$$

*If additionally $T$ is of periodic type then*

$$\liminf_{n\to\infty} \mu(C^{(n)}_\alpha) > 0 \tag{4.6}$$

*and*

$$\varphi^{(h^{(n+1)}_\alpha)}(x) = ((A^t)^{n+1} v)_\alpha \text{ for all } x\in C^{(n)}_\alpha. \tag{4.7}$$

*Proof.* Since $C^{(n)}_\alpha \triangle T C^{(n)}_\alpha \subset T^{h^{(n+1)}_{\alpha_1}} I^{(n+1)}_\alpha \cup I^{(n+1)}_\alpha$, we have

$$\mu(C^{(n)}_\alpha \triangle T C^{(n)}_\alpha) \le 2\mu(I^{(n+1)}_\alpha) \to 0 \text{ as } n\to+\infty.$$

Suppose that $x \in T^i I^{(n+1)}_\alpha$ for some $0\le i < h^{(n)}_{\alpha_1}$. Then

$$T^{h^{(n+1)}_\alpha} x \in T^i T^{h^{(n+1)}_\alpha} I^{(n+1)}_\alpha \subset T^i I^{(n+1)} \subset T^i I^{(n)}_{\alpha_1}.$$

It follows that

$$x, T^{h^{(n+1)}_\alpha} x \in T^i I^{(n)}_{\alpha_1} \subset I_\beta \text{ for some } \beta\in\mathcal{A}. \tag{4.8}$$

Therefore

$$|x - T^{h^{(n+1)}_\alpha} x| \le |I^{(n)}_{\alpha_1}| \text{ for all } x\in C^{(n)}_\alpha.$$

Next, by (4.3), $\varphi^{(h_\alpha^{(n+1)})}(x) = (Q(n+1)^t v)_\alpha$ for every $x \in I_\alpha^{(n+1)}$. Moreover, if $x \in C_\alpha^{(n)}$, say $x = T^i x_0$ with $x_0 \in I_\alpha^{(n+1)}$ and $0 \le i < h_{\alpha_1}^{(n)}$, then

$$\varphi^{(h_\alpha^{(n+1)})}(T^i x_0) - \varphi^{(h_\alpha^{(n+1)})}(x_0) = \sum_{0\le j<i} \varphi(T^{h_\alpha^{(n+1)}} T^j x_0) - \varphi(T^j x_0).$$

By (4.8), $\varphi(T^{h_\alpha^{(n+1)}} T^j x_0) = \varphi(T^j x_0)$ for every $0 \le j < h_{\alpha_1}^{(n)}$, and hence

$$\varphi^{(h_\alpha^{(n+1)})}(x) = \varphi^{(h_\alpha^{(n+1)})}(x_0) = (Q(n+1)^t v)_\alpha \text{ for all } x \in C_\alpha^{(n)}.$$

Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and $A$ is its periodic matrix. Denote by $\rho_1$ the Perron-Frobenius eigenvalue of $A$. Then there exists $C > 0$ such that $\frac{1}{C}\rho_1^n \le \|A^n\| \le C\rho_1^n$. Since $h_{\max}^{(n)} = \|A^n\| = \max_{\alpha\in\mathcal{A}} A_\alpha^n$ and $h_{\min}^{(n)} = \min_{\alpha\in\mathcal{A}} A_\alpha^n$, by (4.2), it follows that

$$\frac{1}{C\nu(A)}\rho_1^n \le h_{\min}^{(n)} < h_{\max}^{(n)} \le C\rho_1^n. \tag{4.9}$$

As $|I_\alpha^{(n+1)}| = \rho_1^{-(n+1)}|I_\alpha^{(0)}|$, we have

$$\mu(C_\alpha^{(n)}) = |I_\alpha^{(n+1)}|h_{\alpha_1}^{(n)} = |I_\alpha^{(0)}|h_{\min}^{(n)}/\rho_1^{n+1} \ge \frac{|I_\alpha^{(0)}|}{C\nu(A)\rho_1} > 0.$$

Multiplying the period of $T$, if necessary, we have $I^{(n+1)} \subset I_{\alpha_1}^{(n)}$ for every natural $n$, and hence

$$\varphi^{(h_\alpha^{(n+1)})}(x) = (Q(n+1)^t v)_\alpha = ((A^t)^{n+1} v)_\alpha \text{ for all } x \in C_\alpha^{(n)}.$$

□

4.2. **Ergodic cocycles in case $\kappa > 1$.**
Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and $\kappa = \kappa(\pi) > 1$. Then $\dim\ker\Omega_\pi = \kappa - 1 > 0$. As we already mentioned $A$ is the identity on $\ker\Omega_\pi$. Let

$$F(T) = \{v \in \mathbb{R}^{\mathcal{A}} : A^t v = v\}.$$

Then $F(T)$ is a linear subspace with $\dim F(T) = k \ge \kappa - 1$. Since

$$\langle v, \lambda\rangle = \langle A^t v, \lambda\rangle = \langle v, A\lambda\rangle = \rho_1\langle v, \lambda\rangle \text{ for each } v \in F(T),$$

we have $F(T) \subset \Gamma_0^{(0)}$. Moreover, we can choose a basis of the linear space $F(T)$ such that each of its element belongs to $\mathbb{Z}^{\mathcal{A}}$. It follows that $\mathbb{Z}^{\mathcal{A}} \cap F(T)$ is a free abelian group of rank $k$.

**Lemma 4.2.** *Let $v_i = (v_{i\alpha})_{\alpha\in\mathcal{A}}$, $1 \le i \le k$, be a basis of the group $\mathbb{Z}^{\mathcal{A}} \cap F(T)$. Then the collection of vectors $w_\alpha = (v_{i\alpha})_{i=1}^k \in \mathbb{Z}^k$, $\alpha \in \mathcal{A}$, generates the group $\mathbb{Z}^k$.* □

**Theorem 4.3.** *Let $v_i = (v_{i\alpha})_{\alpha\in\mathcal{A}}$, $1 \le i \le k$ be a basis of the group $\mathbb{Z}^{\mathcal{A}} \cap F(T)$. Then the cocycle $\varphi : I \to \mathbb{Z}^k$ given by $\varphi = (\varphi_1, \dots, \varphi_k)$ with $\varphi_i = \sum_{\alpha\in\mathcal{A}} v_{i\alpha}\chi_{I_\alpha}$ for $i = 1, \dots, k$ is ergodic.*

*If $R$ is a $(k-1)\times k$-real matrix satisfying (3.10), then the cocycle $\widetilde{\varphi} : I \to \mathbb{R}^{k-1}$ given by $\widetilde{\varphi}(x) = R\varphi(x)$, which is constant over exchanged intervals, is ergodic.*

*Proof.* By (4.7), for every $\alpha \in \mathcal{A}$ we have

$$\varphi^{(h_\alpha^{(n+1)})}(x) = (((A^t)^{n+1}v_1)_\alpha, \ldots, ((A^t)^{n+1}v_k)_\alpha) = ((v_1)_\alpha, \ldots, (v_k)_\alpha) = w_\alpha$$

for $x \in C_\alpha^{(n)}$. In view of Lemma 4.1, we can apply Corollary 2.6. Thus $w_\alpha \in E(\varphi)$ for all $\alpha \in \mathcal{A}$. Since $E(\varphi)$ is a group, by Lemma 4.2, we obtain $E(\varphi) = \mathbb{Z}^k$.

It is easy to show that $RE(\varphi) \subset E(R\varphi)$. Since $E(\varphi) = \mathbb{Z}^k$ and $E(R\varphi)$ is closed, by Remark 3.8, we obtain $E(\widetilde{\varphi}) = E(R\varphi) \supset \overline{R\mathbb{Z}^k} = \mathbb{R}^{k-1}$. □

*Remark* 4.4. Note that Remark 3.8 indicates how to construct matrices $R$ satisfying (3.10).

## 5. Ergodicity of corrected cocycles

In this section, using a method from [24], we present a procedure of correction of functions in $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ by piecewise constant functions (in $\Gamma_0^{(0)}$) in order to obtain better control on the growth of Birkhoff sums. It will allow us to prove the ergodicity of some corrected cocycles.

### 5.1. **Rauzy-Veech induction for cocycles.**

For every cocycle $\varphi : I^{(k)} \to \mathbb{R}$ for the IET $T^{(k)} : I^{(k)} \to I^{(k)}$ and $l > k$ denote by $S(k,l)\varphi : I^{(l)} \to \mathbb{R}$ the renormalized cocycle for $T^{(l)}$ given by

$$S(k,l)\varphi(x) = \sum_{0 \le i < Q_\beta(k,l)} \varphi((T^{(k)})^i x) \text{ for } x \in I_\beta^{(l)}.$$

Note that the operator $S(k,l)$ maps $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ into $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$ and

$$\operatorname{Var} S(k,l)\varphi \le \operatorname{Var}\varphi, \tag{5.1}$$

$$\|S(k,l)\varphi\|_{\sup} \le \|Q(k,l)\|\|\varphi\|_{\sup} \quad \text{and} \tag{5.2}$$

$$\int_{I^{(l)}} S(k,l)\varphi(x)\,dx = \int_{I^{(k)}} \varphi(x)\,dx \tag{5.3}$$

for all $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$. In view of (5.3), $S(k,l)$ maps $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ into $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$.

Recall that $\Gamma^{(k)}$ is the space of functions $\varphi : I^{(k)} \to \mathbb{R}$ which are constant on each interval $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$ and $\Gamma_0^{(k)}$ is the subspace of functions with zero mean. Then

$$S(k,l)\Gamma^{(k)} = \Gamma^{(l)} \quad \text{and} \quad S(k,l)\Gamma_0^{(k)} = \Gamma_0^{(l)}.$$

Moreover, every function $\sum_{\alpha\in\mathcal{A}} h_\alpha \chi_{I_\alpha^{(k)}}$ from $\Gamma^{(k)}$ can be identified with the vector $h = (h_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. Under this identification,

$$\Gamma_0^{(k)} = Ann(\lambda^{(k)}) := \{h = (h_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}} : \langle h, \lambda^{(k)}\rangle = 0\}$$

and the operator $S(k,l)$ is the linear automorphism of $\mathbb{R}^{\mathcal{A}}$ whose matrix in the canonical basis is $Q(k,l)^t$. Moreover, the norm on $\Gamma^{(k)}$ inherited from the supremum norm coincides with the norm of vectors.

5.2. **Correction of functions of bounded variation.**
Suppose now that $T$ is of periodic type. Let us consider the linear subspaces

$$\Gamma^{(k)}_{cs} = \{h \in \Gamma^{(k)} : \limsup_{l\to\infty} \frac{1}{l}\log\|S(k,l)h\| = \limsup_{l\to\infty}\frac{1}{l}\log\|Q(k,l)^t h\| \leq 0\},$$

$$\Gamma^{(k)}_{u} = \{h \in \Gamma^{(k)} : \limsup_{l\to\infty} \frac{1}{l}\log\|S(k,l)h\| = \limsup_{l\to\infty}\frac{1}{l}\log\|Q(k,l)^t h\| > 0\}.$$

Denote by

$$U^{(k)} : \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)/\Gamma^{(k)}_{cs}$$

the projection on the quotient space. Let us consider the linear operator $P^{(k)}_0$ : $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)$ given by

$$P^{(k)}_0\varphi(x) = \varphi(x) - \frac{1}{|I^{(k)}_\alpha|}\int_{I^{(k)}_\alpha}\varphi(t)dt \text{ if } x \in I^{(k)}_\alpha.$$

**Theorem 5.1.** *For every $\varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)$ the sequence*

$$\{U^{(k)}\circ S(k,l)^{-1}\circ P^{(l)}_0 \circ S(k,l)\varphi\}_{l\geq k} \tag{5.4}$$

*converges in the quotient norm on $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)/\Gamma^{(k)}_{cs}$ induced by $\|\cdot\|_{\mathrm{BV}}$.*

*Notations.* Let $P^{(k)} : \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)/\Gamma^{(k)}_{cs}$ stand for the limit operator. Note that if $\varphi \in \Gamma^{(k)}_0$ then $P^{(k)}_0\varphi = 0$, and hence $P^{(k)}\varphi = 0$.

We denote by $\mathrm{BV}^{\lozenge}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the subspace of functions $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ such that $\varphi_-(x) = \varphi_+(x)$ for every $x = T^n l_\alpha$, $\alpha \in \mathcal{A}$, $\pi_0(\alpha) \neq 1$, $n \in \mathbb{Z}\setminus\{0\}$.

Recall that, in general, the growth of $(S(k)\varphi)_{k\geq 1}$ is exponential with exponent $\theta_2/\theta_1$ (see Theorem 2.2). Nevertheless, the growth can be reduced by correcting the function $\varphi$ by a function $h$ constant on the exchanged intervals.

**Theorem 5.2.** *Suppose now that $T = T_{(\pi,\lambda)}$ is of periodic type and $M$ is the maximal size of Jordan blocks in the Jordan decomposition of its periodic matrix. Let $\varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$. There exist $C_1, C_2 > 0$ such that if $\widehat{\varphi} + \Gamma^{(0)}_{cs} = P^{(0)}\varphi$, then $\widehat{\varphi} - \varphi \in \Gamma^{(0)}_0$ and*

$$\|S(k)(\widehat{\varphi})\|_{\sup} \leq C_1 k^M \operatorname{Var}\varphi + C_2 k^{M-1}\|\widehat{\varphi}\|_{\sup} \text{ for every natural } k. \tag{5.5}$$

*For every $\varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ there exists $h \in \Gamma^{(0)}_u \cap \Gamma^{(0)}_0$ such that $\varphi + h + \Gamma^{(0)}_{cs} = P^{(0)}\varphi$. Moreover, the vector $h \in \Gamma^{(0)}_u \cap \Gamma^{(0)}_0$ is unique.*

*If additionally $T$ has non-degenerated spectrum and $\varphi \in \mathrm{BV}^{\lozenge}_0(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ then*

$$\|S(k)(\widehat{\varphi})\|_{\sup} \leq C_1 \operatorname{Var}\varphi + C_2\|\widehat{\varphi}\|_{\sup} \text{ for every natural } k.$$

For completeness the proofs of these theorems will be given in Appendix C.

*Remark* 5.3. If we restrict the choice of $h$ to the subspace $\Gamma^{(0)}_u \cap \Gamma^{(0)}_0$, then the correction $h \in \Gamma^{(0)}_u \cap \Gamma^{(0)}_0$ is unique. In what follows, $\widehat{\varphi}$ will stand for the function $\varphi$ corrected by the unique correction $h \in \Gamma^{(0)}_u \cap \Gamma^{(0)}_0$ (i.e. $\widehat{\varphi} = \varphi + h$).

If $\varphi : I \to \mathbb{R}^\ell$ with $\varphi = (\varphi_1, \ldots, \varphi_\ell)$, we deal with the corrected function $\widehat{\varphi} := (\widehat{\varphi_1}, \ldots, \widehat{\varphi_\ell})$, and we have

$$\|S(k)(\widehat{\varphi})\|_{\sup} \leq C_1 \max_{1\leq i\leq \ell}\operatorname{Var}\varphi_i + C_2\|\widehat{\varphi}\|_{\sup} \text{ for every natural } k.$$

### 5.3. **Ergodicity of corrected step functions.**

We now consider piecewise constant zero mean cocycles $\varphi : I \to \mathbb{R}^\ell$, $\ell \geq 1$ which are also discontinuous in the interior of the exchanged intervals. Suppose that $\gamma_i \in I$, $i = 1, \ldots, s$ are discontinuities of $\varphi$ different from $l_\alpha$, $\alpha \in \mathcal{A}$. Denote by $\bar{d}_i \in \mathbb{R}^\ell$ the vector describing the jumps of coordinate functions of $\varphi$ at $\gamma_i$, this is, $\bar{d}_i = \varphi_+(\gamma_i) - \varphi_-(\gamma_i) \in \mathbb{R}^\ell$. In this section we will prove the ergodicity of $\widehat{\varphi}$ for almost every choice of discontinuities. Note that the corrected cocycle $\widehat{\varphi}$ is also piecewise constant and it is discontinuous at $\gamma_i$ with the jump vector $\bar{d}_i$ for $i = 1, \ldots, s$, and hence it is still non-trivial.

**Theorem 5.4.** *Suppose that $T = T_{(\pi,\lambda)}$ is an IET of periodic type and it has non-degenerated spectrum. There exists a set $D \subset I^s$ of full Lebesgue measure such that if*

(i) $(\gamma_1, \ldots, \gamma_s) \in D$;
(ii) *the subgroup $\mathbb{Z}(\bar{d}_1, \ldots, \bar{d}_s) \subset \mathbb{R}^\ell$ generated by $\bar{d}_1, \ldots, \bar{d}_s$ is dense in $\mathbb{R}^\ell$,*

*then the cocycle $\widehat{\varphi} : I \to \mathbb{R}^\ell$ is ergodic.*

*Proof.* As we already mentioned we can assume that $I^{(n+1)} \subset I^{(n)}_{\alpha_1}$ for every natural $n$, where $\alpha_1 = (\pi_0^{(n)})^{-1}(1) = \pi_0^{-1}(1)$. Fix $\alpha \in \mathcal{A}$ and choose $b_0 < a_1 < b_1 < \ldots < a_s < b_s < a_{s+1}$ so that $[b_0, a_{s+1}) = I_\alpha$. Let

$$F_i^{(n)} = \bigcup_{h^{(n)}_{\alpha_1} \leq j < h^{(n+1)}_\alpha} T^j(a_i/\rho_1^{n+1}, b_i/\rho_1^{n+1}), \text{ for } 1 \leq i \leq s,$$

$$C_i^{(n)} = \bigcup_{0 \leq j < h^{(n)}_{\alpha_1}} T^j(b_i/\rho_1^{n+1}, a_{i+1}/\rho_1^{n+1}), \text{ for } 0 \leq i \leq s$$

($\rho_1$ is the Perron-Frobenius eigenvalue of the periodic matrix $A$ of $T$). Since $[b_0/\rho_1^{n+1}, a_{s+1}/\rho_1^{n+1}) = I^{(n+1)}_\alpha$, the sets $C_i^{(n)}$, $F_i^{(n)}$ are towers for which each level is an interval. Moreover, $C_i^{(n)} \subset C_\alpha^{(n)}$ for $0 \leq i \leq s$ and

$$h^{(n+1)}_\alpha - h^{(n)}_{\alpha_1} \geq \sum_{\beta \in \mathcal{A}} h^{(n)}_\beta - h^{(n)}_{\alpha_1} \geq h^{(n)}_{\min}.$$

In view of (4.9), it follows that

$$\mu(C_i^{(n)}) = (a_{i+1} - b_i)\frac{h^{(n)}_{\alpha_1}}{\rho_1^{n+1}} \geq (a_{i+1} - b_i)\frac{h^{(n)}_{\min}}{\rho_1^{n+1}} \geq \frac{a_{i+1} - b_i}{C\nu(A)\rho_1} > 0,$$

$$\mu(F_i^{(n)}) = (b_i - a_i)\frac{h^{(n+1)}_\alpha - h^{(n)}_{\alpha_1}}{\rho_1^{n+1}} \geq (b_i - a_i)\frac{h^{(n)}_{\min}}{\rho_1^{n+1}} \geq \frac{b_i - a_i}{C\nu(A)\rho_1} > 0.$$

Recall that if $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and $(\Xi_n)_{n\geq 1}$ is a sequence of towers for $T$ for which

$$\liminf_{n\to\infty} \mu(\Xi_n) > 0 \text{ and } \operatorname{height}(\Xi_n) \to \infty,$$

then (see King [20], Lemma 3.4)

$$\mu(B \cap \Xi_n) - \mu(B)\mu(\Xi_n) \to 0 \text{ for all } B \subset \mathcal{B}. \tag{5.6}$$

It follows that, for $\mu$-almost every $x \in X$, the point $x$ belongs to $\Xi_n$ for infinitely many $n$.

Applying this fact for subsequences of $(F_i^{(n)})_{n\geq 1}$ successively for $i=1,\ldots,s$, we conclude that for a.e. $(\gamma_1,\ldots,\gamma_s)\in I^s$ there exists a subsequence $(k_n)_{n\geq 1}$ such that

$$\gamma_i\in F_i^{(k_n)} \text{ for all } 1\leq i\leq s \text{ and } n\geq 1.$$

Denote by $D\subset I^s$ the subset of all such $(\gamma_1,\ldots,\gamma_s)$ for which $\gamma_i$ does not belong to the union of orbits of $l_\alpha$, $\alpha\in\mathcal{A}$, for $i=1,\ldots,s$. Therefore $\varphi\in \mathrm{BV}^{\Diamond}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha,\mathbb{R}^\ell)$.

Suppose that for some $n\geq 1$ we have $\gamma_i\in F_i^{(n)}$ for all $1\leq i\leq s$. Then the sets $T^jC_i^{(n)}$, $0\leq j<h_\alpha^{(n+1)}$, $0\leq i\leq s$ do not contain discontinuities of $\widehat{\varphi}$. Thus similar arguments to those from the proof of (4.7) show that $\widehat{\varphi}^{(h_\alpha^{(n+1)})}$ is constant on each $C_i^{(n)}$ and equals say $\bar{g}_i^{(n)}\in\mathbb{R}^\ell$.

Let $x\in[b_{i-1}/\rho_1^{n+1},a_i/\rho_1^{n+1})$ and $y\in[b_i/\rho_1^{n+1},a_{i+1}/\rho_1^{n+1})$. By assumption, $\gamma_i\in T^{j_0}[a_i/\rho_1^{n+1},b_i/\rho_1^{n+1})$ for some $h_{\alpha_1}^{(n)}\leq j_0<h_\alpha^{(n+1)}$. It follow that $\widehat{\varphi}(T^jx)=\widehat{\varphi}(T^jy)$ for all $0\leq j<h_\alpha^{(n+1)}$, $j\neq j_0$ and $\widehat{\varphi}(T^{j_0}y)-\widehat{\varphi}(T^{j_0}x)=\bar{d}_i$. Consequently,

$$\bar{g}_i^{(n)}-\bar{g}_{i-1}^{(n)}=\widehat{\varphi}^{(h_\alpha^{(n+1)})}(y)-\widehat{\varphi}^{(h_\alpha^{(n+1)})}(x)=\bar{d}_i.$$

It follows that

$$\widehat{\varphi}^{(h_\alpha^{(n+1)})}(x)=\bar{g}_0^{(n)}+\sum_{l=1}^{i}\bar{d}_l \text{ for all } x\in C_i^{(n)},\ 0\leq i\leq s.$$

Since $\varphi\in \mathrm{BV}^{\Diamond}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha,\mathbb{R}^\ell)$, by Theorem 5.2 there exists $C>0$ such that

$$\|\widehat{\varphi}^{(h_\alpha^{(n+1)})}(x)\|=\|S(n+1)\widehat{\varphi}(x)\|\leq C \text{ for all } x\in I_\alpha^{(n+1)},$$

and hence $\|\bar{g}_0^{(n)}\|\leq C$. Therefore for each $(\gamma_1,\ldots,\gamma_s)\in D$ there exists a subsequence $(k_n)_{n\geq 1}$ such that

$$\widehat{\varphi}^{(h_\alpha^{(k_n+1)})}(x)=\bar{g}_0^{(k_n)}+\sum_{l=1}^{i}\bar{d}_l \text{ for all } x\in C_i^{(k_n)},\ 0\leq i\leq s$$

and $\bar{g}_0^{(k_n)}\to\bar{g}_0$ in $\mathbb{R}^\ell$. Since $\liminf\mu(C_i^{(k_n)})>0$ for each $0\leq i\leq s$, Corollary 2.6 implies $\bar{g}_0+\sum_{l=1}^{i}\bar{d}_l\in E(\widehat{\varphi})$ for each $0\leq i\leq s$. Therefore $\bar{d}_l\in E(\widehat{\varphi})$ for each $1\leq l\leq s$. Since $\bar{d}_1,\ldots,\bar{d}_s$ generate a dense subgroup of $\mathbb{R}^\ell$ and $E(\widehat{\varphi})$ is closed, it follows that $E(\widehat{\varphi})=\mathbb{R}^\ell$. □

*Remark* 5.5. Notice that the condition (ii) implies $s>\ell$. On the other hand, if $s>\ell$, in view of Remark 3.8, we can easily find a collection of vectors $\bar{d}_1,\ldots,\bar{d}_s\in\mathbb{R}^\ell$ such that $\overline{\mathbb{Z}(\bar{d}_1,\ldots,\bar{d}_s)}=\mathbb{R}^\ell$.

In order to have a more specific condition on the discontinuities $\gamma_i$, $i=1,\ldots,s$ guaranteeing ergodicity, we can use a periodic type condition.

Let us consider a set $\{\gamma_1,\ldots,\gamma_s\}\subset I\setminus\{l_\alpha:\alpha\in\mathcal{A}\}$. The points $\gamma_1,\ldots,\gamma_s$ together with $l_\alpha$, $\alpha\in\mathcal{A}$ give a new partition of $I$ into $d+s$ intervals. Therefore $T$ can be treated as a $d+s$-IET. Denote by $(\pi',\lambda')$ the combinatorial data of this representation of $T$.

*Definition.* We say that the set $\{\gamma_1,\ldots,\gamma_s\}$ is of *periodic type with respect to* $T_{(\pi,\lambda)}$ if the IET $T_{(\pi',\lambda')}$ is of periodic type as an exchange of $d+s$ intervals.

*Remark* 5.6. By the definition of periodic type, $(\lambda',\pi')$ satisfies the Keane condition. Therefore, each $\gamma_i$ does not belong to the orbit of any $l_\alpha$, $\alpha\in\mathcal{A}$.

In view of Theorem 23 in [27], each admissible interval $I^{(p)}$ ($p$ is a period) for $T_{(\pi',\lambda')}$ is also admissible for $T_{(\pi,\lambda)}$. Therefore $T_{(\pi,\lambda)}$ is of periodic type as an exchange of $d$-intervals as well. It follows that, for every $n \geq 0$ and $i = 1, \ldots, s$ if $\gamma_i \in I_\alpha$, then $\gamma_i = T^j_{(\pi,\lambda)}(\gamma_i/\rho^n)$ for some $0 \leq j < h^{(n)}_\alpha$. Therefore similar arguments to those in the proof of Theorem 5.4 give the following result.

**Theorem 5.7.** *Suppose that $T = T_{(\pi,\lambda)}$ is an IET of periodic type and it has non-degenerated spectrum. Let $\varphi : I \to \mathbb{R}^\ell$ be a zero mean piecewise constant cocycle with additional discontinuity at $\gamma_i \in I \setminus \{l_\alpha : \alpha \in \mathcal{A}\}$ with the jump vectors $\bar{d}_i \in \mathbb{R}^\ell$ for $i = 1, \ldots, s$. If*

(i) *the set $\{\gamma_1, \ldots, \gamma_s\}$ is of periodic type with respect to $T_{(\pi,\lambda)}$;*
(ii) $\overline{\mathbb{Z}(\bar{d}_1, \ldots, \bar{d}_s)} = \mathbb{R}^\ell$,

*then the cocycle $\widehat{\varphi} : I \to \mathbb{R}^\ell$ is ergodic.* □

## 6. Recurrence and ergodicity of extensions of multivalued Hamiltonians

In this section we deal with a class of smooth flows on non-compact manifolds which are extensions of so called multivalued Hamiltonian flows on compact surfaces of higher genus. Each such flow has a special representation over a skew product of an IET and a BV cocycle. This allows us to apply abstract results from previous sections to state some sufficient conditions for recurrence and ergodicity whenever the IET is of periodic type.

### 6.1. Special flows.

In this subsection we briefly recall some basic properties of special flows. Let $T$ be an automorphism of a $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$. Let $f : X \to \mathbb{R}$ be a strictly positive function such that

$$\sum_{n\geq 1} f(T^n x) = +\infty \text{ for a.e. } x \in X. \tag{6.1}$$

By $T^f = (T^f_t)_{t\in\mathbb{R}}$ we will mean the corresponding special flow under $f$ (see e.g. [8], Chapter 11) acting on $(X^f, \mathcal{B}^f, \mu^f)$, where $X^f = \{(x, s) \in X \times \mathbb{R} : 0 \leq s < f(x)\}$ and $\mathcal{B}^f$ ($\mu^f$) is the restriction of $\mathcal{B} \times \mathcal{B}(\mathbb{R})$ ($\mu \times m_\mathbb{R}$) to $X^f$. Under the action of the flow $T^f$ each point in $X^f$ moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. More precisely, for every $(x, s) \in X^f$ we have

$$T^f_t(x, s) = (T^n x, s + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is a unique number such that $f^{(n)}(x) \leq s + t < f^{(n+1)}(x)$.

*Remark* 6.1. If $T$ is conservative then the condition (6.1) holds automatically and the special flow $T^f$ is conservative as well. Moreover, if $T$ is ergodic then $T^f$ is ergodic.

### 6.2. Basic properties of multivalued Hamiltonian flows.

Now we will consider multivalued Hamiltonians and their associated flows, a model which has been developed by S.P. Novikov (see also [2] for the toral case). Let $(M, \omega)$ be a compact symplectic smooth surface and $\beta$ be a Morse closed 1-form on $M$. Denote by $\pi : \widehat{M} \to M$ the universal cover of $M$ and by $\widehat{\beta}$ the pullback of

$\beta$ by $\pi : \widehat{M} \to M$. Since $\widehat{M}$ is simply connected and $\widehat{\beta}$ is also a closed form, there exists a smooth function $\widehat{H} : \widehat{M} \to \mathbb{R}$, called a multivalued Hamiltonian, such that $d\widehat{H} = \widehat{\beta}$. By assumption, $\widehat{H}$ is a Morse function. Suppose additionally that all critical values of $\widehat{H}$ are distinct.

Denote by $X : M \to TM$ the smooth vector field determined by

$$\beta = i_X\omega = \omega(X, \cdot).$$

Let $(\phi_t)_{t\in\mathbb{R}}$ stand for the smooth flow on $M$ associated to the vector field $X$. Since $d\beta = 0$, the flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the symplectic form $\omega$, and hence it preserves the smooth measure $\nu = \nu_\omega$ determined by $\omega$. Since $\beta$ is a Morse form, the flow $(\phi_t)_{t\in\mathbb{R}}$ has finitely many fixed points (equal to zeros of $\beta$ and equal to images of critical points of $\widehat{H}$ by the map $\pi$). The set of fixed points will by denoted by $\mathcal{F}(\beta)$. All of them are centers or non-degenerated saddles. By assumption, any two different saddles are not connected by a separatrix of the flow (called a saddle connection). Nevertheless, the flow $(\phi_t)_{t\in\mathbb{R}}$ can have saddle connections which are loops. Each such saddle connection gives a decomposition of $M$ into two nontrivial invariant subsets.

By Theorem 14.6.3 in [18], the surface $M$ can be represented as the finite union of disjoint $(\phi_t)_{t\in\mathbb{R}}$–invariant sets as follows

$$M = \mathcal{P} \cup \mathcal{S} \cup \bigcup_{\mathcal{T}\in\mathfrak{T}} \mathcal{T},$$

where $\mathcal{P}$ is an open set consisting of periodic orbits, $\mathcal{S}$ is a finite union of fixed points or saddle connections, and each $\mathcal{T} \in \mathfrak{T}$ is open and every positive semi-orbit in $\mathcal{T}$, that is not a separatrix incoming to a fixed point, is dense in $\mathcal{T}$. It follows that $\overline{\mathcal{T}}$ is a transitive component of $(\phi_t)_{t\in\mathbb{R}}$. Each transitive component $\overline{\mathcal{T}}$ is a surface with boundary and the boundary of $\overline{\mathcal{T}}$ is a finite union of fixed points and loop saddle connections.

*Remark* 6.2. Let $X$ be a smooth tangent vector field preserving a volume form $\omega$ on a surface $M$. A parametrization $\gamma : [a, b] \to M$ of a curve is called *induced* if

$$\int_{\gamma(s)}^{\gamma(s')} i_X\omega = s - s' \text{ for all } s, s' \in [a, b].$$

Let $\gamma : [a, b] \to M$ and $\widetilde{\gamma} : [\widetilde{a}, \widetilde{b}] \to M$ be induced parameterizations of two curves. Suppose that for every $x \in [a, b]$ the positive semi-orbit of the flow through $\gamma(x)$ hits the curve $\widetilde{\gamma}$. Denote by $T_{\gamma\widetilde{\gamma}}(x) \in [\widetilde{a}, \widetilde{b}]$ the parameter and by $\tau_{\gamma\widetilde{\gamma}}(x) > 0$ the time of the first hit. Using Stokes' theorem, it is easy to check that $T_{\gamma\widetilde{\gamma}} : [a, b] \to [\widetilde{a}, \widetilde{b}]$ is a translation and $\tau_{\gamma\widetilde{\gamma}} : [a, b] \to \mathbb{R}_+$ is a smooth function.

Fix $\mathcal{T} \in \mathfrak{T}$ and let $J \subset \mathcal{T}$ be a transversal smooth curve for $(\phi_t)_{t\in\mathbb{R}}$ such that the boundary of $J$ consists of two points lying on an incoming and an outgoing separatrix respectively, and the segment of each separatrix between the corresponding boundary point of $J$ and the fixed point has no intersection with $J$. Let $\gamma : [0, a] \to J$ stand for the induced parametrization such that the boundary points $\gamma(0)$ and $\gamma(a)$ lie on the incoming and outgoing separatrixes respectively (see Figure 1). Set $I = [0, a)$. We will identify the interval $I$ with the curve $J$.

Denote by $T := T_{\gamma\gamma}$ the first-return map induced on $J$; $T$ can be seen as a map $T : I \to I$. By Remark 6.2, $T : I \to I$ is an exchange interval transformation. Then

$T = T_{(\pi,\lambda)}$, where $\pi \in \mathcal{S}^0_{\mathcal{A}}$ for some finite set $\mathcal{A}$ and $(\pi,\lambda) \in \mathcal{S}^0_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_{+}$ satisfies Keane's condition. Recall that $l_\alpha$, $\alpha \in \mathcal{A}$ stand for the left end points of the exchanged intervals. Let $\mathcal{Z} = \mathcal{F}(\beta) \cap \overline{\mathcal{T}}$. Since $\overline{\mathcal{T}}$ is a transitive component, each element of $\mathcal{Z}$ is a non-degenerated saddle. Let us decompose the set of fixed points $\mathcal{Z}$ into subsets $\mathcal{Z}_0$, $\mathcal{Z}_+$ and $\mathcal{Z}_-$ of points $z \in \mathcal{Z}$ such that $z$ has no loop connection, has a loop connection with positive orientation and has a loop connection with negative orientation respectively. For each $z \in \mathcal{Z}_+ \cup \mathcal{Z}_-$ denote by $\sigma_{loop}(z)$ the corresponding loop connection.

Denote by $\underline{z} \in \mathcal{Z}$ the fixed point such that $\gamma(0)$ belongs to its incoming separatrix $\sigma^-(\underline{z})$. Then $\gamma(0)$ is the first backward intersection with $J$ of $\sigma^-(\underline{z})$. Set $\underline{\alpha} = \pi_1^{-1}(1) \in \mathcal{A}$. Then each point $\gamma(l_\alpha)$ with $\alpha \neq \underline{\alpha}$ corresponds to the first backward intersection with $J$ of an incoming separatrix of a fixed point, denoted by $z_{l_\alpha} \in \mathcal{Z}$ (see Figure 1). The point $\gamma(l_{\underline{\alpha}})$ corresponds to the second backward intersection with $J$ of $\sigma^-(\underline{z})$ and $Tl_{\underline{\alpha}} = 0$.

Denote by $\tau : I \to \mathbb{R}_+$ the first-return time map of the flow $(\phi_t)_{t\in\mathbb{R}}$ to $J$. This map is well defined and smooth on the interior of each interval $I_\alpha$, $\alpha \in \mathcal{A}$, and $\tau$ has a singularity of logarithmic type at each point $l_\alpha$, $\alpha \in \mathcal{A}$ (see [21]) except for the right-side of $l_{\underline{\alpha}}$; here the right-sided limit of $\tau$ exists. Moreover, the flow $(\phi_t)_{t\in\mathbb{R}}$ on $(\mathcal{T}, \nu|_{\mathcal{T}})$ is measure-theoretical isomorphic to the special flow $T^\tau$. An isomorphism is established by the map $\Gamma : I^\tau \to \overline{\mathcal{T}}$, $\Gamma(x,s) = \phi_s\gamma(x)$.

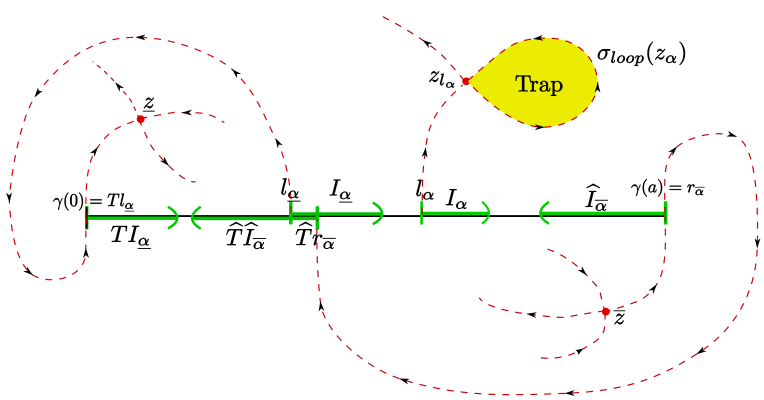


FIGURE 1. Separatrices of $(\phi_t)$

### 6.3. **Extensions of multivalued Hamiltonian flows.**

Let $f : M \to \mathbb{R}^\ell$ be a smooth function. Let us consider a system of differential equations on $M \times \mathbb{R}^\ell$ of the form

$$\begin{cases} \frac{dx}{dt} &= X(x), \\ \frac{dy}{dt} &= f(x), \end{cases}$$

for $(x,y) \in M \times \mathbb{R}^\ell$. Then the associated flow $(\Phi^f_t)_{t\in\mathbb{R}} = (\Phi_t)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}^\ell$ is given by

$$\Phi_t(x,y) = \left(\phi_t x, y + \int_0^t f(\phi_s x)\, ds\right).$$

It follows that $(\Phi_t)_{t\in\mathbb{R}}$ is a skew product flow with the base flow $(\phi_t)_{t\in\mathbb{R}}$ on $M$ and the cocycle $F:\mathbb{R}\times M\to\mathbb{R}^\ell$ given by

$$F(t,x)=\int_0^t f(\phi_s x)\,ds.$$

Therefore $(\Phi_t)_{t\in\mathbb{R}}$ preserves the product measure $\nu\times m_{\mathbb{R}^\ell}$. The deviation of the cocycle $F$ was studied by Forni in [10], [11] for typical $(\phi_t)_{t\in\mathbb{R}}$ with no saddle connections. Recall that the ergodicity of $(\Phi^f_t)_{t\in\mathbb{R}}$ has been already studied in [9] in the simplest case where $M=\mathbb{T}^2$ and $\ell=1$.

In this section we will study recurrence and ergodic properties of the flow $(\Phi^f_t)_{t\in\mathbb{R}}$ for functions $f:M\to\mathbb{R}^\ell$ such that $f(x)=0$ for all $x\in\mathcal{F}(\beta)$. By obvious reason $(\Phi_t)_{t\in\mathbb{R}}$ will be restricted to the invariant set $\overline{\mathcal{T}}\times\mathbb{R}^\ell$, $\mathcal{T}\in\mathfrak{T}$. Let us consider its transversal submanifold $J\times\mathbb{R}^\ell\subset\overline{\mathcal{T}}\times\mathbb{R}^\ell$. Note that every point $(\gamma(x),y)\in\gamma(\operatorname{Int}I_\alpha)\times\mathbb{R}^\ell$ returns to $J\times\mathbb{R}^\ell$ and the return time is $\widehat{\tau}(x,y)=\tau(x)$. Denote by $\varphi:\bigcup_{\alpha\in\mathcal{A}}\operatorname{Int}I_\alpha\to\mathbb{R}^\ell$ the smooth function

$$\varphi(x)=F(\tau(x),\gamma(x))=\int_0^{\tau(x)} f(\phi_s\gamma(x))ds,\quad\text{for}\quad x\in\bigcup_{\alpha\in\mathcal{A}}\operatorname{Int}I_\alpha.$$

Notice that

$$\int_I\varphi(x)\,dx=\int_{\mathcal{T}} f\,d\nu. \tag{6.2}$$

Let us consider the skew product $T_\varphi:(I\times\mathbb{R}^\ell,\mu\times m_{\mathbb{R}^\ell})\to(I\times\mathbb{R}^\ell,\mu\times m_{\mathbb{R}^\ell})$, $T_\varphi(x,y)=(Tx,y+\varphi(x))$ and the special flow $(T_\varphi)^{\widehat{\tau}}$ built over $T_\varphi$ and under the roof function $\widehat{\tau}:I\times\mathbb{R}^\ell\to\mathbb{R}_+$ given by $\widehat{\tau}(x,y)=\tau(x)$.

**Lemma 6.3.** *The special flow $(T_\varphi)^{\widehat{\tau}}$ is measure-theoretical isomorphic to the flow $(\Phi_t)$ on $(\mathcal{T}\times\mathbb{R}^\ell,\nu|_{\mathcal{T}}\times m_{\mathbb{R}^\ell})$.* □

*Remark* 6.4. If $\int_{\mathcal{T}} f\,d\nu\neq 0$ then, by (6.2), the skew product $T_\varphi$ is dissipative. In view of Lemma 6.3, the flow $(\Phi_t)$ on $(\mathcal{T}\times\mathbb{R}^\ell,\nu|_{\mathcal{T}}\times m_{\mathbb{R}^\ell})$ is dissipative, as well.

On the other hand, if $\ell=1$ and $(\phi_t)$ on $(\mathcal{T},\nu|_{\mathcal{T}})$ is ergodic, then $\int_{\mathcal{T}} f\,d\nu=0$ implies the recurrence of $(\Phi_t)$ on $(\mathcal{T}\times\mathbb{R},\nu|_{\mathcal{T}}\times m_{\mathbb{R}})$.

The following lemma will help us to find out further properties of $\varphi$. Since the proof is rather straightforward and the first part follows very closely the proof of Proposition 2 in [14], we leave it to the reader.

**Lemma 6.5.** *Let $g:[-1,1]\times[-1,1]\to\mathbb{R}$ be a $C^1$-function such that $g(0,0)=0$. Then the function $\xi:[0,1]\to\mathbb{R}$,*

$$\xi(s)=\begin{cases}\int_s^1 g\left(u,\frac{s}{u}\right)\frac{1}{u}du, & \text{if}\quad s>0,\\ \int_0^1\left(g(u,0)+g(0,u)\right)\frac{1}{u}du, & \text{if}\quad s=0,\end{cases}$$

*is absolutely continuous. If additionally $g$ is a $C^2$-function, $g'(0,0)=0$, and $g''(0,0)=0$, then $\xi'$ is absolutely continuous.* □

*Remark* 6.6. Note that the second conclusion of the lemma becomes false if the requirement $g''(0,0)=0$ is omitted. Indeed, if $g(x,y)=x\cdot y$ then $\xi(s)=-\log s-1$, $s>0$, is not even bounded.

For each $z\in\mathcal{Z}_+\cup\mathcal{Z}_-$ choose an element $u_z$ of the saddle loop $\sigma_{loop}(z)$.

**Theorem 6.7.** *If $f(x) = 0$ for all $x \in \mathcal{F}(\beta)$, then $\varphi$ is absolutely continuous on each interval $I_\alpha$, $\alpha \in \mathcal{A}$, in particular $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^\ell)$. Moreover,*

$$\int_I \varphi'(x)\,dx = \sum_{z\in\mathcal{Z}_+} \int_{\mathbb{R}} f(\phi_s u_z)\,ds - \sum_{z\in\mathcal{Z}_-} \int_{\mathbb{R}} f(\phi_s u_z)\,ds.$$

*If additionally $f'(x) = 0$ and $f''(x) = 0$ for all $x \in \mathcal{F}(\beta)$, then $\varphi'' \in L^1(I, \mathbb{R}^\ell)$, in particular, $\varphi \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^\ell)$.*

*Proof.* First note that it suffices to consider the case $\ell = 1$. Since $d\beta = 0$, there exists a family of pairwise disjoint open sets $U_z \subset M$, $z \in \mathcal{Z}$ such that $z \in U_z$ and there exists a smooth function $H : \bigcup_{z\in\mathcal{Z}} U_z \to \mathbb{R}$ such that $dH = \beta$ on $U_z$ for every $z \in \mathcal{Z}$. By the Morse Lemma, for every $z \in \mathcal{Z}$ there exist a neighborhood $(0,0) \in V_z \subset \mathbb{R}^2$ and a smooth diffeomorphism $\Upsilon_z : V_z \to U_z$ such that $\Upsilon_z(0,0) = z$ and

$$H_z(x,y) := H \circ \Upsilon_z(x,y) = x \cdot y \text{ for all } (x,y) \in V_z.$$

Denote by $\omega^z \in \Omega^2(V_z)$ the pullback of the form $\omega$ by $\Upsilon_z : V_z \to U_z$. Since $\omega^z$ is non-zero at each point, there exists a smooth non-zero function $p = p_z : V_z \to \mathbb{R}$ such that

$$\omega^z_{(x,y)} = p(x,y)dx \wedge dy.$$

Let $(\phi^z_t)$ stand for the pullback of the flow $(\phi_t)$ by $\Upsilon_z : V_z \to U_z$, i.e. the local flow on $V_z$ given by $\phi^z_t = \Upsilon_z^{-1} \circ \phi_t \circ \Upsilon_z$. Denote by $X^z : V_z \to \mathbb{R}^2$ the vector field corresponding to $(\phi^z_t)$. Then $dH_z = \omega^z(X^z, \cdot\,)$, and hence

$$X^z(x,y) = \frac{\left(\frac{\partial H_z}{\partial y}(x,y), -\frac{\partial H_z}{\partial x}(x,y)\right)}{p(x,y)} = \frac{(x,-y)}{p(x,y)}.$$

Let $\delta$ be a positive number such that $[-\delta,\delta] \times [-\delta,\delta] \subset V_z$ for every $z \in \mathcal{Z}$. Let us consider the $C^\infty$–curves $\gamma_z^{\pm,0}, \gamma_z^{\pm,1} : [-\delta^2,\delta^2] \to M$ given by

$$\gamma_z^{\pm,0}(s) = \Upsilon_z(\pm s/\delta, \pm\delta), \ \ \gamma_z^{\pm,1}(s) = \Upsilon_z(\pm\delta, \pm s/\delta).$$

Notice that $\gamma_z^{\pm,i}$ establishes an induced parametrization for the form $\omega(x,y)$ and the vector field $X$. Indeed, we have for every $s \in [-\delta^2,\delta^2]$ and $i = 0,1$,

$$\int_{\gamma_z^{\pm,i}(0)}^{\gamma_z^{\pm,i}(s)} \beta = \int_{\gamma_z^{\pm,i}(0)}^{\gamma_z^{\pm,i}(s)} dH = H(\gamma_z^{\pm,i}(s)) - H(\gamma_z^{\pm,i}(0)) = \pm s/\delta \cdot \pm\delta = s.$$

We consider the functions $\tau_z^\pm$ and $\varphi_z^\pm$ from $[-\delta^2,0) \cup (0,\delta^2]$ to $\mathbb{R}$, where $\tau_z^\pm(x)$ is the exit time of the point $\gamma_z^{\pm,0}(x)$ for the flow $(\phi_t)$ from the set $\Upsilon_z([-\delta,\delta]\times[-\delta,\delta])$ and

$$\varphi_z^\pm(x) = \int_0^{\tau_z^\pm(x)} f(\phi_s \gamma_z^{\pm,0} x)ds.$$

Note that $\tau_z^\pm(x)$ is the passage time from $(\pm x/\delta, \pm\delta)$ to $(\pm\operatorname{sgn}(x)\delta, \pm\operatorname{sgn}(x)x/\delta)$ for the local flow $(\phi^z_t)$. Let $f_z : V_z \to \mathbb{R}$ be given by $f_z = f \circ \Upsilon_z$. By assumption, $f_z$ is a smooth function such that $f_z(0,0) = 0$. Furthermore,

$$\varphi_z^\pm(x) = \int_0^{\tau_z^\pm(x)} f_z(\phi^z_s(\pm x/\delta, \pm\delta))ds.$$

Let $(x_s, y_s) = \phi^z_s(\pm x/\delta, \pm\delta)$. Then

$$\left(\frac{d}{ds}x_s, \frac{d}{ds}y_s\right) = X^z(x_s, -y_s) = \frac{(x_s, -y_s)}{p(x_s,y_s)}, \tag{6.3}$$

and hence

$$x_s \cdot y_s = H_z(x_s, y_s) = H_z(x_0, y_0) = H_z(\pm x/\delta, \pm\delta) = x.$$

Since $x \neq 0$, it follows that $x_s \neq 0$ for all $s \in \mathbb{R}$. By using the substitution $u = x_s$, we obtain $du = \frac{d}{ds}x_s ds = \frac{x_s}{p(x_s,y_s)}ds$ and

$$\begin{aligned}\varphi_z^{\pm}(x) &= \int_0^{\tau_z^{\pm}(x)} f_z(x_s, y_s)ds = \int_0^{\tau_z^{\pm}(x)} f_z\left(x_s, \frac{x}{x_s}\right) ds \\ &= \int_{\pm x/\delta}^{\pm\,\mathrm{sgn}(x)\delta} f_z\left(u, \frac{x}{u}\right) p\left(u, \frac{x}{u}\right) \frac{du}{u}.\end{aligned}$$

By Lemma 6.5, the functions $\varphi_z^{\pm} : [-\delta^2, 0) \cup (0, \delta^2] \to \mathbb{R}$ is absolutely continuous and

$$\lim_{x\to 0^+} \varphi_z^{\pm}(x) = \int_0^{\pm\delta} f_z(u, 0)p(u, 0)\frac{du}{u} + \int_0^{\pm\delta} f_z(0, u)p(0, u)\frac{du}{u}.$$

It follows that

$$\lim_{x\to 0^+} \varphi_z^{\pm}(x) = \int_0^{+\infty} f(\phi_s\gamma_z^{\pm,0}0)ds + \int_{-\infty}^0 f(\phi_s\gamma_z^{\pm,1}0)ds. \tag{6.4}$$

Similar arguments to those above show that

$$\lim_{x\to 0^-} \varphi_z^{\pm}(x) = \int_0^{+\infty} f(\phi_s\gamma_z^{\pm,0}0)ds + \int_{-\infty}^0 f(\phi_s\gamma_z^{\mp,1}0)ds. \tag{6.5}$$

In view of Remark 6.2, we conclude that $\varphi : I \to \mathbb{R}$ is absolutely continuous on each interval $I_\alpha$, $\alpha \in \mathcal{A}$ and

$$\varphi_+(l_\alpha) = \int_0^{+\infty} f(\phi_s\gamma(l_\alpha))ds + \int_{-\infty}^0 f(\phi_s\gamma(Tl_\alpha))ds, \tag{6.6}$$

whenever $\alpha \neq \underline{\alpha}$ and $z_{l_\alpha} \in \mathcal{Z}_- \cup \mathcal{Z}_0$. If $\alpha \neq \underline{\alpha}$ and $z_{l_\alpha} \in \mathcal{Z}_+$, then computing $\varphi_+(l_\alpha)$ we have cover also a distance along the loop separatrix $\sigma_{loop}(z_{l_\alpha})$, so that

$$\varphi_+(l_\alpha) = \int_0^{+\infty} f(\phi_s\gamma(l_\alpha))ds + \int_{-\infty}^0 f(\phi_s\gamma(Tl_\alpha))ds + \int_{-\infty}^{+\infty} f(\phi_s u_{z_{l_\alpha}})ds. \tag{6.7}$$

Moreover, if $f'(z_{l_\alpha}) = 0$ and $f''(z_{l_\alpha}) = 0$ then the derivative $\varphi''$ is integrable on a neighborhood of $l_\alpha$. It follows that if $f'(z) = 0$ and $f''(z) = 0$ for each $z \in \mathcal{F}(\beta)$ then $\varphi''$ is integrable.

If $\alpha = \underline{\alpha}$, then, by definition, the positive semi-orbit through $\gamma(l_\alpha)$ returns to $\gamma$ before approaching the fixed point $\underline{z}$. It follows that

$$\varphi_+(l_\alpha) = \int_0^{\tau_{\gamma\gamma}(l_\alpha)} f(\phi_s\gamma(l_\alpha))ds. \tag{6.8}$$

Let $\overline{\alpha} = \pi_0^{-1}(d)$, i.e. $r_{\overline{\alpha}} = |I|$. Similar arguments to those used for the right-sided limits show that for every $\alpha \neq \overline{\alpha}$ we have

$$\varphi_-(r_\alpha) = \int_0^{+\infty} f(\phi_s\gamma(r_\alpha))ds + \int_{-\infty}^0 f(\phi_s\gamma(\widehat{T}r_\alpha))ds \text{ if } z_{r_\alpha} \in \mathcal{Z}_0 \cup \mathcal{Z}_+, \tag{6.9}$$

$$\varphi_-(r_\alpha) = \int_0^{+\infty} f(\phi_s\gamma(r_\alpha))ds + \int_{-\infty}^0 f(\phi_s\gamma(\widehat{T}r_\alpha))ds + \int_{-\infty}^{+\infty} f(\phi_s u_{z_{r_\alpha}})ds, \tag{6.10}$$

if $z_{r_\alpha} \in \mathcal{Z}_-$. Moreover, since $\gamma(r_{\overline{\alpha}})$ lies on an outgoing separatrix, the positive semi-orbit through $\gamma(r_\alpha)$ returns to the curve $\gamma$, so that

$$\varphi_-(r_{\overline{\alpha}}) = \varphi_-(|I|) = \int_0^{\tau_{\gamma\gamma}(r_{\overline{\alpha}})} f(\phi_s\gamma(r_{\overline{\alpha}}))ds. \tag{6.11}$$

In view of (6.6) - (6.11), we have

$$\begin{aligned}
\int_I \varphi'(x)\,dx &= \sum_{\alpha\in\mathcal{A}} \int_{I_\alpha} \varphi'(x)\,dx = \sum_{\alpha\in\mathcal{A}} (\varphi_-(r_\alpha) - \varphi_+(l_\alpha)) \\
&= \sum_{z\in\mathcal{Z}_-} \int_{-\infty}^{+\infty} f(\phi_s u_z)ds - \sum_{z\in\mathcal{Z}_+} \int_{-\infty}^{+\infty} f(\phi_s u_z)ds \\
&\quad + \sum_{\alpha\in\mathcal{A},\alpha\neq\overline{\alpha}} \int_{-\infty}^{0} f(\phi_s\gamma(\widehat{T}r_\alpha))ds - \sum_{\alpha\in\mathcal{A},\alpha\neq\underline{\alpha}} \int_{-\infty}^{0} f(\phi_s\gamma(Tl_\alpha))ds \\
&\quad + \sum_{\alpha\in\mathcal{A},\alpha\neq\overline{\alpha}} \int_{0}^{+\infty} f(\phi_s\gamma(r_\alpha))ds - \sum_{\alpha\in\mathcal{A},\alpha\neq\underline{\alpha}} \int_{0}^{+\infty} f(\phi_s\gamma(l_\alpha))ds \\
&\quad + \int_0^{\tau_{\gamma\gamma}(r_{\overline{\alpha}})} f(\phi_s\gamma(r_{\overline{\alpha}}))ds - \int_0^{\tau_{\gamma\gamma}(l_{\underline{\alpha}})} f(\phi_s\gamma(l_{\underline{\alpha}}))ds.
\end{aligned} \tag{6.12}$$

Since $\underline{\alpha} = \pi_1^{-1}(1)$ and $\overline{\alpha} = \pi_0^{-1}(d)$, in view of (2.1), (2.2), we have

$$\begin{aligned}
\{r_\alpha : \alpha\in\mathcal{A},\ \alpha\neq\overline{\alpha}\} &= \{r_\alpha : \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq d\} = \{l_\alpha : \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq 1\}, \\
\{Tl_\alpha : \alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}\} &= \{Tl_\alpha : \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq 1\} = \{\widehat{T}r_\alpha : \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq d\}.
\end{aligned}$$

Moreover, $l_{\pi_0^{-1}(1)} = 0 = Tl_{\underline{\alpha}}$ and $\widehat{T}r_{\pi_1^{-1}(d)} = |I| = r_{\overline{\alpha}}$. It follows that

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{A},\alpha\neq\overline{\alpha}} \int_{-\infty}^{0} f(\phi_s\gamma(\widehat{T}r_\alpha))ds - \sum_{\alpha\in\mathcal{A},\alpha\neq\underline{\alpha}} \int_{-\infty}^{0} f(\phi_s\gamma(Tl_\alpha))ds \\
&\qquad = \int_{-\infty}^{0} f(\phi_s\gamma(r_{\overline{\alpha}}))ds - \int_{-\infty}^{0} f(\phi_s\gamma(\widehat{T}r_{\overline{\alpha}}))ds, \\
&\sum_{\alpha\in\mathcal{A},\alpha\neq\overline{\alpha}} \int_{0}^{+\infty} f(\phi_s\gamma(r_\alpha))ds - \sum_{\alpha\in\mathcal{A},\alpha\neq\underline{\alpha}} \int_{0}^{+\infty} f(\phi_s\gamma(l_\alpha))ds \\
&\qquad = \int_{0}^{+\infty} f(\phi_s\gamma(l_{\underline{\alpha}}))ds - \int_{0}^{+\infty} f(\phi_s\gamma(Tl_{\underline{\alpha}}))ds.
\end{aligned}$$

Since the negative semi-orbit of $\widehat{T}r_{\overline{\alpha}}$ visits $r_{\overline{\alpha}}$ before approaching the fixed point $\overline{z}$ and the positive semi-orbit of $l_{\underline{\alpha}}$ visits $Tl_{\underline{\alpha}}$ before approaching the fixed point $\underline{z}$ (see Figure 1), we have

$$\begin{aligned}
\int_{-\infty}^{0} f(\phi_s\gamma(\widehat{T}r_{\overline{\alpha}}))ds - \int_{-\infty}^{0} f(\phi_s\gamma(r_{\overline{\alpha}}))ds &= \int_0^{\tau_{\gamma\gamma}(r_{\overline{\alpha}})} f(\phi_s\gamma(r_{\overline{\alpha}}))ds, \\
\int_{0}^{+\infty} f(\phi_s\gamma(l_{\underline{\alpha}}))ds - \int_{0}^{+\infty} f(\phi_s\gamma(Tl_{\underline{\alpha}}))ds &= \int_0^{\tau_{\gamma\gamma}(l_{\underline{\alpha}})} f(\phi_s\gamma(l_{\underline{\alpha}}))ds.
\end{aligned}$$

In view of (6.12), it follows that

$$\int_I \varphi'(x)\,dx = \sum_{z\in\mathcal{Z}_-}\int_{-\infty}^{+\infty} f(\phi_s u_z)ds - \sum_{z\in\mathcal{Z}_+}\int_{-\infty}^{+\infty} f(\phi_s u_z)ds.$$

□

*Remark* 6.8. Notice that, in view of Remark 6.6, the assumption on the vanishing of derivatives of $f$ at fixed points is necessary to control the smoothness of $\varphi_f$.

**Theorem 6.9.** *Suppose that the IET $T$ is of periodic type. Let $f : M \to \mathbb{R}^\ell$ be a smooth function such that $f(x) = 0$ for all $x \in \mathcal{F}(\beta)$ and $\int_{\mathcal{T}} f\,d\nu = 0$. If $\theta_2(T)/\theta_1(T) < 1/\ell$ then the flow $(\Phi_t)$ on $\mathcal{T}\times\mathbb{R}^\ell$ is conservative.*

*Proof.* By Lemma 6.3, Theorem 6.7 and (6.2), the flow $(\Phi_t)$ on $\mathcal{T}\times\mathbb{R}$ is isomorphic to a special flow built over the skew product $T_\varphi$, where $\varphi : I \to \mathbb{R}^\ell$ is a function of bounded variation with zero mean. In view of Corollary 2.4, the skew product is conservative. Now the conservativity of $(\Phi_t)$ follows from Remark 6.1. □

Let $g$ be a Riemann metric on $M$. Let us consider 1-form $\vartheta^\beta \in \Omega^1(M \setminus \mathcal{F}(\beta))$ on $M \setminus \mathcal{F}(\beta)$ defined by

$$\vartheta^\beta_x Y = \frac{g_x(Y, X(x))}{g_x(X(x), X(x))}.$$

Then $\vartheta^\beta_x X(x) = 1$, and hence

$$\int_{\{\phi_s x: s\in[a,b]\}} f\cdot\vartheta^\beta = \int_a^b f(\phi_s x)\cdot\vartheta^\beta_{\phi_s x}(X(\phi_s x))ds = \int_a^b f(\phi_s x)ds.$$

It follows that

$$\int_{\partial\mathcal{T}} f\cdot\vartheta^\beta = \sum_{z\in\mathcal{Z}_-}\int_{-\infty}^{+\infty} f(\phi_s u_z)ds - \sum_{z\in\mathcal{Z}_+}\int_{-\infty}^{+\infty} f(\phi_s u_z)ds = \int_I \varphi'(x)\,dx.$$

**Theorem 6.10.** *Suppose that the IET $T$ is of periodic type. Let $f : M \to \mathbb{R}$ be a smooth function such that $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$ for all $x \in \mathcal{F}(\beta)$,*

$$\int_{\mathcal{T}} f\,d\nu = 0 \text{ and } \int_{\partial\mathcal{T}} f\cdot\vartheta^\beta \neq 0.$$

*Then the corresponding flow $(\Phi_t)$ on $\mathcal{T}\times\mathbb{R}$ is ergodic.*

*Proof.* By Lemma 6.3, Theorem 6.7 and (6.2), the flow $(\Phi_t)$ on $\mathcal{T}\times\mathbb{R}$ is isomorphic to a special flow built over the skew product $T_\varphi$, where $\varphi \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ has zero mean and

$$s(\varphi) = \int_I \varphi'(x)dx = \int_{\partial\mathcal{T}} f\cdot\vartheta^\beta \neq 0.$$

By Lemma 3.2, the cocycle $\varphi$ is cohomologous to a cocycle $\varphi_{pl} \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\int \varphi_{pl}(x)\,dx = 0$ and $s(\varphi_{pl}) = s(\varphi) \neq 0$. In view of Theorem 3.4, the skew product $T_{\varphi_{pl}}$ is ergodic. Consequently, the skew product $T_\varphi$ and hence the flow $(\Phi_t)$ on $\mathcal{T}\times\mathbb{R}$ are ergodic, by Remark 6.1. □

Suppose that the IET $T$ is of periodic type and $\theta_2(T)/\theta_1(T) < 1/\ell$ ($\ell \geq 2$). Let $f : M \to \mathbb{R}^\ell$ be a smooth function such that $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$ for all $x \in \mathcal{F}(\beta)$,

$$\int_{\mathcal{T}} f\,d\nu = 0 \text{ and } \mathbb{R}^\ell \ni v = \int_{\partial\mathcal{T}} f\cdot\vartheta^\beta \neq 0.$$

Let $a_2, \ldots, a_\ell$ be a basis of the subspace $\{v\}^\perp$ and let $f_a : M \to \mathbb{R}^{\ell-1}$ be given by $f_a = (\langle a_2, f\rangle, \ldots, \langle a_\ell, f\rangle)$.

**Theorem 6.11.** *If the flow $(\Phi_t^{f_a})$ on $\mathcal{T} \times \mathbb{R}^{\ell-1}$ is ergodic then $(\Phi_t^f)$ on $\mathcal{T} \times \mathbb{R}^\ell$ is ergodic.*

*Proof.* Without loss of generality we can assume that $v = (1, 0, \ldots, 0)$, $a_2 = (0, 1, 0, \ldots, 0), \ldots, a_\ell = (0, \ldots, 0, 1)$. Then $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 : I \to \mathbb{R}$, $\varphi_2 : I \to \mathbb{R}^\ell$ are functions with $\int_I \varphi_1'(x)\,dx \neq 0$ and $\int_I \varphi_2'(x)\,dx = 0$. Applying Proposition 3.1 we can pass to cohomological cocycles which are piecewise linear with constant slope. Now we can apply Theorem 3.5 to prove the ergodicity of $T_\varphi$ which implies the ergodicity of the flow $(\Phi_t)$ on $\mathcal{T} \times \mathbb{R}^\ell$. □

## 7. Examples of ergodic extensions of multivalued Hamiltonian flows

In this section we will apply Theorems 5.7, 6.10 and 6.11 to construct explicit examples of recurrent ergodic extensions of multivalued Hamiltonian flows.

### 7.1. Construction of multivalued Hamiltonians.

Let $T = T_{(\pi,\lambda)} : I \to I$ be an arbitrary IET satisfying the Keane condition. We begin this section by recalling a recipe for constructing multivalued Hamiltonians such that the corresponding flows have special representation over $T$. Let us start from any translation surface $(M, \alpha)$ built over $T$ by applying the zipped rectangles procedure (see [30] or [33]). Denote by $\Sigma = \{p_1, \ldots, p_\kappa\}$ the set of singular point of $(M, \alpha)$. Let $J \subset M \setminus \Sigma$ be a curve transversal to the vertical flow and such that the first return map to $J$ is $T$. We will constantly identify $J$ with the interval $I$. Denote by $S \subset M$ the union of segments of all separatrices connecting singular points with $J$.

We will consider so called regular adapted coordinates on $M \setminus \Sigma$, this is coordinates $\zeta$ relatively to which $\alpha_\zeta = d\zeta$. If $p \in \Sigma$ is a singular point with multiplicity $m \geq 1$ then we consider singular adapted coordinates around $p$, this is coordinates $\zeta$ relatively to which $\alpha_\zeta = d\frac{\zeta^{m+1}}{m+1} = \zeta^m\,d\zeta$. Then all changes of regular coordinates are given by translations. If $\zeta'$ is a regular adapted coordinate and $\zeta$ is a singular adapted coordinate, then $\zeta' = \zeta^{m+1}/(m+1) + c$. Let us consider the vertical vector field $Y$ and the associated vertical flow $(\psi_t)_{t\in\mathbb{R}}$ on $(M, \alpha)$, this is $\alpha_x Y(x) = i$ and $\frac{d}{dt}\psi_t x = Y(\psi_t x)$ for $x \in M \setminus \Sigma$. Then for a regular adapted coordinate $\zeta$ we have $Y(\zeta) = i$ and $\psi_t\zeta = \zeta + it$. Moreover, for a singular adapted coordinate $\zeta$ we have $\zeta^m Y(\zeta) = i$, and hence $Y(\zeta) = \frac{i\bar{\zeta}^m}{|\zeta|^{2m}}$.

For each $\varepsilon > 0$ and $p \in \Sigma$ denote by $B_\varepsilon(p)$ the $\varepsilon$ open ball of center $p$ and let $g = g_\varepsilon : [0, +\infty) \to [0, 1]$ be a monotonic $C^\infty$-function such that $g(x) = x$ for $x \in [0, \varepsilon]$ and $g(x) = 1$ for $x \geq 2\varepsilon$. Fix $\varepsilon > 0$ small enough. In what follows, we will deal with regular adapted coordinates on $M \setminus \bigcup_{p\in\Sigma} B_{2\varepsilon}(p)$ and singular adapted coordinates on $B_{3\varepsilon}(p)$ for $p \in \Sigma$. Let us consider a tangent $C^\infty$-vector field $\tilde{Y}$ on $M$ such that in adapted coordinates $\zeta$ we have

$$\tilde{Y}(\zeta) = \begin{cases} Y(\zeta) = i, & \text{on} \quad M \setminus \bigcup_{p\in\Sigma} B_{2\varepsilon}(p), \\ \frac{g(|\zeta|)^{2m} i\bar{\zeta}^m}{|\zeta|^{2m}}, & \text{on} \quad B_{3\varepsilon}(p),\ p \in \Sigma. \end{cases}$$

Denote by $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ the associated $C^\infty$-flow on $M$. Then $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ on $M\setminus\Sigma$ is obtained by a $C^\infty$ time change in the vertical flow $(\psi_t)_{t\in\mathbb{R}}$, and $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ coincides with $(\psi_t)_{t\in\mathbb{R}}$ on $M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p)$.

Denote by $\tilde{\omega}$ the symplectic $C^\infty$-form on $M$ such that in adapted coordinates $\zeta=x+iy$ we have

$$\tilde{\omega}_\zeta=\begin{cases} dx\wedge dy, & \text{on} \quad M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p),\\ \frac{|\zeta|^{2m}}{g(|\zeta|)^{2m}}dx\wedge dy, & \text{on} \quad B_{3\varepsilon}(p),\ p\in\Sigma.\end{cases}$$

Let us consider the $C^\infty$ 1-form on $M$ given by $\tilde{\beta}=i_{\tilde{Y}}\tilde{\omega}$. Then in adapted coordinates $\zeta=x+iy$ we have

$$\tilde{\beta}_\zeta=\begin{cases} -dx, & \text{on} \quad M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p),\\ -\Re\zeta^m dx+\Im\zeta^m dy, & \text{on} \quad B_{3\varepsilon}(p),\ p\in\Sigma.\end{cases}$$

By Cauchy-Riemann equations, $\frac{\partial}{\partial y}\Re\zeta^m+\frac{\partial}{\partial x}\Im\zeta^m=0$, and hence $d\tilde{\beta}=0$. Therefore $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ is a multivalued Hamiltonian $C^\infty$-flow whose orbits on $M\setminus\Sigma$ coincide with orbits of the vertical flow. It follows that $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ has a special representation over the IET $T_{(\pi,\lambda)}$. If the multiplicity of a singularity $p\in\Sigma$ is equal to $m=1$ then in singular adapted coordinates $\zeta=x+iy$ on $B_\varepsilon(p)$ we have $\tilde{\beta}=-xdx+ydy$, and hence the multivalued Hamiltonian $\widehat{H}$ is equal to $\widehat{H}(x,y)=(y^2-x^2)/2+const$, so $p$ is a non-degenerated critical point of $\widehat{H}$.

Let us consider the symplectic form $\nu=ce^{2x}dx\wedge dy$, $c\neq 0$ on the disk $D=\{(x,y)\in\mathbb{R}^2:(x-1/2)^2+y^2\leq(3/2)^2\}$ and the Hamilton differential equation

$$\frac{dx}{dt}=-y,\quad \frac{dy}{dt}=x(x-1)+y^2.$$

Then the function $-ce^{2x}((x-1)^2+y^2)/2$ is the corresponding Hamiltonian. Denote by $(h_t)$ the associated local Hamiltonian flow. It has two critical points: $z_0=(0,0)$ is a non-degenerated saddle and $(1,0)$ is a center. The point $(0,0)$ has a loop saddle connection which coincides with the curve $e^{2x}((x-1)^2+y^2)=1$, $x\geq 0$. Inside this loop connection all trajectories of $(h_t)$ are periodic (see Figure 2). Such domains are called traps. It is easy to show that the corresponding Hamiltonian vector field

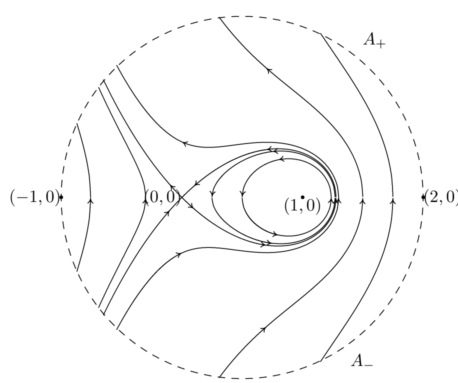


FIGURE 2. The phase portrait of the Hamiltonian flow for $c>0$

$Z$ does not vanish on $\partial D$, and that it has two contact points $(2,0)$ and $(-1,0)$

and two arcs $A_+$ and $A_-$ connecting them with the same length (with respect to $\nu$). Let us cut out from $M \setminus S$ a disk $B_\delta(q)$, $\delta > 0$ such that $B_{2\delta}(q)$ is disjoint from the transversal curve $J$ and from each $B_{3\varepsilon}(p)$, $p \in \Sigma$. The vector field $\tilde{Y}$ does not vanish on $\partial B_\delta(q)$, has two contact points and two arcs $\tilde{A}_+$ and $\tilde{A}_-$ connecting them with the same length (with respect to $\tilde{\omega}$). Choose $c \neq 0$ such that all four arcs $A_+$, $A_-$, $\tilde{A}_+$ and $\tilde{A}_-$ have the same length. Note that $c$ is unique up to sign. Therefore, by Lemma 1 in [4], there exists a $C^\infty$-diffeomorphism $f : \partial D \to \partial B_\delta(q)$, a symplectic $C^\infty$-form $\omega$ on $(M \setminus B_\delta(q)) \cup_f D$ and a tangent $C^\infty$ vector field $X$ such that

- $\mathcal{L}_X \omega = 0$;
- $\omega = \tilde{\omega}$ and $X = \tilde{Y}$ on $M \setminus B_{2\delta}(q)$;
- $\omega = \nu$ and $X = Z$ on $D$;
- the orbits of $X$ on $M \setminus B_{2\delta}(q)$ are pieces of orbits of the flow $(\tilde{\psi_t})$.

Of course, $(M \setminus B_\delta(q)) \cup_f D$ is diffeomorphic to $M$, and so the vector field $X$ and the symplectic form $\omega$ can be considered on $M$. Since $d(i_X\omega) = \mathcal{L}_X\omega = 0$, $X$ is a Hamiltonian vector field with respect to $\omega$. Denote by $(\phi_t)_{t\in\mathbb{R}}$ the Hamilton flow associated to $X$. Since the dynamics of $(\phi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ coincide on $M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup B_{2\delta}(q))$ and $J \subset M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup B_{2\delta}(q))$, the first return map to $J$ for $(\phi_t)_{t\in\mathbb{R}}$ is $T$. Denote by $\gamma \in I$ the first backward intersection with $J$ of the separatrix incoming to $z_0$. Note that $\gamma$ may be an arbitrary point of $I$ different from the ends of the exchanged intervals. It suffices to choose the point $q \in M \setminus S$ and $\delta > 0$ carefully enough. Recall that the saddle point $z_0$ has a loop connection which will be denoted by $\sigma_{loop}(z_0)$. Then the orientation of $\sigma_{loop}(z_0)$ is positive if $c > 0$ and negative if $c < 0$.

*Remark* 7.1. We can repeat the procedure of producing new loop connections (positively or negatively oriented) as many times as we want. Therefore for any collection of distinct points $\{\gamma_1, \ldots, \gamma_s\} \subset \bigcup_{\alpha\in\mathcal{A}} \operatorname{Int} I_\alpha$ and $\delta > 0$ small enough we can construct a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ on $M$ which has $s$ non-degenerated saddle critical points $z_1, \ldots, z_s$ such that each $z_i$ has a loop connection $\sigma_{loop}(z_i)$ included in $B_\delta(z_i)$ for $i = 1, \ldots, s$. Moreover, $(\psi_t)_{t\in\mathbb{R}}$ and $(\phi_t)_{t\in\mathbb{R}}$ coincide on $M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup \bigcup_{i=1}^s B_{2\delta}(z_i)))$ and $\gamma_i \in I$ corresponds to the first backward intersection with $J$ of the separatrix incoming to $z_i$ for $i = 1, \ldots, s$.

We denote by $Trap_i$ the trap corresponding to $z_i$, by $\epsilon(z_i) \in \{-, +\}$ the sign of the orientation of $\sigma_{loop}(z_i)$ for $i = 1, \ldots, s$, and by $\mathcal{T}$ the surface $M$ with the interior of the traps $Trap_i$, $i = 1, \ldots, s$ removed.

*Remark* 7.2. Choose $0 < \delta' < \delta$ such that $\sigma_{loop}(z_i) \cap (M \setminus B_{\delta'}(z_i)) \neq \emptyset$ for $i = 1, \ldots, s$. Let $f : M \to \mathbb{R}$ be a $C^\infty$-function with $\int_{\mathcal{T}} f\omega = 0$ and such that $f$ vanishes on each $B_{2\varepsilon}(p)$, $p \in \Sigma$ and $B_\delta(z_i)$, $i = 1, \ldots, s$. Then the corresponding function

$$\varphi_f : I \to \mathbb{R}, \quad \varphi_f(x) = \varphi(x) = \int_0^{\tau(x)} f(\phi_t x)\, dt$$

($\tau : I \to \mathbb{R}_+$ is the first-return time map of the flow $(\phi_t)_{t\in\mathbb{R}}$ to $J$) can be extended to a $C^\infty$-function on the closure of any interval of the partition $\mathcal{P}(\{l_\alpha : \alpha \in \mathcal{A}\} \cup \{\gamma_i : i = 1, \ldots, s\})$. Moreover,

$$d_i(f) := \varphi_+(\gamma_i) - \varphi_-(\gamma_i) = \epsilon(z_i) \int_{-\infty}^{+\infty} f(\phi_t u_{z_i})\, dt \text{ and } s(\varphi) = \sum_{i=1}^{s} d_i(f), \tag{7.1}$$

where $u_{z_i}$ is an arbitrary point of $\sigma_{loop}(z_i)$ for $i = 1, \ldots, s$.

**Lemma 7.3.** *For every $(d_1, \ldots, d_s) \in \mathbb{R}^s$ there exists a $C^\infty$-function $f : M \to \mathbb{R}$ which vanishes on a neighborhood of each fixed point of $(\phi_t)$ such that $\int_{\mathcal{T}} f\omega = 0$ and $(d_1(f), \ldots, d_s(f)) = (d_1, \ldots, d_s)$.*

*Proof.* Let us start from $f \equiv 0$. Since $\sigma_{loop}(z_i) \cap (B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)) \neq \emptyset$, we can modify $f$ smoothly on $B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)$ such that

$$\epsilon(z_i) \int_{-\infty}^{+\infty} f(\phi_t u_{z_i})\, dt = d_i \text{ and } \int_{(B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)) \setminus Trap_i} f\omega = 0$$

for $i = 1, \ldots, s$. In view of (7.1), it follows that $d_i(f) = d_i$ for $i = 1, \ldots, s$. Moreover,

$$\int_{\mathcal{T}} d\omega = \sum_{i=1}^{s} \int_{(B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)) \setminus Trap_i} f\omega = 0.$$

□

**Lemma 7.4.** *For every $h \in H_\pi$ there exists a $C^\infty$-function $f : M \to \mathbb{R}$ such that $\varphi_f = \sum_{\alpha \in \mathcal{A}} h_\alpha \chi_{I_\alpha}$ (cf. Remark 7.2). If $h \in H_\pi \cap \Gamma_0$ then $\int_{\mathcal{T}} f\omega = 0$.*

*Proof.* Following [33], for every $\alpha \in \mathcal{A}$ denote by $[v_\alpha] \in H_1(M, \mathbb{R})$ the homology class of any closed curve $v_\alpha$ formed by a segment of the orbit for $(\psi_t)_{t\in\mathbb{R}}$ starting at any point $x \in \operatorname{Int} I_\alpha$ and ending at $Tx$ together with the segment of $J$ that joins $Tx$ and $x$. Let $\Psi : H^1(M, \mathbb{R}) \to \mathbb{R}^{\mathcal{A}}$ be given by $\Psi([\varrho]) = (\int_{v_\alpha} \varrho)_{\alpha\in\mathcal{A}}$. By Lemma 2.19 in [33], the map $\Psi : H^1(M, \mathbb{R}) \to H_\pi$ establishes the isomorphism of linear spaces. Therefore for every $h \in H_\pi$ there exists a closed 1-form $\varrho$ such that $\Psi([\varrho]) = h$ and $\varrho$ vanishes on an open neighborhood of $J$. Let $f : M \to \mathbb{R}$ be given by $f(x) = \varrho_x X_x$ for $x \in M$.

For every $x \in \operatorname{Int} I_\alpha$ let $v_x$ be the closed curve formed by the segment of orbit for $(\phi_t)_{t\in\mathbb{R}}$ starting at $x$ and ending at $Tx$ together with the segment of $J$ that joins $Tx$ and $x$. Then $[v_x] = [v_\alpha]$. Therefore, $h_\alpha = \int_{v_\alpha} \varrho = \int_{v_x} \varrho$.

Since the form $\rho$ vanishes on $J$, we have

$$\int_{v_x} \varrho = \int_0^{\tau(x)} \varrho_{\phi_t x} X(\phi_t x)\, dt = \int_0^{\tau(x)} f(\phi_t x)\, dt = \varphi_f(x).$$

Consequently, $\varphi_f(x) = h_\alpha$ for all $x \in \operatorname{Int} I_\alpha$ and $\alpha \in \mathcal{A}$. If we assume that $h \in H_\pi \cap \Gamma_0$, then

$$0 = \langle \lambda, h \rangle = \int_I \varphi_f(x) dx = \int_{\mathcal{T}} f\omega.$$

□

7.2. **Examples.**
Let us consider an IET $T = T_{(\pi,\lambda)}$ and a set $\{\gamma_1, \ldots, \gamma_s\} \subset I \setminus \{l_\alpha : \alpha \in \mathcal{A}\}$, $s \geq 3$. Set $\ell = s - 1$. Suppose that

(7.2) $\{\gamma_1, \ldots, \gamma_s\}$ is of periodic type with respect to $T$ and $\theta_2(T)/\theta_1(T) < 1/\ell$.

Recall that $T$ has to be of periodic type as well. An explicit example of such data for $s = 3$ is given at the end of this section.

By Remark 7.1, there exists a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ with $s$ traps (determined by saddle points $z_i$, $i = 1, \ldots, s$) on a symplectic surface $(M, \omega)$ such that $(\phi_t)_{t\in\mathbb{R}}$ on $\mathcal{T}$ has a special representation over $T_{(\pi,\lambda)}$ and $\gamma_i$ corresponds to the first backward intersection with the transversal curve of the separatrix incoming to $z_i$ for $i=1,\ldots,s$.

By Lemma 7.3 and (7.1), there exists a $C^\infty$-function $f_1 : M \to \mathbb{R}$ such that $\int_{\mathcal{T}} f_1\omega = 0$ and $s(\varphi_{f_1}) \neq 0$. In view of Theorem 6.10, the flow $(\Phi_t^{f_1})_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}$ is ergodic.

Let $\bar{d}_1,\ldots,\bar{d}_s$ be vectors in $\mathbb{R}^{\ell-1}$ such that $\overline{\mathbb{Z}(\bar{d}_1,\ldots,\bar{d}_s)} = \mathbb{R}^{\ell-1}$ and $\sum_{i=1}^s \bar{d}_i = \bar{0}$. Since $s = (\ell-1)+2$, the existence of such collection follows directly from Remark 3.8. By Lemma 7.3, there exists a $C^\infty$-function $f_2' : M \to \mathbb{R}^{\ell-1}$ such that $f_2'$ vanishes on a neighborhood of each fixed point of $(\phi_t)$, $\int_{\mathcal{T}} f_2'\omega = \bar{0}$ and $(\varphi_{f_2'})_+(\gamma_i) - (\varphi_{f_2'})_-(\gamma_i) = \bar{d}_i$ for $i=1,\ldots,s$. Then $\varphi_{f_2'}$ has zero mean and, by (7.1), $s(\varphi_{f_2'}) = \sum_{i=1}^s \bar{d}_i = \bar{0}$.

Denote by $\bar{\varphi} : I \to \mathbb{R}^\ell$ the piecewise constant function with zero mean whose discontinuities are $\gamma_i$, $i=1,\ldots,s$ and $\bar{\varphi}_+(\gamma_i) - \bar{\varphi}_-(\gamma_i) = \bar{d}_i$ for $i=1,\ldots,s$. In view of (7.2) and Remark 5.6, $\bar{\varphi} \in \mathrm{BV}_0^\lozenge(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$. By Remark 7.2, $\varphi_{f_2'}$ can be extended to a $C^\infty$-function on the closure of any interval of the partition $\mathcal{P}(\{l_\alpha : \alpha\in\mathcal{A}\}\cup\{\gamma_i : i=1,\ldots,s\})$. It follows that $\varphi_{f_2'} - \bar{\varphi} \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}}, \mathbb{R}^{\ell-1})$. Moreover, $\varphi_{f_2'} - \bar{\varphi}$ has zero mean and $s(\varphi_{f_2'} - \bar{\varphi}) = s(\varphi_{f_2'}) - s(\bar{\varphi}) = 0$. Therefore, by Proposition 3.1, $\varphi_{f_2'} - \bar{\varphi}$ is cohomologous to $\bar{h}^1 = (h_1^1,\ldots,h_{\ell-1}^1)$, where $h_i^1 \in \Gamma_0$ for $i=1,\ldots,\ell-1$.

In view of Theorem 5.2 applied to the coordinate functions of the function $\bar{\varphi} + \bar{h}^1 \in \mathrm{BV}_0^\lozenge(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$, there exists $\bar{h}^2 = (h_2^1,\ldots,h_{\ell-1}^2)$ with $h_i^2 \in \Gamma_u \cap \Gamma_0$ for $i=1,\ldots,\ell-1$ such that $\bar{\varphi} + \bar{h}^1 + \bar{h}^2 = \widehat{\bar{\varphi}+\bar{h}^1}$. Moreover, by Theorem 5.7, the cocycle $\bar{\varphi} + \bar{h}^1 + \bar{h}^2 = \widehat{\bar{\varphi}+\bar{h}^1}$ is ergodic. As $\varphi_{f_2'} + \bar{h}^2$ is cohomologous to $\bar{\varphi} + \bar{h}^1 + \bar{h}^2$, it is ergodic as well. By Lemma 7.4, there exists a $C^\infty$-function $f_2'' : M \to \mathbb{R}^{\ell-1}$ with $\int_{\mathcal{T}} f_2''\omega = \bar{0}$ such that $\varphi_{f_2''} = \bar{h}^2$. Setting $f_2 = f_2' + f_2''$, we have $\int_{\mathcal{T}} f_2\omega = \bar{0}$, $\varphi_{f_2} = \varphi_{f_2'} + \bar{h}^2$, and $s(\varphi_{f_2}) = s(\varphi_{f_2'}) = \sum_{i=1}^s \bar{d}_i = \bar{0}$. It follows that the flow $(\Phi_t^{f_2})_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}^{\ell-1}$ is ergodic. Finally applying Theorem 6.11 to $f = (f_1, f_2) : I \to \mathbb{R}^\ell$ we have the ergodicity of the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}^\ell$.

*Example* 1. Let us consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \end{pmatrix}$ and a corresponding pair $\pi'$. On the Rauzy graph $\mathcal{R}(\pi')$ let us consider the loop starting from $\pi'$ and passing through the edges labeled consecutively by

$$1,0,1,1,1,1,1,1,0,1,1,0,1,1,1,0,0,1,1,1,1,0,1,0,0,0,0,1,1,1.$$

Then the resulting matrix is

$$A' := \begin{pmatrix} 9 & 8 & 20 & 20 & 15 & 5 & 5 \\ 1 & 2 & 4 & 4 & 3 & 2 & 2 \\ 2 & 2 & 6 & 5 & 4 & 1 & 1 \\ 2 & 2 & 5 & 6 & 4 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 & 3 & 2 & 1 \\ 1 & 1 & 3 & 3 & 2 & 1 & 2 \end{pmatrix}$$

and $(A')^2$ has positive entries. Let $\lambda' \in \mathbb{R}_+^7$ be a Perron-Frobenius eigenvector of $A'$. Then $T_{(\pi',\lambda')}$ is of periodic type and $A'$ is its periodic matrix. Of course, $T_{(\pi',\lambda')}$ is an exchange of 4 intervals, more precisely, $T_{(\pi',\lambda')} = T_{(\pi_4^{sym},\lambda)}$, where $\lambda_1 = \lambda_1' + \lambda_2'$,

$\lambda_2 = \lambda'_3 + \lambda'_4$, $\lambda_3 = \lambda'_5$ and $\lambda_4 = \lambda'_6 + \lambda'_7$. As we already noticed in Section 5.3, $T_{(\pi_4^{sym},\lambda)}$ has also periodic type and the family $\gamma_1 = \lambda'_1$, $\gamma_2 = \lambda'_1 + \lambda'_2 + \lambda'_3$, $\gamma_3 = \lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 + \lambda'_5 + \lambda'_6$ is of periodic type with respect to $T_{(\pi_4^{sym},\lambda)}$. Moreover,

$$A = \begin{pmatrix} 10 & 24 & 18 & 7 \\ 4 & 11 & 8 & 2 \\ 1 & 2 & 2 & 0 \\ 3 & 7 & 5 & 3 \end{pmatrix}$$

is the periodic matrix of $T_{(\pi_4^{sym},\lambda)}$, so

$$\rho_1 = \frac{13}{2} + \frac{1}{2}\sqrt{115} + \frac{1}{2}\sqrt{280 + 26\sqrt{115}}, \quad \rho_2 = \frac{13}{2} - \frac{1}{2}\sqrt{115} + \frac{1}{2}\sqrt{280 - 26\sqrt{115}}.$$

Hence $\theta_2/\theta_1 \approx 0.164 < 1/2$, so $T_{(\pi_4^{sym},\lambda)}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfy (7.2) with $s = 3$.

*Remark* 7.5. Similar examples can be constructed by matching the set $\{\gamma_1, \dots, \gamma_s\}$ for a fixed IET $T = T_{\pi,\lambda}$ of periodic type. Let $p \geq 1$ be the period of $T$ and let $\rho > 1$ be the Perron-Frobenius eigenvalue of the periodic matrix $A$ of $T$. For every $x \in I$ let $k(x) = \inf\{k \geq 0 : T^{-k}x \in I^{(p)}\}$. Let us consider the map $S : I \to I$, $S(x) = \rho \cdot T^{-k(x)}x$. Note that for every $\alpha \in \mathcal{A}$ the map $S$ has at least $A_{\alpha\alpha} - 2$ fixed points in the interior of $I_\alpha$. Therefore, multiplying the period of $T$, if necessary, for every $s \geq 1$ we can find $s$ distinct fixed points $\gamma_1, \dots, \gamma_s$ different from $l_\alpha$, $\alpha \in \mathcal{A}$. In view of Theorem 23 in [27], the set $\{\gamma_1, \dots, \gamma_s\}$ is of periodic type with respect to $T$.

Denote by $M_2$ a compact $C^\infty$-surface of genus 2. We can apply the above constructions to the sequence of IETs $T$ with arbitrary small values of the ratio $\theta_2(T)/\theta_1(T)$ from Appendix B to obtain the following result:

**Corollary 7.6.** *For every $\ell \geq 1$ there exists a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ on $M_2$ and a $C^\infty$-function $f : M_2 \to \mathbb{R}^\ell$ for which the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ on $\mathcal{T} \times \mathbb{R}^\ell$ is ergodic.* □

## Appendix A. Deviation of cocycles: proofs

Let $T : I \to I$ be an arbitrary IET satisfying Keane's condition. For every $x \in I$ and $n \geq 0$ set

$$m(x, n, T) = \max\{l \geq 0 : \#\{0 \leq k \leq n : T^k x \in I^{(l)}\} \geq 2\}.$$

**Proposition A.1** (see [36] or [33])**.** *For every $x \in I$ and $n > 0$ we have*

$$\min_{\alpha\in\mathcal{A}} Q_\alpha(m) \leq n \leq d \max_{\alpha\in\mathcal{A}} Q_\alpha(m+1) = d\|Q(m+1)\|, \text{ where } m = m(x, n, T). \ \square$$

*Remark* A.2. Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and $A$ is its periodic matrix. Then there exists $C > 0$ such that $e^{\theta_1 k}/C \leq \|A^k\| \leq Ce^{\theta_1 k}$ for every $k \geq 1$, where $\theta_1$ is the greatest Lyapunov exponent of $A$. Let $m = m(x, n, T)$. Since $\|A^n\| = \max_{\alpha\in\mathcal{A}} A^n_\alpha$, by Proposition A.1 and (4.2), it follows that

$$n \geq \min_{\alpha\in\mathcal{A}} Q_\alpha(m) = \min_{\alpha\in\mathcal{A}} A^m_\alpha \geq \frac{1}{\nu(A)} \max_{\alpha\in\mathcal{A}} A^m_\alpha = \frac{\|A^m\|}{\nu(A)} \geq \frac{e^{\theta_1 m}}{C\nu(A)}.$$

Thus

$$m \leq \frac{1}{\theta_1} \log(C\nu(A)n). \tag{A.1}$$

**Proposition A.3** (see [24]). *For each bounded function $\varphi : I \to \mathbb{R}$, $x \in I$ and $n > 0$ we have*

$$|\varphi^{(n)}(x)| \leq 2\sum_{l=0}^{m} \|Z(l+1)\|\|S(l)\varphi\|_{\sup}, \text{ where } m = m(x,n,T). \tag{A.2}$$

*If additionally $\varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then*

$$\|S(l)\varphi\|_{\sup} \leq \sum_{1\leq j\leq l} \|Z(j)\|\|S(j,l)|_{\Gamma_0^{(j)}}\| \operatorname{Var}\varphi. \ \square \tag{A.3}$$

*Proof of Theorem 2.2.* Since $\lambda$ is a positive Perron-Frobenius eigenvector of $A$, by Proposition 5 in [36], the restriction of $A^t$ to the invariant space $Ann(\lambda) = \{h \in \mathbb{R}^{\mathcal{A}} : \langle h, \lambda\rangle = 0\}$ has the following Lyapunov exponents:

$$\theta_2 \geq \theta_3 \geq \ldots \geq \theta_g \geq 0 = \ldots = 0 \geq -\theta_g \geq \ldots \geq -\theta_3 \geq -\theta_2 > -\theta_1.$$

Thus there exists $C > 0$ such that for every $k \in \mathbb{N}$ we have

$$\|(A^t)^k h\| \leq Ck^{M-1}\exp(k\theta_2)\|h\| \text{ for all } h \in Ann(\lambda).$$

Since $\Gamma_0^{(j)} = Ann(\lambda)$ and $S(j,l) = Q^t(j,l) = (A^t)^{l-j}$ on $\Gamma_0^{(j)}$, by (A.3),

$$\begin{aligned}\|S(l)\varphi\|_{\sup} &\leq \sum_{1\leq j\leq l} \|A\|\|(A^t)^{l-j}|_{Ann(\lambda)}\| \operatorname{Var}\varphi \\ &\leq \sum_{0\leq k<l} \|A\|Ck^{M-1}\exp(k\theta_2)\operatorname{Var}\varphi \leq \|A\|Cl^M \exp(l\theta_2)\operatorname{Var}\varphi.\end{aligned}$$

In view of (A.2), it follows that

$$\begin{aligned}|\varphi^{(n)}(x)| &\leq 2\sum_{l=0}^{m}\|A\|\|S(l)\varphi\|_{\sup} \leq 2\sum_{l=0}^{m}\|A\|^2 Cl^M\exp(l\theta_2)\operatorname{Var}\varphi \\ &\leq 2\|A\|^2 Cm^{M+1}\exp(m\theta_2)\operatorname{Var}\varphi,\end{aligned}$$

where $m = m(x,n,T)$. Consequently, by (A.1),

$$|\varphi^{(n)}(x)| \leq 2\frac{\|A\|^2C^2\nu(A)}{\theta_1^{M+1}}\log^{M+1}(C\nu(A)n)n^{\theta_2/\theta_1}\operatorname{Var}\varphi.$$

$\square$

## Appendix B. Possible values of $\theta_2/\theta_1$

In this section we will show that for each symmetric pair $\pi_4^{sym}$ there are IETs of periodic type such that $\theta_2/\theta_1$ is arbitrary small and the spectrum of the periodic matrix is non-degenerated. As it was shown in [24] for every natural $n$ the matrix

$$M(n) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ n & n+1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ n+1 & n+2 & 2 & 2 \end{pmatrix}$$

is a resulting matrix corresponding to a loop in the Rauzy class of $\pi_4^{sym}$ and starting from $\pi_4^{sym}$. Since $M(n)$ is primitive, there exists an IET of periodic type for which

$M(n)$ is its periodic matrix. The eigenvalues $\rho_1(n) > \rho_2(n) > 1 > \rho_3(n) > \rho_4(n) > 0$ of $M(n)$ are of the form

$$\rho_1(n) = \frac{1}{2}\left(a_n^+ + \sqrt{(a_n^+)^2 - 4}\right), \quad \rho_2(n) = \frac{1}{2}\left(a_n^- + \sqrt{(a_n^-)^2 - 4}\right),$$
$$\rho_3(n) = \frac{1}{2}\left(a_n^- - \sqrt{(a_n^-)^2 - 4}\right), \quad \rho_4(n) = \frac{1}{2}\left(a_n^+ - \sqrt{(a_n^+)^2 - 4}\right),$$

where

$$a_n^\pm = \frac{1}{2}(n + 6 \pm \sqrt{n^2+4}).$$

Since $a_n^+ \to +\infty$ and $a_n^- \to 3$ as $n \to +\infty$, it follows that

$$\frac{\theta_2(n)}{\theta_1(n)} = \frac{\log \rho_2(n)}{\log \rho_1(n)} \to 0 \text{ as } n \to +\infty.$$

## Appendix C. Deviation of corrected functions

*Proof of Theorem 5.1.* First note that for every natural $k$ the subspace $\Gamma_{cs}^{(k)} \subset \mathbb{R}^{\mathcal{A}}$ is the direct sum of invariant subspaces associated to Jordan blocks of $A^t$ with non-positive Lyapunov exponents. It follows that there exists $C > 0$ such that

$$\|(A^t)^n h\| \leq C n^{M-1}\|h\| \text{ for all } h \in \Gamma_{cs}^{(k)} \text{ and } n \geq 0. \tag{C.1}$$

It is easy to show that $\Gamma_{cs}^{(k)} \subset \Gamma_0^{(k)}$.

Next note that $S(k,l)\Gamma_{cs}^{(k)} = \Gamma_{cs}^{(l)}$ and the quotient linear transformation

$$S_u(k,l) : \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})/\Gamma_{cs}^{(k)} \to \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})/\Gamma_{cs}^{(l)}$$

is invertible. Moreover,

$$S_u(k,l) \circ U^{(k)}\varphi = U^{(l)} \circ S(k,l)\varphi \text{ for } \varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}). \tag{C.2}$$

Since $\Gamma_u^{(k)} \subset \mathbb{R}^{\mathcal{A}}$ the direct sum of invariant subspaces associated to Jordan blocks of $A^t$ with positive Lyapunov exponents, $\mathbb{R}^{\mathcal{A}} = \Gamma^{(k)} = \Gamma_{cs}^{(k)} \oplus \Gamma_u^{(k)}$ is an invariant decomposition. Moreover, there exist $\theta_+ > 0$ and $C > 0$ such that

$$\|(A^t)^{-n}h\| \leq C\exp(-n\theta_+)\|h\| \text{ for all } h \in \Gamma_u^{(k)} \text{ and } n \geq 0.$$

Since the linear operators $A^t : \Gamma_u^{(k)} \to \Gamma_u^{(k)}$ and $A^t : \Gamma^{(k)}/\Gamma_{cs}^{(k)} \to \Gamma^{(k)}/\Gamma_{cs}^{(k)}$ are isomorphic, there exists $C' > 0$ such that

$$\|(A^t)^{-n}(h + \Gamma_{cs}^{(k)})\| \leq C'\exp(-n\theta_+)\|h + \Gamma_{cs}^{(k)}\|$$

for all $h + \Gamma_{cs}^{(k)} \in \Gamma^{(k)}/\Gamma_{cs}^{(k)}$ and $n \geq 0$. Consequently,

$$\|(S_u(k,l))^{-1}(h + \Gamma_{cs}^{(k)})\| \leq C'\exp(-(l-k)\theta_+)\|h + \Gamma_{cs}^{(k)}\| \tag{C.3}$$

for all $h + \Gamma_{cs}^{(k)} \in \Gamma^{(k)}/\Gamma_{cs}^{(k)}$ and $0 \leq k < l$.

Let us consider the linear operator $C^{(k)} : \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \Gamma_0^{(k)}$ given by

$$C^{(k)}\varphi(x) = \frac{1}{|I_\alpha^{(k)}|}\int_{I_\alpha^{(k)}} \varphi(t)dt \text{ if } x \in I_\alpha^{(k)}.$$

Then $P_0^{(k)}\varphi = \varphi - C^{(k)}\varphi$ and

$$\|C^{(k)}\varphi\| \leq \|\varphi\|_{\sup}, \tag{C.4}$$

$$\|P_0^{(k)}\varphi\|_{\sup} \leq \mathrm{Var}\, P_0^{(k)}\varphi = \mathrm{Var}\,\varphi. \tag{C.5}$$

Let $\varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Note that for $0 \le k \le l$ we have

$$\begin{aligned} & P_0^{(k)}\varphi - S(k,l)^{-1}\circ P_0^{(l)}\circ S(k,l)\varphi \\ &= \sum_{k\le r<l} (S(k,r)^{-1}\circ P_0^{(r)}\circ S(k,r) - S(k,r+1)^{-1}\circ P_0^{(r+1)}\circ S(k,r+1))\varphi \\ &= \sum_{k\le r<l} S(k,r+1)^{-1}\circ (S(r,r+1)\circ P_0^{(r)} - P_0^{(r+1)}\circ S(r,r+1))\circ S(k,r)\varphi. \end{aligned}$$

Next observe that

$$(S(r,r+1)\circ P_0^{(r)} - P_0^{(r+1)}\circ S(r,r+1))\psi = C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi \in \Gamma_0^{r+1}$$

for $\psi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(r)})$. Indeed, if $\psi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(r)})$ then $\psi = P_0^{(r)}\psi + C^{(r)}\psi$ and

$$P_0^{(r+1)}\circ S(r,r+1)\psi = P_0^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi + P_0^{(r+1)}\circ S(r,r+1)\circ C^{(r)}\psi.$$

Since $S(r,r+1)\circ C^{(r)}\psi \in \Gamma_0^{(r+1)}$, we obtain $P_0^{(r+1)}\circ S(r,r+1)\circ C^{(r)}\psi = 0$; hence

$$\begin{aligned} & S(r,r+1)\circ P_0^{(r)}\psi - P_0^{(r+1)}\circ S(r,r+1)\psi \\ &= S(r,r+1)\circ P_0^{(r)}\psi - P_0^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi \\ &= C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi. \end{aligned}$$

Therefore

$$\begin{aligned} & P_0^{(k)}\varphi - S(k,l)^{-1}\circ P_0^{(r)}\circ S(k,l)\varphi \\ &= \sum_{k\le r<l} S(k,r+1)^{-1}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi \in \Gamma_0^{(k)}. \end{aligned}$$

In view of (C.2),

$$\begin{aligned} & (U^{(k)}\circ P_0^{(k)} - U^{(k)}\circ S(k,l)^{-1}\circ P_0^{(r)}\circ S(k,l))\varphi \\ &= \sum_{k\le r<l} S_u(k,r+1)^{-1}\circ U^{(r+1)}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi. \end{aligned}$$

Moreover, using (C.4), (5.2), (C.5) and (5.1) successively we obtain

$$\begin{aligned} & \|C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi\| \le \|S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi\|_{\sup} \\ &\le \|Z(r+1)\|\|P_0^{(r)}\circ S(k,r)\varphi\|_{\sup} \le \|A\|\operatorname{Var} S(k,r)\varphi \le \|A\|\operatorname{Var}\varphi. \end{aligned}$$

Next let consider the series in $\Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$

$$\sum_{r\ge k}(S_u(k,r+1))^{-1}\circ U^{(r+1)}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi. \tag{C.6}$$

Since $\|U^{(r+1)}\| = 1$ and $U^{(r+1)}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi \in \Gamma_0^{(r+1)}/\Gamma_{cs}^{(r+1)}$, by (C.3), the norm of the $r$-th element of the series (C.6) is bounded from above by $C'\|A\|\exp(-(r-k+1)\theta_+)\operatorname{Var}\varphi$. As

$$\sum_{r\ge k} C'\|A\|\exp(-(r-k+1)\theta_+)\operatorname{Var}\varphi < +\infty,$$

the series (C.6) converges in $\Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$. Denote by $\Delta P^{(k)}\varphi \in \Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$ the sum of (C.6). Then there exists $K > 0$ such that

$$\|\Delta P^{(k)}\varphi\| \le K\operatorname{Var}\varphi, \text{ for every } \varphi \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \text{ and } k \ge 0. \tag{C.7}$$

It follows that the sequence (5.4) converges in $\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})/\Gamma_{cs}^{(k)}$ and

$$P^{(k)} = U^{(k)}\circ P_0^{(k)} - \Delta P^{(k)}. \tag{C.8}$$

□

**Lemma C.1.** *For all* $0\le k\le l$ *and* $\varphi\in\mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$ *we have*

$$S_u(k,l)\circ P^{(k)}\varphi \;=\; P^{(l)}\circ S(k,l)\varphi, \tag{C.9}$$

$$\|P^{(k)}\varphi\|_{\sup/\Gamma_{cs}^{(k)}} \;\le\; (1+K)\operatorname{Var}\varphi. \tag{C.10}$$

*Proof.* By definition and by (C.2),

$$\begin{aligned}
S_u(k,l)\circ P^{(k)}\varphi &= S_u(k,l)\lim_{r\to\infty}U^{(k)}\circ S(k,r)^{-1}\circ P_0^{(r)}\circ S(k,r)\varphi\\
&= \lim_{r\to\infty}U^{(l)}\circ S(l,r)\circ S(k,r)^{-1}\circ P_0^{(r)}\circ S(k,r)\varphi\\
&= \lim_{r\to\infty}U^{(l)}\circ S(l,r)^{-1}\circ P_0^{(r)}\circ S(l,r)\circ S(k,l)\varphi = P^{(l)}\circ S(k,l)\varphi.
\end{aligned}$$

Moreover, by (C.8), (C.5) and (C.7),

$$\|P^{(k)}\varphi\|_{\sup/\Gamma_{cs}^{(k)}} \le \|P_0^{(k)}\varphi\|_{\sup} + \|\Delta P^{(k)}\varphi\| \le (1+K)\operatorname{Var}\varphi.$$

□

Let $p:\{0,1,\ldots,d,d+1\}\to\{0,1,\ldots,d,d+1\}$ stand for the permutation

$$p(j)=\begin{cases}\pi_1\circ\pi_0^{-1}(j) & \text{if} \quad 1\le j\le d\\ j & \text{if} \quad j=0,d+1.\end{cases}$$

Following [30, 31], denote by $\sigma=\sigma_\pi$ the corresponding permutation on $\{0,1,\ldots,d\}$,

$$\sigma(j)=p^{-1}(p(j)+1)-1 \text{ for } 0\le j\le d.$$

Then $\widehat{T}_{(\pi,\lambda)}r_{\pi_0^{-1}(j)} = T_{(\pi,\lambda)}r_{\pi_0^{-1}(\sigma j)}$ for all $j\ne 0,p^{-1}(d)$. Denote by $\Sigma(\pi)$ the set of orbits for the permutation $\sigma$. Let $\Sigma_0(\pi)$ stand for the subset of orbits that do not contain zero. Then $\Sigma(\pi)$ corresponds to the set of singular points of any translation surface associated to $\pi$ and hence $\#\Sigma(\pi)=\kappa(\pi)$. For every $\mathcal{O}\in\Sigma(\pi)$ denote by $b(\mathcal{O})\in\mathbb{R}^{\mathcal{A}}$ the vector given by

$$b(\mathcal{O})_\alpha = \chi_{\mathcal{O}}(\pi_0(\alpha)) - \chi_{\mathcal{O}}(\pi_0(\alpha)-1) \text{ for } \alpha\in\mathcal{A}.$$

**Lemma C.2** (see [31])**.** *For every irreducible pair* $\pi$ *we have* $\sum_{\mathcal{O}\in\Sigma(\pi)}b(\mathcal{O})=0$*, the vectors* $b(\mathcal{O})$*,* $\mathcal{O}\in\Sigma_0(\pi)$ *are linearly independent and the linear subspace generated by them is equal to* $\ker\Omega_\pi$*. Moreover,* $h\in H_\pi$ *if and only if* $\langle h,b(\mathcal{O})\rangle=0$ *for every* $\mathcal{O}\in\Sigma(\pi)$*.* □

*Remark* C.3. Let $\Lambda^\pi:\mathbb{R}^{\mathcal{A}}\to\mathbb{R}^{\Sigma_0(\pi)}$ stand for the linear transformation given by $(\Lambda^\pi h)_{\mathcal{O}}=\langle h,b(\mathcal{O})\rangle$ for $\mathcal{O}\in\Sigma_0(\pi)$. By Lemma C.2, $H_\pi=\ker\Lambda^\pi$ and if $\mathbb{R}^{\mathcal{A}}=F\oplus H_\pi$ is a direct sum decomposition then $\Lambda^\pi:F\to\mathbb{R}^{\Sigma_0(\pi)}$ establishes an isomorphism of linear spaces. It follows that there exists $K_F>0$ such that

$$\|h\|\le K_F\|\Lambda^\pi h\| \quad \text{for all} \quad h\in F.$$

**Lemma C.4** (see [31])**.** *Suppose that* $T_{(\tilde\pi,\tilde\lambda)}=\mathcal{R}(T_{(\pi,\lambda)})$*. Then there exists a bijection* $\xi:\Sigma(\pi)\to\Sigma(\tilde\pi)$ *such that* $\Theta(\pi,\lambda)^{-1}b(\mathcal{O})=b(\xi\mathcal{O})$ *for* $\mathcal{O}\in\Sigma(\pi)$*.* □

Let $T = T_{(\pi,\lambda)}$ be an IET satisfying Keane's condition. For every $\mathcal{O} \in \Sigma(\pi)$ and $\varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ let

$$\mathcal{O}(\varphi) = \sum_{\alpha\in\mathcal{A},\pi_0(\alpha)\in\mathcal{O}} \varphi_-(r_\alpha) - \sum_{\alpha\in\mathcal{A},\pi_0(\alpha)-1\in\mathcal{O}} \varphi_+(l_\alpha).$$

Note that if $h \in \Gamma^{(0)}$ (i.e. $h$ is a function constant on exchanged intervals), then

$$\mathcal{O}(h) = \sum_{\pi_0(\alpha)\in\mathcal{O}} h_\alpha - \sum_{\pi_0(\alpha)-1\in\mathcal{O}} h_\alpha = \sum_{\alpha\in\mathcal{A}} (\chi_{\mathcal{O}}(\pi_0(\alpha)) - \chi_{\mathcal{O}}(\pi_0(\alpha)-1))h_\alpha = \langle h, b(\mathcal{O})\rangle.$$

Moreover,

$$|\mathcal{O}(\varphi)| \leq 2d\|\varphi\|_{\sup} \text{ for every } \varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \text{ and } \mathcal{O} \in \Sigma(\pi). \tag{C.11}$$

Let us consider $T_{(\tilde\pi,\tilde\lambda)} = \mathcal{R}(T_{(\pi,\lambda)})$ and the renormalized cocycle $\tilde\varphi : \tilde I \to \mathbb{R}$, this is

$$\tilde\varphi(x) = \sum_{0\leq i<\Theta_\beta(\pi,\lambda)} \varphi(T^i_{(\pi,\lambda)}x) \text{ for } x \in \tilde I_\beta.$$

The proof of the following lemma is straightforward and we leave it to the reader.

**Lemma C.5.** *If $\varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then $\tilde\varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} \tilde I_\alpha)$ and $(\xi\mathcal{O})(\tilde\varphi) = \mathcal{O}(\varphi)$ for each $\mathcal{O} \in \Sigma(\pi)$.* □

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type and let $A$ be its periodic matrix. By Lemma C.4, there exists a bijection $\xi : \Sigma(\pi) \to \Sigma(\pi)$ such that $A^{-1}b(\mathcal{O}) = b(\xi\mathcal{O})$ for $\mathcal{O} \in \Sigma(\pi)$. Since $\xi^N = Id_{\Sigma(\pi)}$ for some $N \geq 1$, multiplying the period of $T$ by $N$, we can assume that $\xi = Id_{\Sigma(\pi)}$. Therefore $Ab(\mathcal{O}) = b(\mathcal{O})$ for each $\mathcal{O} \in \Sigma(\pi)$, and hence $A|_{\ker\Omega_\pi} = Id$. It follows that the dimension of $\Gamma^{(0)}_c = \{h \in \mathbb{R}^{\mathcal{A}} : A^t h = h\}$ is greater or equal than $\kappa - 1$. Denote by $\Gamma^{(0)}_s \subset \mathbb{R}^{\mathcal{A}}$ the direct sum of invariant subspaces associated to Jordan blocks of $A^t$ with negative Lyapunov exponents.

Assume that $T$ has non-degenerated spectrum, i.e. $\theta_g > 0$. Then $\dim\Gamma^{(0)}_s = \dim\Gamma^{(0)}_u = g$. Since $2g + \kappa - 1 = d$ and $\dim\Gamma^{(0)}_c = \kappa - 1$,

$$\mathbb{R}^{\mathcal{A}} = \Gamma^{(0)} = \Gamma^{(0)}_s \oplus \Gamma^{(0)}_c \oplus \Gamma^{(0)}_u$$

is an $A^t$–invariant decompositions. It follows that $\Gamma^{(0)}_s \oplus \Gamma^{(0)}_c = \Gamma^{(0)}_{cs} \subset \Gamma^{(0)}_0$. Therefore

$$\Gamma^{(0)}_0 = \Gamma^{(0)}_s \oplus \Gamma^{(0)}_c \oplus (\Gamma^{(0)}_u \cap \Gamma^{(0)}_0).$$

Recall that $\Gamma^{(0)}_s \oplus \Gamma^{(0)}_u \subset H_\pi$. As $T$ has non-degenerated spectrum, these subspaces have the same dimension, and so they are equal. Denote by $\Gamma^{(k)}_s$, $\Gamma^{(k)}_c$ and $\Gamma^{(k)}_u$ the subspaces of functions on $I^{(k)}$ constant on intervals $I^{(k)}_\alpha$, $\alpha \in \mathcal{A}$ corresponding to the vectors from $\Gamma^{(0)}_s$, $\Gamma^{(0)}_c$ and $\Gamma^{(0)}_u$ respectively. Then

$$\Gamma^{(k)} = \Gamma^{(k)}_s \oplus \Gamma^{(k)}_c \oplus \Gamma^{(k)}_u, \quad H_\pi = \Gamma^{(k)}_s \oplus \Gamma^{(k)}_u, \quad \Gamma^{(k)}_0 = \Gamma^{(k)}_s \oplus \Gamma^{(k)}_c \oplus (\Gamma^{(k)}_u \cap \Gamma^{(k)}_0) \tag{C.12}$$

for $k \geq 0$ is a family of decomposition invariant with respect to the renormalization operators $S(k,l)$ for $0 \leq k < l$.

As $\xi = Id_{\Sigma(\pi)}$, by Lemma C.5, for every $\varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)$ and $l \geq k$ we have

$$S(k,l)\varphi \in \mathrm{BV}^{\diamondsuit}(\sqcup_{\alpha\in\mathcal{A}} I^{(l)}_\alpha) \text{ and } \mathcal{O}(S(k,l)\varphi) = \mathcal{O}(\varphi) \text{ for each } \mathcal{O} \in \Sigma(\pi). \tag{C.13}$$

*Proof of Theorem 5.2.* Since

$$U^{(0)}\widehat{\varphi} = P^{(0)}\varphi = U^{(0)} \circ P_0^{(0)}\varphi - \Delta P^{(0)}\varphi = U^{(0)}\varphi - U^{(0)} \circ C^{(0)}\varphi - \Delta P^{(0)}\varphi,$$

we have

$$\varphi - \widehat{\varphi} \in U^{(0)} \circ C^{(0)}\varphi + \Delta P^{(0)}\varphi \subset \Gamma_0^{(0)}.$$

In view of (C.2) and (C.9),

$$U^{(k)} \circ S(k)\widehat{\varphi} = S_u(k) \circ U^{(0)}\widehat{\varphi} = S_u(k) \circ P^{(0)}\varphi = P^{(k)} \circ S(k)\varphi.$$

Therefore, by (C.10) and (5.1), we have

$$\begin{aligned} \|U^{(k)} \circ S(k)\widehat{\varphi}\|_{\sup/\Gamma_{cs}^{(k)}} &= \|P^{(k)}(S(k)\varphi)\|_{\sup/\Gamma_{cs}^{(k)}} \\ &\leq (1+K)\operatorname{Var}(S(k)\varphi) \leq (1+K)\operatorname{Var}\varphi. \end{aligned}$$

It follows that for every $k \geq 0$ there exists $\varphi_k \in \mathrm{BV}_0(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $h_k \in \Gamma_{cs}^{(k)}$ such that

$$S(k)\widehat{\varphi} = \varphi_k + h_k \text{ and } \|\varphi_k\|_{\sup} \leq (1+K)\operatorname{Var}\varphi. \tag{C.14}$$

As

$$\varphi_{k+1} + h_{k+1} = S(k+1)\widehat{\varphi} = S(k,k+1)S(k)\widehat{\varphi} = S(k,k+1)\varphi_k + A^t h_k, \tag{C.15}$$

setting $\Delta h_{k+1} = h_{k+1} - A^t h_k$ ($\Delta h_0 = h_0$) we have $\Delta h_{k+1} = -\varphi_{k+1} + S(k,k+1)\varphi_k$. Moreover, by (C.14),

$$\begin{aligned} \|\Delta h_{k+1}\| &= \|\varphi_{k+1} - S(k,k+1)\varphi_k\|_{\sup} \\ &\leq \|\varphi_{k+1}\|_{\sup} + \|S(k,k+1)\varphi_k\|_{\sup} \leq (1+\|A\|)(1+K)\operatorname{Var}\varphi. \end{aligned}$$

and

$$\|\Delta h_0\| = \|\widehat{\varphi} - \varphi_0\|_{\sup} \leq \|\widehat{\varphi}\|_{\sup} + (1+K)\operatorname{Var}\varphi.$$

Since $h_k = \sum_{0\leq l\leq k}(A^t)^l \Delta h_{k-l}$ and $\Delta h_l \in \Gamma_{cs}^{(l)}$, by (C.1),

$$\begin{aligned} \|h_k\| &\leq \sum_{0\leq l\leq k} \|(A^t)^l \Delta h_{k-l}\| \leq \sum_{0\leq l\leq k} C l^{M-1}\|\Delta h_{k-l}\| \\ &\leq Ck^M(1+\|A\|)(1+K)\operatorname{Var}\varphi + Ck^{M-1}\|\widehat{\varphi}\|_{\sup}. \end{aligned}$$

In view of (C.14), it follows that

$$\|S(k)\widehat{\varphi}\|_{\sup} \leq \|\varphi_k\|_{\sup} + \|h_k\| \leq Ck^M(2+\|A\|)(1+K)\operatorname{Var}\varphi + Ck^{M-1}\|\widehat{\varphi}\|_{\sup}.$$

Since $\widehat{\varphi} - \varphi \in \Gamma_0^{(0)} = (\Gamma_u^{(0)} \cap \Gamma_0^{(0)}) \oplus \Gamma_{cs}^{(0)}$, there exist $h \in (\Gamma_u^{(0)} \cap \Gamma_0^{(0)})$ and $h' \in \Gamma_{cs}^{(0)}$ such that $\varphi + h = \widehat{\varphi} + h'$. Hence

$$\varphi + h + \Gamma_{cs}^{(0)} = \widehat{\varphi} + \Gamma_{cs}^{(0)} = P^{(0)}\varphi.$$

Suppose that $h_1, h_2 \in \Gamma_u^{(0)} \cap \Gamma_0^{(0)}$ are vectors such that

$$\varphi + h_1 + \Gamma_{cs}^{(0)} = \varphi + h_2 + \Gamma_{cs}^{(0)} = P^{(0)}\varphi.$$

In view of (5.5), $\|S(k)(\varphi + h_1)\|_{\sup}$ and $\|S(k)(\varphi + h_2)\|_{\sup}$ have at most polynomial growth. Therefore, $\|(A^t)^k(h_1 - h_2)\| = \|S(k)(h_1 - h_2)\|$ has at most polynomial growth, as well. Since $h_1 - h_2 \in \Gamma_u^{(0)}$, it follows that $h_1 = h_2$.

Assume that $T$ has non-degenerated spectrum. Then $\Gamma_{cs}^{(k)} = \Gamma_c^{(k)} \oplus \Gamma_s^{(k)}$. Suppose $\varphi_k$, $h_k \in \Gamma_{cs}^{(k)}$ satisfy (C.14). Let us decompose $h_k = h_k^s + h_k^c$, where $h_k^c \in \Gamma_c^{(k)}$ and

$h^s_k \in \Gamma^{(k)}_s \subset H_\pi$. By Remark C.3, $\Lambda^\pi(h^s_k) = 0$. In view of (C.14) and (C.13), it follows that

$$\mathcal{O}(\widehat{\varphi}) = \mathcal{O}(S(k)\widehat{\varphi}) = \mathcal{O}(\varphi_k) + \mathcal{O}(h^c_k) \text{ for every } \mathcal{O} \in \Sigma(\pi).$$

Moreover, by (C.11) and (C.14),

$$|\mathcal{O}(\varphi_k)| \le 2d\|\varphi_k\|_{\sup} \le 2d(1+K)\operatorname{Var}\varphi \text{ and } |\mathcal{O}(\widehat{\varphi})| \le 2d\|\widehat{\varphi}\|_{\sup}$$

for every $\mathcal{O} \in \Sigma(\pi)$. Therefore

$$|\langle h^c_k, b(\mathcal{O})\rangle| = |\mathcal{O}(h^c_k)| \le 2d((1+K)\operatorname{Var}\varphi + \|\widehat{\varphi}\|_{\sup}) \text{ for every } \mathcal{O} \in \Sigma(\pi),$$

so that

$$\|\Lambda^\pi(h^c_k)\| \le 2d((1+K)\operatorname{Var}\varphi + \|\widehat{\varphi}\|_{\sup}). \tag{C.16}$$

By (C.12), we have $\mathbb{R}^{\mathcal{A}} = \Gamma^{(k)} = \Gamma^{(k)}_c \oplus H_\pi$, so in view of Remark C.3, there exists $K' \ge 1$ such that $\|h\| \le K'\|\Lambda^\pi h\|$ for every $h \in \Gamma^{(k)}_c$. By (C.16), it follows that

$$\|h^c_k\| \le 2dK'((1+K)\operatorname{Var}\varphi + \|\widehat{\varphi}\|_{\sup}). \tag{C.17}$$

Let $\Delta h^s_{k+1} = h^s_{k+1} - A^t h^s_k$ for $k \ge 0$ and $\Delta h^s_0 = h^s_0$. Then from (C.15), we have

$$\Delta h^s_{k+1} = -\varphi_{k+1} + S(k,k+1)\varphi_k - h^c_{k+1} + A^t h^c_k = -\varphi_{k+1} + S(k,k+1)\varphi_k - h^c_{k+1} + h^c_k.$$

Therefore, by (C.14) and (C.17),

$$\begin{aligned}
\|\Delta h^s_{k+1}\| &\le \|\varphi_{k+1}\|_{\sup} + \|A\|\|\varphi_k\|_{\sup} + \|h^c_{k+1}\| + \|h^c_k\| \\
&\le (1+\|A\|+4dK')(1+K)\operatorname{Var}\varphi + 4dK'\|\widehat{\varphi}\|_{\sup}, \\
\|\Delta h^s_0\| &= \|\widehat{\varphi} - \varphi_0 - h^c_0\|_{\sup} \le (1+2dK')(\|\widehat{\varphi}\|_{\sup} + (1+K)\operatorname{Var}\varphi).
\end{aligned}$$

Notice that for every $0 < \theta_- < \theta_g$ there exists $C \ge 1$ such that

$$\|(A^t)^n h\| \le C\exp(-n\theta_-)\|h\| \text{ for all } h \in \Gamma^{(k)}_s \text{ and } n \ge 0.$$

Since $h^s_k = \sum_{0\le l\le k}(A^t)^l \Delta h^s_{k-l}$ and $\Delta h^s_l \in \Gamma^{(l)}_s$, it follows that

$$\begin{aligned}
\|h^s_k\| &\le \sum_{0\le l\le k}\|(A^t)^l\Delta h^s_{k-l}\| \le \sum_{0\le l\le k} C\exp(-l\theta_-)\|\Delta h^s_{k-l}\| \\
&\le \frac{C(1+\|A\|+4dK')}{1-\exp(-\theta_-)}((1+K)\operatorname{Var}\varphi + \|\widehat{\varphi}\|_{\sup}).
\end{aligned}$$

In view of (C.14) and (C.17), it follows that

$$\begin{aligned}
\|S(k)\widehat{\varphi}\|_{\sup} &\le \|\varphi_k\|_{\sup} + \|h^c_k\| + \|h^s_k\| \\
&\le \frac{C(2+\|A\|+6dK')}{1-\exp(-\theta_-)}((1+K)\operatorname{Var}\varphi + \|\widehat{\varphi}\|_{\sup}),
\end{aligned}$$

which completes the proof. □

**Theorem C.6.** *There exist $C_3, C_4 > 0$ such that*

$$\|\widehat{\varphi}^{(n)}\|_{\sup} \le C_3\log^{M+1} n\operatorname{Var}\varphi + C_4\log^M n\|\widehat{\varphi}\|_{\sup}$$

*for every natural $n$. If additionally $T$ has non-degenerated spectrum then*

$$\|\widehat{\varphi}^{(n)}\|_{\sup} \le C_3\log n\operatorname{Var}\varphi + C_4\log n\|\widehat{\varphi}\|_{\sup}.$$

*Proof.* By Proposition A.3 and Theorem 5.2, for every $x \in I$ we have

$$\begin{aligned}\|\widehat{\varphi}^{(n)}(x)\| &\leq 2\|A\|\sum_{k=0}^{m}(C_1 k^M \operatorname{Var}\varphi + C_2 k^{M-1}\|\widehat{\varphi}\|_{\sup})\\ &\leq 2\|A\|(C_1 m^{M+1}\operatorname{Var}\varphi + C_2 m^M\|\widehat{\varphi}\|_{\sup}),\end{aligned}$$

where $m = m(x,n,T)$. Now the assertion follows directly from (A.1). □

## Appendix D. Example of non-regular step cocycle

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type with periodic matrix is $A$. Then there exists $C > 0$ and $\theta > 0$ such that

$$\|(A^t)^n h\| \leq C\exp(-n\theta)\|h\| \text{ for all } h \in \Gamma_s^{(0)} \text{ and } n \geq 0.$$

**Lemma D.1.** *Suppose that $h \in \Gamma_0^{(0)}$ and $\varphi : I \to \mathbb{R}$ is the associated step cocycle. If $h \in \Gamma_s^{(0)}$ then $\varphi$ is a coboundary. If $h \notin \Gamma_{cs}^{(0)}$ then $\varphi$ is not a coboundary.*

*Proof.* Assume that $h \in \Gamma_s^{(0)}$. Since

$$\|S(l)\varphi\|_{\sup} = \|(A^t)^l h\| \leq C\exp(-l\theta)\|h\|,$$

by Proposition A.3, we have

$$\|\varphi^{(n)}\|_{\sup} \leq 2\sum_{l=0}^{\infty}\|Z(l+1)\|\|S(l)\varphi\|_{\sup} \leq 2C\|A\|\|h\|\sum_{l=0}^{\infty}\exp(-l\theta) = \frac{2C\|A\|\|h\|}{1-\exp(-\theta)}$$

for every natural $n$. But each bounded cocycle in $\mathbb{R}^\ell$ is a coboundary.

Now suppose that $h \in \Gamma_0^{(0)}$ and $\varphi$ is a coboundary. Set

$$\varepsilon = \inf\{\mu(C_\alpha^{(n)}) : n \geq 0, \alpha \in \mathcal{A}\}$$

(see Section 4 for the definition of the tower $C_\alpha^{(n)}$). In view of (4.6), $\varepsilon > 0$. Since $\varphi$ is a coboundary, there exist $M > 0$ and a sequence $(B_k)_{k\geq 0}$ of measurable sets with $\mu(B_k) > 1-\varepsilon$ for $n \geq 0$ such that $|\varphi^{(k)}(x)| \leq M$ for all $x \in B_k$ and $k \geq 0$. Recall that for every $x \in C_\alpha^{(n)}$ we have $\varphi^{(h_\alpha^{(n+1)})}(x) = ((A^t)^{n+1}h)_\alpha$. Since $C_\alpha^{(n)} \cap B_{h_\alpha^{(n+1)}} \neq \emptyset$, it follows that $|((A^t)^{n+1}h)_\alpha| \leq M$ for every $n \geq 0$ and $\alpha \in \mathcal{A}$. Thus $\|(A^t)^{n+1}h\| \leq M$ for every $n \geq 0$, and hence $h \in \Gamma_{cs}^{(0)}$. □

*Example* 2. Let us consider an IET $T = T_{(\pi_5^{sym},\lambda)}$ of periodic type whose periodic matrix is equal to

$$A = \begin{pmatrix} 18 & 28 & 31 & 38 & 18\\ 10 & 16 & 8 & 9 & 6\\ 13 & 20 & 36 & 46 & 18\\ 2 & 3 & 16 & 22 & 6\\ 39 & 61 & 63 & 77 & 37 \end{pmatrix}.$$

The existence of such IET was shown in [29]. The Perron-Frobenius eigenvalue of $A$ is $55 + 12\sqrt{21}$ and $\lambda$ is equal to

$$(1+\sqrt{21}, 2, 1+\sqrt{21}, 2, 7+\sqrt{21})$$

up to multiplication by a positive constant. Moreover, the eigenvalues and eigenvectors of $A^t$ are as follows:

$$\begin{array}{ll}\rho_1 = 55 + 12\sqrt{21}, & v_1 = (-1+\sqrt{21}, 1+\sqrt{21}, 3+\sqrt{21}, 5+\sqrt{21}, 4)\\ \rho_2 = 9 + 4\sqrt{5}, & v_2 = (-2, -1-1\sqrt{5}, 2, 1+\sqrt{5}, 0)\\ \rho_3 = 1, & v_3 = (-1, -2, 0, -1, 1)\\ \rho_4 = 9 - 4\sqrt{5}, & v_4 = (-2, -1+1\sqrt{5}, 2, 1-\sqrt{5}, 0)\\ \rho_5 = 55 - 12\sqrt{21}, & v_5 = (-1-\sqrt{21}, 1-\sqrt{21}, 3-\sqrt{21}, 5-\sqrt{21}, 4).\end{array}$$

Note that $v_2, v_3, v_4, v_5 \in \Gamma_0^{(0)}$. Denote by $\varphi_i : I \to \mathbb{R}$ the step function corresponding to $v_i$ for $1 < i \leq 5$. Since $|\rho_2| > 1 > |\rho_4|$, by Lemma D.1, $\varphi_4$ is a coboundary and $\varphi_2$ is not a coboundary.

We will show that $\varphi_2$ is a non-regular cocycle. Note that the cocycles $\varphi_2 + \varphi_4$ and $\varphi_2 - \varphi_4$ take values in $\mathbb{Z}$ and $\sqrt{5}\mathbb{Z}$ respectively. Since $\varphi_4$ is a coboundary, it follows that $E(\varphi_2) \subset \mathbb{Z}$ and $E(\varphi_2) \subset \sqrt{5}\mathbb{Z}$, and hence $E(\varphi_2) = \{0\}$. Since $\varphi_2$ is not a coboundary, $\overline{E}(\varphi_2) = \{0, \infty\}$, and hence it is non-regular.

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