Limit law for some modified ergodic sums

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Abstract

An example due to Erdös and Fortet shows that, for a lacunary sequence of integers (q_n) and a trigonometric polynomial φ, the asymptotic distribution of $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(q_k x)$ can be a mixture of gaussian laws. Here we give a generalization of their example interpreted as the limiting behavior of some modified ergodic sums in the framework of dynamical systems.

Introduction

Let (q_n) be a lacunary sequence of natural integers. The stochastic-like behavior of the sums $\sum_{k=0}^{n-1}\varphi(q_k x)$ for φ a regular 1-periodic real function has been the subject of several works (Fortet, Kac, Salem, Zygmund,...). Before the second war and in the forties, different particular cases were treated for which the Central Limit Theorem (CLT) can be shown. The fact that the CLT is not always satisfied in its standard form was already noticed by Fortet et Erdös. Gaposhkin [Ga65], Berkes [Be76], recently Berkes and Aisleitner [AiBe08] gave arithmetic conditions on the sequence (q_n) which imply the CLT.

The counter-example of Fortet and Erdös is very simple. Let us take $q_n = 2^n - 1$ and $\varphi(x) = \cos(2\pi x) + \cos(4\pi x)$. Then the limit law of the distribution of the normalized sums $n^{-\frac{1}{2}}\sum_{k=0}^{n-1}\varphi((2^k-1)x)$ is not the gaussian one, but a mixture of gaussian laws, explicited in [Ka47], [Ka49][1].

This fact is related to the arithmetic properties of the sequence $2^n - 1$. But it can also be interpreted from other points of view. It can be viewed as a consequence of non ergodicity for stationary martingales which gives asymptotically a mixture of gaussian laws.

[1]The proof announced by Kac, as well as an article by Erdos, Ferrand, Fortet and Kac on the sums $\sum_{k=0}^{n-1}\varphi(q_n x)$ mentioned in [Ka49], were never published as far as we know. For a proof based on a result of Salem and Zygmund, [SaZy48]) for lacunary sequences see Berkes and Aisleitner [AiBe08]. Here we give a slightly different proof and generalizations of it.

As we will show in Section 2, one can equally view it as a special case of a general phenomenon for isometric perturbation of dynamical systems of hyperbolic type.

For instance, let $A \in SL(d, \mathbb{Z})$ be a matrix without eigenvalue root of the unity, let B be a matrix with integral coefficients and $\det B \neq 0$. Denoting by λ the Lebesgue measure on the d-dimensional torus $\mathbb{T}^d$, we have, for every centered Hölder function φ on $\mathbb{T}^d$, for every $t \in \mathbb{R}$,

$$\lambda\{x : \frac{1}{\sqrt{n}} \sum_{0}^{n-1} \varphi((A^k - B)x) < t\} \longrightarrow \int_{\mathbb{T}^d} \frac{1}{\sqrt{2\pi}\sigma_y} \int_{-\infty}^{t} \exp(\frac{-s^2}{2\sigma_y^2}) ds \; d\lambda(y),$$

where $\sigma(\varphi_y)$ is the asymptotic variance of the translated function: $\varphi_y(x) := \varphi(x+y)$.

In Section 3 we will also use a different method, based on a property of multiple decorrelation, to extend the results to a large class of chaotic dynamical systems and their modified ergodic sums.

In what follows (X, d) will be a metric space with its Borel σ-algebra $\mathcal{B}$ and a probability measure μ on $\mathcal{B}$, T a measure preserving transformation on $(X, \mathcal{B}, \mu)$, and φ a real function on X with some regularity. We denote by $S_n\varphi$ (or $S_n(\varphi)$) the ergodic sums of φ:

$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(T^k x).$$

Contents

1 Generalization of an example of Fortet and Erdös

In order to explain the method on a simple example, we begin with the counter-example of Fortet and Erdös mentioned in the introduction. Then we indicate how it can be extended to modified ergodic sums for expanding maps of the interval.

1.1 The example of Fortet and Erdös

The space X is the circle $\mathbb{R}/\mathbb{Z}$, μ is the Lebesgue measure and T is the transformation $x \to 2x \bmod 1$.

Notations 1.1 *We denote by $R_n\varphi(x,y)$ the translated modified ergodic sums*

$$R_n\varphi(x,y) := \sum_{k=0}^{n-1} \varphi(2^k x - x + y). \tag{1}$$

Let φ be an Hölderian function on $\mathbb{R}/\mathbb{Z}$ such that $\int_0^1 \varphi \, d\mu = 0$. We denote by $c_p(\varphi)$ its Fourier coefficient of order $p \in \mathbb{Z}$. We have:

$$n^{-1}\|S_n\varphi\|_2^2 = \sum_{p\neq 0}[|c_p(\varphi)|^2 + 2\sum_{1\leq j\leq n-1}(1-\frac{j}{n})\, c_{2^j p}(\varphi)\overline{c}_p(\varphi)].$$

The variance of φ is well defined and given by

$$\sigma^2(\varphi) := \lim_n \frac{\|S_n\varphi\|_2^2}{n} = \sum_{p\neq 0}[|c_p(\varphi)|^2 + 2\sum_{j\geq 1} c_{2^j p}(\varphi)\overline{c}_p(\varphi)].$$

For $\varphi(x) = \cos(2\pi x) + \cos(4\pi x)$, we have the following convergence for every $t \in \mathbb{R}$:

$$\lim_n \mu\{x : \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi((2^k-1)x) \leq t\} = \frac{1}{\sqrt{2\pi}}\int_0^1 (\int_{-\infty}^{t/|\cos y|} e^{-s^2/2} ds)\, dy, \tag{2}$$

or in terms of characteristic function:

$$\lim_n \mathbb{E}(e^{it\frac{1}{\sqrt{n}}\sum_{k=1}^n \varphi((2^k-1).)}) = \int_0^1 e^{-\frac{1}{2}(\cos y)^2\, t^2}\, dy. \tag{3}$$

Before we give generalizations of this result, for the reader's convenience we recall a proof of (3) (cf. [AiBe08]) based on the following general statement:

Lemma 1.2 *Let (Z_n) be a sequence of real random variables on $[0,1]$. Let $\mathcal{L}$ be a probability distribution on $\mathbb{R}$, with characteristic function $\Phi(t) = \int e^{itx}\, \mathcal{L}(dx)$. The following conditions are equivalent:*

a) for every probability density ρ, the sequence (Z_n) under the measure $\rho\mu$ converges in distribution to $\mathcal{L}$;
b) for every interval $I \subset [0,1]$,

$$\lim_n \frac{1}{\mu(I)} \mu\{x \in I : Z_n(x) \leq t\} = \mathcal{L}(]-\infty, t]), \ \forall t \in \mathbb{R}; \tag{4}$$

c) for every Riemann integrable function ψ, the sequence (ψZ_n) converges in distribution to a limit distribution with characteristic function $\int_0^1 \Phi(\psi(y)\, t)\, dy$.
In particular if $\mathcal{L} = \mathcal{N}(0,1)$, under the previous conditions the sequence (φZ_n) converges in distribution to a limit distribution with characteristic function $\int_0^1 e^{-\frac{1}{2}\varphi^2(y)\, t^2}\, dy$.

<u>Proof</u> Assume b). Let ψ be a step function, $\psi = \sum_{j=0}^p c_j 1_{[a_j, a_{j+1}[}$, with $a_0 = 0 < a_1 < ... < a_{p+1} = 1$. We have:

$$\mathbb{E}_\mu(e^{it\psi(.)Z_n(.)}) = \sum_j \int_{a_j}^{a_{j+1}} e^{itc_j Z_n(.)}\, d\mu \to \sum_j \mu(I_j)\Phi(c_j t) = \int_0^1 \Phi(\psi(y)\, t)\, dy.$$

The general case c) follows by approaching ψ by step functions.

Conversely, let $\rho = \sum_{j=0}^p c_j 1_{[a_j, a_{j+1}[}$, with $\rho \geq 0$ and $\int_0^1 \rho\, dx = 1$. Under Condition b) or c), we have:

$$\mathbb{E}_{\rho\mu}(e^{itZ_n(.)}) = \sum_j c_j \int_{a_j}^{a_{j+1}} e^{itZ_n(.)}\, d\mu \to [\sum_j c_j(a_{j+1} - a_j)]\Phi(t) = \Phi(t).$$

As above we obtain the general case by approximation. Condition a) follows. □

Salem and Zygmund proved in [SaZy48] the CLT with Condition b) of Lemma 1.2 for $\varphi = \cos$ and any lacunary sequence (q_k). The convergence (3) follows from their result, from Lemma 1.2, and from the trigonometric identity ($n \geq 2$):

$$\begin{aligned}&\sum_{k=1}^n [\cos(2\pi(2^k - 1)\, x) + \cos(4\pi(2^k - 1)\, x)] \\ &= \cos(2\pi x) + \cos(2\pi(2^{n+1} - 2)\, x) + 2\cos(\pi x) \sum_{k=2}^n \cos(2\pi(2^k - 3/2)\, x).\end{aligned}$$

Now we give an analogous result for more general functions. The method of proof is slightly different from the previous one and will be applied to various examples in the sequel of the paper.

First of all we need a well known improved version of the CLT for regular functions. In the special case of this section, it can be proved using the classical method of quasi-compact operator. We first sketch the idea for the transformation $x \to 2x \bmod 1$

on $\mathbb{R}/\mathbb{Z}$. For this map, the dual (Perron-Frobenius) operator P is given by

$$P\varphi(x) = \frac{1}{2}[\varphi(\frac{x}{2}) + \varphi(\frac{x}{2} + \frac{1}{2})].$$

Let ρ be a probability density on $X = \mathbb{R}/\mathbb{Z}$. Then for every integer $\ell \geq 0$, we have:

$$\begin{aligned}
& |\int_X e^{i\frac{t}{\sqrt{n}}S_n\varphi(x)} \ \rho(x)\ dx - \int_X e^{i\frac{t}{\sqrt{n}}S_n\varphi(x)} \ \ dx| \\
= \ & |\int_X [e^{i\frac{t}{\sqrt{n}}S_n\varphi(x)} - e^{i\frac{t}{\sqrt{n}}S_n\varphi(T^\ell x)}]\ \rho(x)\ dx + \int_X e^{i\frac{t}{\sqrt{n}}S_n\varphi(x)} \ (P^\ell\rho(x) - 1)\ dx| \\
\leq \ & 2\frac{|t|}{\sqrt{n}}\ell\|\varphi\|_\infty + \|P^\ell\rho - 1\|_1.
\end{aligned}$$

If ρ is a function of bounded variation or a Hölder function, then $\|P^\ell\rho - 1\|_1$ converges to 0 when ℓ tends to ∞ with an exponential rate. If ρ is only in L^1, convergence holds a priori without rate.

Therefore, in the CLT, we can replace the Lebesgue measure by a measure which is absolutely continuous with respect to the Lebesgue measure when the density is regular. It allows us to apply Lemma 1.2 or similar results. Actually this principle holds for dynamical systems in a very general situation as we will see later. Applied here to the density $\mu(D)^{-1}1_{D-y}\mu$ where D is an interval in $[0,1[$, it yields the following result:

Lemma 1.3 *For every y, every interval D, every regular 1-periodic function φ,*

$$\mu(D)^{-1}\mu\{x : n^{-\frac{1}{2}}\sum_{k=0}^{n-1}\varphi(2^k x + y) \leq t \text{ and } x \in D - y\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\,ds,$$

where $\sigma(\varphi_y)$ is the asymptotic variance of the translated function: $\varphi_y(x) := \varphi(x+y)$.

We will also use a property of regularity of the variance with respect to translations:

Lemma 1.4 *If $C(\varphi) := \sum_p |p|\,|c_p(\varphi)| < +\infty$, then we have:*

$$n^{-\frac{1}{2}}\|R_n\varphi(.,y) - R_n\varphi(.,y')\|_2 \leq C(\varphi)|y - y'|. \tag{5}$$

<u>Proof</u> For every $p \neq 0$, we have $\frac{1}{n}\int|\sum_{k=0}^{n-1} e^{2\pi i 2^k p x}|^2\ dx = 1$.

Writing $\varphi(2^k x - x + y) = \sum_{p\neq 0} c_p(\varphi)e^{2i\pi p(y-x)}e^{2i\pi p 2^k x}$, the following uniform bound holds:

$$\begin{aligned}
\frac{1}{\sqrt{n}}\|R_n\varphi(.,y) - R_n\varphi(.,y')\|_2 \ &\leq \ |y - y'|\sum_p |p|\,|c_p(\varphi)|\|\frac{1}{\sqrt{n}}\|S_n e^{2\pi i p.}\|_2 \\
&= \ |y - y'|\sum_p |p|\,|c_p(\varphi)| = C(\varphi)|y - y'|.
\end{aligned}$$

□

By convention in the sequel, when the variance σ is 0, $\int_{-\infty}^{t/\sigma} e^{-u^2/2}\, du$ is interpreted as the repartition function of the limit law, the Dirac mass at 0.

Theorem 1.5 *If φ satisfies $\sum_p |p|\, |c_p(\varphi)| < +\infty$, we have:*

$$\mu\{x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi(2^k x - x) \leq t\} \to \frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy. \tag{6}$$

Proof It is enough to check the convergence when t is a continuity point of the limit distribution. By integrating with respect to y and applying Lebesgue theorem, it follows from Lemma 1.3:

$$\frac{1}{\mu(D)} \mu\{(x,y) : n^{-\frac{1}{2}} \sum_{k=0}^{n-1} \varphi(2^k x + y) \leq t \text{ and } x \in D - y\} \to \frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy.$$

The change of variable $(x, y) \to (x, y + x)$ leaves the measure $dx \times dy$ invariant. We get that the difference:

$$\frac{1}{\mu(D)} \mu\{(x,y) : n^{-\frac{1}{2}} \sum_{k=0}^{n-1} \varphi(2^k x - x + y) \leq t \text{ and } y \in D\} - \frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy \tag{7}$$

converges to 0. For $\gamma > 0$, we obtain by (5):

$$\begin{aligned} &\mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, 0) \leq t\} \\ &\leq \mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, y) \leq t + \gamma\} + \mu\{x : \frac{1}{\sqrt{n}} |R_n\varphi(x, 0) - R_n\varphi(x, y)| \geq \gamma\}, \end{aligned}$$

hence:

$$\begin{aligned} &\mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, 0) \leq t\} \\ &\leq \frac{1}{\mu(D)} \int_X \mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, y) \leq t + \gamma\}\ 1_D(y)\ dy + C(\varphi)^2 \frac{\delta^2}{\gamma^2}, \end{aligned} \tag{8}$$

$$\begin{aligned} &\frac{1}{\mu(D)} \int_X \mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, y) \leq t - \gamma\}\ 1_D(y)\ dy - C(\varphi)^2 \frac{\delta^2}{\gamma^2} \\ &\leq \mu\{x : \frac{1}{\sqrt{n}} R_n\varphi(x, 0) \leq t\}. \end{aligned} \tag{9}$$

Let $\varepsilon > 0$. First we take γ such that

$$|\frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t \pm \gamma/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy - \frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy| < \varepsilon,$$

then δ such that $C(\varphi)^2 \frac{\delta^2}{\gamma^2} < \varepsilon$ and finally n_0 such that the difference in (7) (with $t \pm \gamma$ in place of t) is less then ε.

Applying (8) and (9), we get

$$|\mu\{x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi(2^k x - x) \leq t\} - \frac{1}{\sqrt{2\pi}} \int_0^1 (\int_{-\infty}^{t/\sigma(\varphi_y)} e^{-s^2/2}\, ds)\, dy| < 6\varepsilon.$$

□

Non-degeneracy of the limit distribution

For a regular function φ, the set $\{y : \sigma(\varphi_y) = 0\}$ is closed, since $\sigma(\varphi_y)$ is continuous as a function of y. This set coincides with $E(\varphi, T) := \{y : \varphi_y \text{ is a } T\text{−coboundary}\}$. Since φ is regular, if $\varphi_y = \psi_y \circ T - \psi_y$ for a function ψ_y, then ψ_y is also regular.

If the set $E(\varphi, T)$ has measure 1, it coincides with $[0, 1]$. The functions φ_y vanish for every y on the fixed point of T, which implies that φ is identically zero. Therefore the limiting distribution in (6) is non degenerated when φ is non identically zero.

1.2 First generalizations, expanding maps

Expanding maps

The previous example can be extended to other dynamical systems or "sequential dynamical systems" in different directions. For instance we can consider a lacunary sequence of positive integers (q_n) and a modified version of the corresponding ergodic sums, like $\sum_{q=0}^{n-1} \varphi(q_k x - x)$.

An other extension is in the class of expanding maps on the interval, like the β-transformations, for which the spectral properties of the transfer operator can be used.

We will not develop these specific extensions, but we will consider the following general framework. The example of Erdös and Fortet can be view as a special case of the following general construction.

Let us consider a dynamical system (X, T, μ), a space $\mathcal{F}$ of real valued functions on X, and a map $\theta : x \to \theta_x$ from X into the set of Borel maps from X to X which preserve $\mathcal{F}$ under composition. Thus, for each $x \in X$, a map θ_x (also denoted by $\theta(x)$) is given.

Suppose that for $\varphi \in \mathcal{F}$ the sums $\sum_{k=0}^{n-1} \varphi(T^k x)$ after normalization have a distribution limit, for instance convergence in distribution of $n^{-\frac{1}{2}} \sum_{k=0}^{n-1} \varphi(T^k x)$ toward a gaussian law. Now let us consider the modified sums:

$$\sum_{k=0}^{n-1} \varphi(\theta_x(T^k x)). \tag{10}$$

In the previous section, the system was $x \to 2x \bmod 1$ on the circle, $\mathcal{F}$ the space of Hölder functions or the space of functions φ satisfying $\sum_p |p|\,|c_p(\varphi)| < +\infty$, and the map $\theta_x : y \to y - x$. In the sequel of the paper we will describe different cases where a limit law for the modified sums (10) can be obtained.

A simple situation is the following. Let us consider a map θ taking a finite number of values. More precisely, suppose that there is a finite partition $(A_j, j \in J)$ of X in measurable sets A_j such that $\theta_x = \theta_j$ on A_j, where θ_j is a Hölderian map from X to X.

Applying a result of Zweimüller ([Zw07]), we have:

$$\begin{aligned}
\mathbb{E}[e^{it\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(\theta(.)(T^k.))}] &= \sum_{j\in J}\mu(A_j)\int e^{it\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(\theta_j(T^k x))}\ \mu(A_j)^{-1}1_{A_j}\,d\mu \\
&\to \sum_{j\in J}\mu(A_j)\,e^{-\frac{1}{2}\sigma^2(\varphi\circ\theta_j)\ t^2}.
\end{aligned}$$

We would like to extend such a convergence to maps θ taking a continuum of values like in the example of Erdös and Fortet. In the next section, this will be done when there is a compact group G acting on X and when θ is a regular map with values in G. If we denote the variance $\sigma^2(\varphi \circ \theta_x)$ by σ_x^2, our aim is to prove that the limit distribution of the normalized modified sums (10) has for characteristic function:

$$\int_X e^{-\frac{1}{2}\sigma_x^2 t^2}\ d\mu(x).$$

2 Generalization to group actions

2.1 A general result

Let (X, T, μ) be a dynamical system. Suppose that a compact group K acts on X and preserves the measure μ. Denote by m the Haar measure on K, and by $(k, x) \to kx$ the action of K on X.

Let $\theta : X \to K$ be a Borel function. The modified sums that we consider here have the form

$$\sum_{k=0}^{n-1} \varphi(\theta(x)\,T^k x).$$

Let μ_θ the measure on K defined by $\mu_\theta(B) = \mu(\theta^{-1}B)$, for B Borel set in K. For a function φ in $L^2(\mu)$, we introduce the two following properties:

Property 2.1 *(Central limit theorem with density for $\varphi(k.)$) There exists $\sigma_k \geq 0$ such that, for every ρ density with respect to μ on X, for every interval A in $\mathbb{R}$, every $k \in K$,*

$$(\rho\mu)\{x \,:\, \frac{1}{\sqrt{n}} S_n\varphi(kx) \in A\} \longrightarrow \frac{1}{\sqrt{2\pi}\sigma_k} \int_A \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2})\; ds. \tag{11}$$

If $\sigma_k = 0$ in (11), the limit law is the Dirac measure at 0.

Property 2.2 *(Continuity of the variance of the modified sums with respect to the translations)*

$$\|\frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \varphi(\theta(\cdot)^{-1}k\, T^\ell \cdot) - \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \varphi(\theta(\cdot)^{-1} T^\ell \cdot)\|_2 \leq C\sqrt{n} d(k,e).$$

Proposition 2.3 *If the function φ satisfies 2.1 and 2.2, the following convergence holds:*

$$\mu\{x :\; \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \varphi(\theta(x)\, T^\ell x) \in A\} \longrightarrow \int_K \frac{1}{\sqrt{2\pi}\sigma_k} \int_A \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2}) ds \; d\mu_\theta(k).$$

Proof First, let us fix D a compact neighborhood of the identity in K. For every element k of K, the regularity of the action of K and Property 2.1 for the function $\varphi_k(\cdot) = \varphi(k\cdot)$ give the convergence

$$\mu\{x \,:\, \frac{1}{\sqrt{n}} S_n\varphi_k(x) < u,\, \theta(x)^{-1}k \in D\} \longrightarrow \frac{1}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^u \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2}) ds\, \mu\{x \,:\, \theta(x)^{-1}k \in D\},$$

where σ_k^2 is the asymptotic variance associated to the function φ_k.

By taking the integral over K, we obtain:

$$\begin{aligned}(\mu \otimes m)\{(x,k) \,:\, &\frac{1}{\sqrt{n}} S_n\varphi_k(x) < u,\; \theta(x)^{-1}k \in D\} \\ &\longrightarrow \int_K \frac{1}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^u \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2}) ds\; \mu(\{x \,:\, \theta(x)^{-1}k \in D\})\; dk.\end{aligned}$$

The measure $\mu \otimes m$ is preserved by the change of variable $x = x,\; k' = \theta(x)^{-1}k$. So, by dividing by $m(D)$, we get:

$$\begin{aligned}\frac{1}{m(D)}\mu \otimes m\{(x,k') \,:\, &\frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \varphi(\theta(x)\, k'\, T^\ell x) < u,\; k' \in D\} \\ &\longrightarrow \int_K \frac{1}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^u \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2}) ds\; \frac{\mu(\{x :\; \theta(x)^{-1}k \in D\})}{m(D)}\; dk.\end{aligned}$$

The measure

$$\frac{\mu(\{x:\ \theta(x)^{-1}k \in D\})}{m(D)}\ dk$$

tends to μ_θ (the image of μ by θ) when the diameter of D tends to 0. Property 2.2 now allows us to conclude. Let ε be a positive number. Let $R_n\varphi(x,k)$ be the sum

$$R_n\varphi(x,k) = \sum_{\ell=0}^{n-1} \varphi(\theta(x)kT^\ell x).$$

We have

$$\begin{aligned}
&\mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\}) \\
&\quad \le \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,k) \le u+\varepsilon\}) + \mu(\{x\ :\ \frac{1}{\sqrt{n}}|R_n\varphi(x,k) - R_n\varphi(x,e)| > \varepsilon\}) \\
&\quad \le \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,k) \le u+\varepsilon\}) + \frac{Var(R_n\varphi(\cdot,k) - R_n\varphi(\cdot,e))}{n\varepsilon^2} \\
&\quad \le \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,k) \le u+\varepsilon\}) + \frac{C^2 n d(k,e)^2}{n\varepsilon^2}.
\end{aligned}$$

Denoting by $\delta(D)$ the diameter of D, the average taken on D yields:

$$\begin{aligned}
&\mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\}) \\
&\quad \le \frac{1}{m(D)}\mu \otimes m\{(x,k)\ :\ \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\varphi(\theta(x)kT^l x) < u+\varepsilon,\ k' \in D\} + \frac{C^2\delta(D)^2}{\varepsilon^2}.
\end{aligned}$$

and, because of the above convergence,

$$\begin{aligned}
&\limsup_{n\to\infty} \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\}) \\
&\quad \le \int_K \frac{1}{\sqrt{2\pi}\sigma_k}\int_{-\infty}^{u+\varepsilon} \exp(\frac{-s^2}{2\sigma_k^2})ds\ \frac{\mu(\{x\ :\ \theta(x)^{-1}k \in D\})}{m(D)}\ dk + \frac{C^2\delta(D)^2}{\varepsilon^2},
\end{aligned}$$

thus, letting $\delta(D)$ tends to 0,

$$\limsup_{n\to\infty} \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\})\ \le\ \int_K \frac{1}{\sqrt{2\pi}\sigma_k}\int_{-\infty}^{u+\varepsilon} \exp(\frac{-s^2}{2\sigma_k^2})ds\ d\mu_\theta(k).$$

Similarly we have

$$\liminf_{n\to\infty} \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\})\ \ge \int_K \frac{1}{\sqrt{2\pi}\sigma_k}\int_{-\infty}^{u-\varepsilon} \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2})ds\ d\mu_\theta(k).$$

Therefore, excepted maybe for $u = 0$ if $\sigma_k = 0$ for a set of positive μ_θ-measure of elements k, we have the convergence:

$$\lim_{n\to\infty} \mu(\{x\ :\ \frac{1}{\sqrt{n}}R_n\varphi(x,e) \le u\})\ = \int_K \frac{1}{\sqrt{2\pi}\sigma_k}\int_{-\infty}^{u} \exp(-\frac{1}{2}\frac{s^2}{\sigma_k^2})ds\ d\mu_\theta(k).$$

□

A typical situation in which one might apply this result is when the central limit theorem holds for regular functions and the action of K is regular. Let (X, d) be a metric space. For a real number $\eta > 0$, on the space of Hölder continuous functions of order η we define the η-variation and the η-Hölder norm by

$$[\varphi]_\eta = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\eta}, \ \|\varphi\|_\eta = \|\varphi\|_\infty + [\varphi]_\eta. \tag{12}$$

We say that the action of K is Hölder continuous if, for every $k \in K$, $x \to kx$ is Hölder continuous. For many chaotic systems it has been proved that the central limit theorem holds for Hölder continuous functions. If the action of K is Hölder continuous, then the central limit theorem holds for $\varphi(k.)$. Moreover, because of the theorem of Eagleson ([Ea76], [Zw07]), the theorem is true for measures absolutely continuous with respect to μ. In this case a Hölder continuous function φ satisfies Property 2.1.

Definition 2.4 We say that (X, T, μ) is *exponentially mixing* if, for every $\eta > 0$, there exist $C > 0$ and $\alpha \in]0, 1[$ such that, for every centered η-Hölder continuous functions φ, ψ,

$$|\langle T^n\varphi, \psi\rangle| = |\int_X \varphi \circ T^n \ \psi \ d\mu| \leq C \, \|\varphi\|_\eta \, \|\psi\|_2 \, \alpha^n.$$

Proposition 2.5 *Let (X, T, μ) be a dynamical system where X is a riemannian manifold. Assume that there is a measure preserving action of a compact Lie group K on (X, μ) which is C^∞. Assume that the central limit theorem holds for differentiable functions on X and that (X, T, μ) is exponentially mixing. Then for every C^∞ function φ on X, for every $t \in \mathbb{R}$,*

$$\mu\{x : \ \frac{1}{\sqrt{n}} \sum_0^{n-1} \varphi(\theta(x)T^k x) < t\} \longrightarrow \int_K \frac{1}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^t \exp(-\frac{1}{2}\frac{s2}{\sigma_k 2}) \, ds \ d\mu_\theta(k).$$

<u>Proof</u> We will use harmonic analysis on K. We briefly summarize what we need. For more details see [Bo82].

The action of K on X defines a unitary representation U of K on $L^2(\mu)$ by $k \longmapsto \varphi(k^{-1}x)$ which can be decomposed as a sum of irreducible representations. Let $\hat{K}$ be the set of equivalence classes of irreducible representations of K and let δ be an element of $\hat{K}$.

Let us fix a base R of the root system of $\mathcal{K}$ the Lie algebra of K. Let us call W the associated Weyl chamber. To each irreducible representation of K is uniquely associated a linear form belonging to a lattice in W: the dominant weight of the representation. Let δ be an element of $\hat{K}$ and let γ_δ be the corresponding dominant weight.

The Weyl formula gives the dimension, d_δ, the irreducible representation associated to δ as a function of γ :

$$d_\delta = \prod_{\alpha \in R_+} \frac{\langle \alpha, \gamma_\delta + \rho \rangle}{\langle \alpha, \rho \rangle},$$

where R_+ is the set of positive roots and ρ the half sum of the positive roots.

For every $\delta \in \hat{K}$, let ξ_δ be the character of δ, $\chi_\delta = d_\delta \xi_\delta$, and

$$P_\delta = U(\overline{\chi_\delta}) = d_\delta \int_K \overline{\xi_\delta(k)} U(k)\ dk.$$

The operator P_δ is the projection of $L^2(\mu)$ on the isotypic part $\mathcal{F}_\delta := P_\delta(L^2(\mu))$. One has

$$L^2(\mu) = \bigoplus_{\delta \in \hat{K}} \mathcal{F}_\delta.$$

For a given vector v in $L^2(\mu)$ let us denote v_δ the element P_δ. An element v of $\mathcal{F}_\delta$ is K-finite:

$$\dim \text{Vect} K v \leq d_\delta^2. \tag{13}$$

One says that v is C^∞ if the map $k \mapsto U(k)v$ is C^∞. One defines the derived representation of U on the space of C^∞ elements; it is a representation of the Lie algebra $\mathcal{K}$ of K and that one can be extended to a representation of the universal enveloping algebra of $\mathcal{K}$. We use the same later U to denote these three representations.

Let $X_1, \ldots, X_n$ be an orthonormal basis for an invariant scalar product on $\mathcal{K}$. The operator $\Omega = 1 - \sum_{i=1}^n X_i^2$ belongs to the center of the universal enveloping algebra of $\mathcal{K}$. So, by Schur's Lemma, if μ_δ is a representation of type δ, there exists c_δ such that

$$\mu_\delta(\Omega) = c_\delta \mu_\delta(1).$$

The operators $\Omega(X_i)$ being hermitian, c_δ is positive. One can show (*cf.* [Bo82]) that there exists a scalar product Q such that

$$c_\delta = Q(\gamma_\delta + \rho) - Q(\rho).$$

If v is C^∞, one has

$$P_\delta U(\Omega) v = c_\delta P_\delta v = c_\delta v_\delta,$$

thus, for every non negative integer m, for every δ in $\hat{K}$, one has

$$v_\delta = c_\delta^{-m}\ (U(\Omega^m) v)_\delta,$$

with large c_δ for large γ_δ. From this equality and the definition of P_δ, one deduces that

$$\|v_\delta\|_\infty \leq \frac{d_\delta^2}{c_\delta^m} \|U(\Omega^m) v\|_\infty.$$

In particular, the series $\sum_{\delta\in\hat{K}} v_\delta$ converges uniformly toward v. This allows us to write, for every $x \in X$,

$$\begin{aligned}\varphi(k^{-1}x) &= \sum_{\delta\in\hat{K}} U(k)(\varphi_\delta)(x),\\ \varphi(\theta(x)kx) &= \sum_{\delta\in\hat{K}} U((\theta(x)k)^{-1})(\varphi_\delta)(x).\end{aligned}$$

We want to study the quantity:

$$\|\frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)kT^\ell\cdot) - \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)T^\ell\cdot)\|_2.$$

With the notations introduced above we can write:

$$\begin{aligned}\sum_{\ell=0}^{n-1}\varphi(\theta(x)kT^\ell x) - \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(x)T^\ell x) &= \sum_{\ell=0}^{n-1}((U(k^{-1})-Id)U(\theta(x)^{-1})\varphi)(T^\ell x)\\ &= \sum_{\delta\in\hat{K}}\sum_{\ell=0}^{n-1}((U(k^{-1})-Id)U(\theta(x)^{-1})\varphi_\delta)(T^\ell x)\end{aligned}$$

On one hand, since v_δ is K-finite, there exists a finite set of C^∞ functions $\{\varphi_{\delta,j},\ j = 1\dots q\}$ (with $\varphi_{\delta,j} = U(k_j)\varphi_\delta$ for some k_j and $q \le d_\delta^2$ because of (13)) and uniformly bounded functions $u_{\delta,j}$ such that

$$(U(k^{-1})-Id)U(\theta(x)^{-1})\varphi_\delta = \sum_{j=1}^{q} u_{\delta,j}(x)\varphi_{\delta,j}.$$

From this we deduce that

$$\begin{aligned}&|\langle T^p(U(k^{-1})-Id)U(\theta(\cdot)^{-1})\varphi_\delta, T^q(U(k^{-1})-Id)U(\theta(\cdot)^{-1})\varphi_\delta\rangle|\\ &\le Cd_\delta^4\|\varphi_\delta\|_2\|\varphi_\delta\|_\eta\alpha^{|p-q|} \le Cd_\delta^4\|\varphi_\delta\|_2\|\varphi\|_\eta\alpha^{|p-q|}.\end{aligned} \tag{14}$$

On the other hand, one has

$$\|(U(k^{-1})-Id)v_\delta\| \le \|\gamma_\delta\|^r d(k,e)\|v_\delta\|, \tag{15}$$

where $\|\gamma_\delta\|$ denotes an usual norm of a point in a lattice (the norm of the operator $(U(k^{-1})-Id)$ on $\mathcal{F}_\delta$ is said to be moderately increasing function of δ ([Bo82], p. 82-83)). From (14) and (15) we deduce that

$$\begin{aligned}&\mathbb{E}((\sum_{l=0}^{n-1}((U(k^{-1})-Id)U(\theta(x)^{-1})\varphi_\delta)(T^l x))^2)\\ &\le \sum_{1\le p\le q\le n}|\langle T^p(U(k^{-1})-Id)U(\theta(\cdot)^{-1})\varphi_\delta, T^q(U(k^{-1})-Id)U(\theta(\cdot)^{-1})\varphi_\delta\rangle|\\ &\le C\sum_{1\le p\le q\le n}\min(d_\delta 4\|\varphi_\delta\|_2\|\varphi\|_\eta\alpha^{|p-q|}, \|\gamma_\delta\|^{2r}d(k,e)^2\|\varphi_\delta\|^2).\end{aligned}$$

By cutting this sum in two pieces ($|p-q| < M\ln_+(d(k,e))$ and $|p-q| \geq M\ln_+(d(k,e))$) one obtains that there exists $L > 0$ such that:

$$\mathbb{E}((\sum_{l=0}^{n-1}((U(k^{-1}) - Id)U(\theta(x)^{-1})\varphi_\delta)(T^l x))^2)$$
$$\leq C\|\gamma_\delta\|^L\|\varphi_\delta\|_2\|\varphi\|_\eta\ n\ d(k,e)^2\ln_+(d(k,e)),$$

that is

$$\|\sum_{\ell=0}^{n-1}((U(k^{-1}) - Id)U(\theta(x)^{-1})\varphi_\delta)(T^\ell x)\|_2$$
$$\leq C\|\gamma_\delta\|^L\|\varphi_\delta\|_2^{1/2}\|\varphi\|_\eta^{1/2}\ \sqrt{n}\ d(k,e)\ln_+(d(k,e))^{1/2}.$$

The triangular inequality now gives

$$\|\sum_{\delta\in\hat{K}}\sum_{\ell=0}^{n-1}((U(k^{-1}) - Id)U(\theta(\cdot)^{-1})\varphi_\delta)(T^\ell\cdot)\|_2$$
$$\leq (\sum_{\delta\in\hat{K}} C\|\gamma_\delta\|^L\|\varphi_\delta\|_2^{1/2})\|\varphi\|_\eta^{1/2}\ \sqrt{n}\ d(k,e)\ln_+(d(k,e))^{1/2}.$$

Since φ is C^∞, the series $\sum_{\delta\in\hat{K}} C\|\gamma_\delta\|^L\|\varphi_\delta\|_2^{1/2}$ converges and we have proved that

$$\|\frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)kT^\ell\cdot) - \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)T^\ell\cdot)\|_2 \leq C(\varphi)d(k,e)\ln_+(d(k,e))^{1/2}.$$

□

We have treated the case of exponentially mixing dynamical systems to simplify the exposition. This hypothesis can be weakened: summable correlations for regular functions suffice to obtain an inequality like

$$\|\frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)kT^\ell\cdot) - \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}\varphi(\theta(\cdot)T^\ell\cdot)\|_2 \leq C(\varphi)\beta(d(k,e)),$$

where β is a function such that $\lim_{t\to 0}\beta(t) = 0$.

2.2 Examples

Automorphisms of the torus(*cf.* [Le60], [LB99])

Let $A \in SL(d,\mathbb{Z})$ be a matrix without eigenvalue root of the unity. It defines an ergodic automorphism of the d-dimensional torus $\mathbb{T}^d$ for the Lebesgue measure λ.

Let B be a matrix with integral coefficients. For every centered C^∞ function φ, for every $t \in \mathbb{R}$, we have:

$$\lambda\{x : \frac{1}{\sqrt{n}} \sum_0^{n-1} \varphi((A^k - B)x) < t\} \longrightarrow \int_{\mathbb{T}^d} \frac{1}{\sqrt{2\pi}\sigma_y} \int_{-\infty}^t \exp(\frac{-s^2}{2\sigma_y^2}) ds \, d\lambda_B(y).$$

If $\det B \neq 0$ then $\lambda_B = \lambda$.

Non-degeneracy of the limit

If T is a hyperbolic automorphism of the torus $\mathbb{T}^d$, then for any Hölderian φ, the set $E(\varphi, T) := \{y : \varphi(. + y) \text{ is a } T\text{−coboundary}\}$ is closed because $\varphi(. + y)$ is a coboundary if and only $\sigma(\varphi_y) = 0$ and $y \mapsto \sigma(\varphi_y)$ is a continuous function (this is a consequence of the mixing properties of T ; see section 4.2 below). On the other hand if $\varphi(. + y)$ is a $T-$coboundary then it is a coboundary inside the set of Hölder continuous functions: there exists a Hölder continuous function ψ_y such that $\varphi(. + y) = T\psi_y - \psi_y$. Thus if B is surjective and $\sigma_y = 0$ λ-almost surely, then $\sigma_y = 0$ everywhere and $\varphi(0 + y) = T\psi_y(0) - \psi_y(0) = \psi_y(0) - \psi_y(0) = 0$. So, if B is surjective, unless $\varphi = 0$, the limit law is not degenerated.

If T is an ergodic non-hyperbolic automorphism of the torus $\mathbb{T}^d$ then one has to reinforce the regularity hypothesis on φ in order to apply the second part of the previous reasoning: if φ is d-times differentiable, has a Hölder continuous d^{th}-differential and is a measurable coboundary then it is a coboundary in the space of Hölder continuous functions ([Ve86], [LB99]).

Automorphisms on nilmanifolds (*cf.* [CoLB02])

Let X be the 3-dimensional nilmanifold defined as the homogeneous space N/Γ, where N is the Heisenberg group of triangular matrices

$$\begin{pmatrix} 1 & x_1 & z \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and Γ the discrete subgroup of integral points in N. Let μ be the N-invariant measure on N/Γ induced by the Haar measure on N. We identify N and $\mathbb{R}^3$ equipped with the law

$$(x_1, x_2, z).(x_1', x_2', z') = (x_1 + x_1', x_2 + x_2', z + z' + x_1 x_2' - x_1' x_2).$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a hyperbolic matrix in $Sl(2, \mathbb{Z})$. It defines a transformation T on N/Γ by

$$T : (x_1, x_2, z)\Gamma \to (ax_1 + bx_2, cx_1 + dx_2, z)\Gamma.$$

The group of isometries of the manifold X can be seen as the circle. Let θ be a Borel map defined from the quotient torus $\mathbb{T}^2$ to $\mathbb{R}/\mathbb{Z}$: $\theta(x_1, x_2, z) = \theta(x_1, x_2)$. Then we have for every Hölder function φ:

$$\mu\{x:\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(A^k(x_1,x_2),\theta(x_1,x_2)+z)<t\}\longrightarrow\int_{\mathbb{R}/\mathbb{Z}}\frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{t}\exp(-\frac{1}{2}\frac{s^2}{\sigma_y^2})ds\ d\mu_\theta(y),$$

where σ_y^2 is the asymptotic variance associated to the function $\varphi_y(x_1,x_2,z) = \varphi(x_1,x_2,z+y)$.

Diagonal flows on compact quotients of $SL(d,\mathbb{R})$(*cf.* [LBP05])

Let G be the group $SL(d,\mathbb{R})$, let Γ be a cocompact lattice of G, and let μ be the probability on G/Γ deduced from the Haar measure. Let g_0 be a diagonal matrix in G different from the identity. It defines a transformation T on G/Γ: $x=g\Gamma\longmapsto Tx=g_0g\Gamma$.

Let φ be a centered C^∞ function from G/Γ to $\mathbb{R}$ and let θ be a Borel map from G/Γ to $SO(d,\mathbb{R})$. We have

$$\mu\{x:\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(\theta(x)T^kx)<t\}\longrightarrow\int_{SO(d,\mathbb{R})}\frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{t}\exp(-\frac{1}{2}\frac{s^2}{\sigma_y^2})ds\ d\mu_\theta(y),$$

where σ_y^2 is the asymptotic variance associated to the function $\varphi_y(x)=\varphi(y.x)$.

3 Generalization to multiple decorrelation

3.1 Multiple decorrelation and gaussian laws

Let (X,T,μ) be a dynamical system defined on a manifold X. Let d be a riemannian distance on X. The Hölder norm is defined as in (12). The expectation $\mathbb{E}$ is the integral with respect to μ. In some proofs, we will denote by the same letter C a constant which may vary in the proof. Now we introduce the following multiple decorrelation property:

Property 3.1 *There exist $C>0$ and $\delta\in]0,1[$ such that, for all integers m and m', all Hölder continuous functions $(\varphi_i)_{i=1}^{m+m'}$ defined on X, all integers $0\le\ell_1\le\ldots\le\ell_m\le k_1\le\ldots\le k_{m'}, N>0$,*

$$\mathrm{Cov}(\prod_{i=1}^{m}T^{\ell_i}\,\varphi_i,\prod_{j=1}^{m'}T^{k_j+N}\varphi_j)\le C(\prod_{i=1}^{m+m'}\ \|\varphi_i\|_\infty+\sum_{j=1}^{m+m'}[\varphi_j]_\eta\prod_{i\neq j}\|\varphi_i\|_\infty)\delta^N.$$

This property has many interesting consequences. Following C. Jan's method ([Ja00]), we will show that the normalized ergodic sums behave in some sense like gaussian variables even for some non invariant measures on X absolutely continuous with respect to μ and which can vary with the time.

Let us first state a very simple consequence of this property. There exist $C > 0$, $\zeta \in]0,1[$, such that, for every centered Hölder continuous functions φ and ψ defined on X with zero average with respect to μ, we have:

$$|\langle \varphi, T^n \psi \rangle| \leq C \|\varphi\|_\eta \, \|\psi\|_2 \, \zeta^n, \;\; |\langle \varphi, T^n \psi \rangle| \leq C \|\varphi\|_2 \, \|\psi\|_\eta \, \zeta^n. \tag{16}$$

The first inequality above is a consequence of 3.1, the second one is proved by using the Cauchy-Schwarz inequality and distinguishing two cases $\|\varphi\|_2 \leq \|\varphi\|_\eta \delta^{n/2}$ and $\|\varphi\|_2 \geq \|\varphi\|_\eta \delta^{n/2}$. In particular the asymptotic variance of the normalized ergodic sums of a Hölder continuous function is well defined and given by:

$$\sigma(\varphi)^2 = \mu(\varphi^2) + 2 \sum_{k=1}^{\infty} \langle \varphi, T^k \varphi \rangle.$$

Theorem 3.2 *Let (X, T, μ) be a dynamical system defined on a manifold X satisfying Property 3.1. Let (ρ_n) be a sequence of density functions with respect to μ with norms $\|\rho_n\|_\eta$ bounded by Cn^L, for some constants $\eta > 0, C, L$. Then for every sequence (φ_n) of centered Hölder continuous functions with Hölder norms uniformly bounded, we have*

$$\mu(\rho_n \exp(\frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} T^\ell \varphi_n)) - \exp(-\frac{1}{2} \sigma(\varphi_n)^2 t^2) \to 0.$$

Proof Let $\alpha \in]0, 1/2[$. The difference $\frac{1}{\sqrt{n}} S_n \varphi_n(x) - \frac{1}{\sqrt{n}} S_n \varphi_n(x) \circ T^{n^\alpha}$ tends uniformly toward 0. When $\|\rho_n\|_\eta \leq Cn^L$, Property 3.1 gives:

$$\begin{aligned} &|\mu(\rho_n \exp(\frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} T^{\ell+n^\alpha} \varphi_n)) - \mu(\rho_n) \, \mu(\exp \frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} T^{\ell+n^\alpha} \varphi_n)| \\ &\quad \leq C(\|\rho_n\|_\infty + n[\varphi_n]_\eta \|\rho_n\|_\infty + [\rho_n]_\eta) \delta^{n^\alpha} \leq C n^{L+1} \delta^{n^\alpha}. \end{aligned}$$

Thus we only have to study $\mu(\exp(\frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} T^{\ell+n^\alpha} \varphi_n)) = \mu(\exp \frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} T^\ell \varphi_n)$.

On a probability space $(\Omega, \mathbb{P})$ containing (X, μ), we can construct a sequence $(X_{k,n})$ of centered independent bounded random variables of variance $\sigma(\varphi_n)^2$ with distribution $\frac{1}{2}(\delta_{-\sigma(\varphi_n)} + \delta_{\sigma(\varphi_n)})$, independent from the variables φ_n. Since the sequence $(\|\varphi_n\|_\eta)_n$ is bounded, the sequence $(\sigma(\varphi_n))_n$ is also bounded and one easily check that

$$\mathbb{E}(\exp(\frac{it}{n^{1/2}} \sum_{\ell=0}^{n-1} X_{l,n})) - \exp(-\frac{1}{2} \sigma(\varphi_n)^2 t^2) \to 0.$$

We claim that the difference between the characteristic functions of $\frac{1}{n^{1/2}}\sum_{k=0}^{n-1} T^k\varphi_n$ and $\frac{1}{n^{1/2}}\sum_{k=0}^{n-1} X_{k,n}$ tends to 0. To show it, let us write

$$B_{\ell,n} = \exp(\frac{it}{n^{1/2}}T^\ell\varphi_n), \quad C_{\ell,n} = \exp(\frac{it}{n^{1/2}}X_{\ell,n}).$$

We have:

$$\exp\frac{it}{n^{1/2}}\sum_{\ell=0}^{n-1}T^\ell\varphi_n - \exp\frac{it}{n^{1/2}}\sum_{\ell=0}^{n-1}X_{\ell,n} = \prod_{\ell=0}^{n-1}B_{\ell,n} - \prod_{\ell=0}^{n-1}C_{\ell,n}, \tag{17}$$

and

$$\prod_{\ell=0}^{n-1}B_{\ell,n} - \prod_{\ell=0}^{n-1}C_{\ell,n} = \sum_{\ell=0}^{n-1}(\prod_{k=0}^{\ell-1}C_{k,n})\underbrace{(B_{\ell,n} - C_{\ell,n})(\prod_{k=\ell+1}^{n-1}B_{k,n})}_{\Delta_\ell},$$

where the products with an empty set of indexes are conventionally taken to be 1.

The variables $\Delta_\ell = (B_{\ell,n} - C_{\ell,n})\prod_{k=\ell+1}^{n^{1+\alpha}-1}B_{\ell,n}$ and $\prod_{k=0}^{\ell-1}C_{\ell,n}$ are independent. We will show that most of the n terms $|\mathbb{E}(\Delta_\ell)|$ are bounded by some constant times $n^{-3/2}\ln n$. This will imply the result.

Consider a sequence $(\chi(m))$ that will fixed later. When $\ell + 3\chi(n) + 1 < n$, we split the product Δ_ℓ in three blocks:

$$\begin{aligned}\Delta_\ell &= \underbrace{(B_{\ell,n} - C_{\ell,n})\prod_{k=\ell+1}^{\ell+\chi(n)}B_{\ell,n}}_{\mathcal{A}}\underbrace{\prod_{k=\ell+\chi(n)+1}^{\ell+2\chi(n)}B_{k,n}}_{\mathcal{B}} \\ &\quad \underbrace{\prod_{k=\ell+2\chi(n)+1}^{\ell+3\chi(n)}B_{k,n}}_{\mathcal{C}}\underbrace{\prod_{k=\ell+3\chi(n)+1}^{n-1}B_{k,n}}_{\mathcal{D}}.\end{aligned}$$

Let us now study

$$\mathbb{E}(\Delta_\ell) = \mathbb{E}(\mathcal{ABCD}) = \mathbb{E}(\mathcal{A}(\mathcal{B}-1)(\mathcal{C}-1)\mathcal{D}) + \mathbb{E}(\mathcal{ABD}) + \mathbb{E}(\mathcal{ACD}) - \mathbb{E}(\mathcal{AD}).$$

The mean value theorem shows that $\mathcal{A}$ is bounded by some constant times $tn^{-1/2}\|\varphi_n\|_\infty\|X_{\ell,n}\|_\infty$ and $(\mathcal{B}-1), (\mathcal{C}-1)$ are both bounded by

$$\frac{2t}{n^{1/2}}\sum_{k=\ell+\chi(n)+1}^{\ell+2\chi(n)}\|\varphi_n\|_\infty,$$

thus $\mathbb{E}(\mathcal{A}(\mathcal{B}-1)(\mathcal{C}-1)\mathcal{D}) \leq C\|\varphi_n\|_\infty^3 \frac{t^3}{n^{3/2}}\chi(n)^2$.

The three other terms can be treated in the following way.

Consider for example : $\mathbb{E}(\mathcal{A}\mathcal{B}\mathcal{D}) = Cov(\mathcal{A}\mathcal{B}, \mathcal{D}) + \mathbb{E}(\mathcal{A}\mathcal{B})\mathbb{E}(\mathcal{D})$. Property 3.1 gives $Cov(\mathcal{A}\mathcal{B}, \mathcal{D}) \leq C\ \delta^{\chi(n)}$. Let us now study

$$\begin{aligned}\mathcal{A}\mathcal{B} &= (B_{\ell,n} - C_{\ell,n}) \prod_{k=\ell+1}^{\ell+2\chi(n)} B_{k,n} \\ &= \left(\exp\frac{itT^\ell\varphi_n}{n^{1/2}} - \exp\frac{itX_{\ell,n}}{n^{1/2}}\right) \exp\left(itn^{-1/2}\sum_{k=\ell+1}^{\ell+2\chi(n)} T^k\varphi_n\right)\end{aligned}$$

The expansion of the two terms at order 1 and order 2 yields

$$\begin{aligned}\mathcal{A}\mathcal{B} &= \frac{it}{n^{1/2}}(T^\ell\varphi_n - X_{\ell,n}) - \frac{t^2}{2n}(T^\ell\varphi_n^2 - X_{\ell,n}^2) \\ &\quad -\frac{t^2}{n}\sum_{k=\ell+1}^{\ell+2\chi(n)} T^k\varphi_n\ T^\ell\varphi_n \quad + \frac{t^2}{n}X_{\ell,n}\sum_{k=\ell+1}^{\ell+2\chi(n)} T^k\varphi_n + D,\end{aligned}$$

where D satisfies

$$|D| \leq C[\frac{t^2}{n^{3/2}}(\chi(n)^2 + \frac{t^3}{n^{3/2}} + \frac{t^4}{n^2}\chi(n)] \leq C\frac{t^2}{n^{3/2}}\chi(n)^2.$$

By taking the expectation, one obtains:

$$\begin{aligned}&|\mathbb{E}(\mathcal{A}\mathcal{B})| \leq \\ &\frac{t^2}{2n}\mathbb{E}(X_{\ell,n}^2) - \frac{t^2}{2n}(\mathbb{E}(T^\ell\varphi_n^2) + 2\sum_{k=\ell+1}^{\ell+2\chi(n)} \mathbb{E}(T^k\varphi_n\ T^\ell\varphi_n)) + C\frac{t^2}{n^{3/2}}\chi(n)^2. \quad (18)\end{aligned}$$

By hypothesis we have

$$\begin{aligned}\mathbb{E}(X_{\ell,n}^2) &= \sigma^2(\varphi_n) = E(T^\ell\varphi_n^2) + 2\sum_{k=\ell+1}^{\infty} \mathbb{E}(T^k\varphi_n\ T^\ell\varphi_n) \\ &= E(T^\ell\varphi_n^2) + 2\sum_{k=\ell+1}^{\ell+2\chi(n)} \mathbb{E}(T^k\varphi_n\ T^\ell\varphi_n)\ + 2\sum_{k=\ell+2\chi(n)+1}^{\infty} \mathbb{E}(T^k\varphi_n\ T^\ell\varphi_n).\end{aligned}$$

Replacing $\mathbb{E}(X_{\ell,n}^2)$ by this expression in (18) we obtain

$$|\mathbb{E}(\mathcal{A}\mathcal{B})| \leq C(t^2\sum_{k=\ell+2\chi(n)+1}^{\infty} \mathbb{E}(T^k\varphi_n\ T^\ell\varphi_n)n^{-1} + \chi(n)^2n^{-3/2}),$$

and because of the decay of correlations (16):

$$|\mathbb{E}(\mathcal{AB})| \leq C(t^2\zeta^{\chi(n)}n^{-1} + \chi(n)^2 n^{-3/2}).$$

Since, on the other hand, $|\mathbb{E}(\mathcal{D})| \leq 1$, we have

$$\mathbb{E}(\Delta_\ell) \leq C(t^2(\zeta^{\chi(n)}n^{-1} + \chi(n)^2 n^{-3/2}) + \ \zeta^{\chi(n)} + \|\varphi_n\|_\infty^3 \| \frac{t^3}{n^{3/2}}\chi(n)^3).$$

Now we can bound (17):

$$\begin{aligned} |\mathbb{E}(\prod_0^{n-1} B_{\ell,n} - \prod_0^{n-1} C_{\ell,n})| &= |\sum_{\ell=0}^{n-1} \mathbb{E}(\prod_{k=0}^{\ell-1} C_{k,n})\mathbb{E}(\Delta_\ell)| \leq \sum_{\ell=0}^{n-1} |\mathbb{E}(\Delta_\ell)| \\ &\leq \sum_{\ell=0}^{n-1-4\chi(n)-1} C(t^2(\zeta^{\chi(n)}n^{-1} + \chi(n)^2 n^{-3/2}) + \ \zeta^{\chi(n)} \\ &\quad + \|\varphi_n\|_\infty^3 \| \frac{t^3}{n^{3/2}}\chi(n)^3) + \sum_{\ell=n-1-4\chi(n)}^{n-1} \mathbb{E}(\Delta_\ell). \end{aligned}$$

From the mean value theorem one also deduces

$$\mathbb{E}(\Delta_\ell) \leq Cn^{-1/2}.$$

If we take $\chi(n) = C \ln n$ with C large enough, then we are done. □

3.2 Application

Let (X, T, μ) be a dynamical system defined on a manifold X. Suppose that Property 3.1 holds. Let $\theta : x \to \theta_x$ be a map from X into the set of the Hölder continuous maps from X to X. We will use also the notation $\theta(x)$ instead of θ_x. We suppose that θ satisfies for some $\eta > 0$

$$\sup_{z\in X} d(\theta_x(z), \theta_y(z)) \leq Cd(x, y)^\eta. \tag{19}$$

Let φ be a Hölder continuous function from X to $\mathbb{R}$. We want to study the behavior of

$$\sum_{k=0}^{n-1} \varphi(\theta_x(T^k x)).$$

We can assume that the exponent of regularity in (19) and the exponent for φ are the same. Since the maps θ_x do not necessarily preserve the measure μ, one has to center these modified ergodic sums. We write

$$\overline{\varphi}_{\theta_x} := \mathbb{E}(\varphi(\theta_x(.))) = \int_X \varphi(\theta_x(y))d\mu(y),$$

$$\varphi_{\theta_x} := \varphi(\theta_x \cdot) - \overline{\varphi}_{\theta_x}, \quad \sigma^2_{\theta_x} := \sigma^2(\varphi_{\theta_x}).$$

For μ-almost every x (the generic points of the dynamical system),

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi(\theta_x(T^k x)) \to \overline{\varphi}_{\theta_x}.$$

We claim that both functions $\overline{\varphi}_{\theta_x}$ and $\sigma^2_{\theta_x}$ are Hölder continuous. Let us show it for $\sigma^2_{\theta_x}$. On one hand, in view of (16), the absolute value of

$$\sigma^2(\varphi_{\theta_x}) - \sigma^2(\varphi_{\theta_y}) = \mu(\varphi^2_{\theta_x}) - \mu(\varphi^2_{\theta_y}) + 2\sum_1^\infty (\mu(\varphi_{\theta_x}.T^k\varphi_{\theta_x}) - \mu(\varphi_{\theta_y}.T^k\varphi_{\theta_y}))$$

is bounded by

$$2\|\varphi\|_\infty \|\varphi_{\theta_x} - \varphi_{\theta_y}\|_2 + 2\sum_1^\infty |\mu(\varphi_{\theta_x}(T^k\varphi_{\theta_x} - T^k\varphi_{\theta_y})| + 2\sum_1^\infty |\mu((\varphi_{\theta_x} - \varphi_{\theta_y})T^k\varphi_{\theta_y})|$$
$$\leq 2\|\varphi\|_\infty \|\varphi_{\theta_x} - \varphi_{\theta_y}\|_2 + 4C\sum_1^\infty C(\|\varphi_{\theta_x}\|_\eta + \|\varphi_{\theta_y}\|_\eta)\|\varphi_{\theta_x} - \varphi_{\theta_y}\|_2\, \zeta^k$$
$$\leq C(\|\varphi_{\theta_x}\|_\eta + \|\varphi_{\theta_y}\|_\eta)\|\varphi_{\theta_x} - \varphi_{\theta_y}\|_2,$$

and, on the other hand,

$$\|\varphi_{\theta_x} - \varphi_{\theta_y}\|_2 \leq \|\varphi_{\theta_x} - \varphi_{\theta_y}\|_\infty \leq Cd(x,y)^{\eta^2}.$$

Now suppose that there exists $(\mathcal{P}_n)$ a sequence of partitions with the diameter of the elements of $\mathcal{P}_n$ smaller than n^{-2/η^2} such that, for positive constants C, L, for every $n \in \mathbb{N}$, every P in $\mathcal{P}_n$, there exists a density function $\rho_{n,P}$ such that

$$\|\rho_{n,P}\|_\eta \leq Cn^L, \ \|\rho_{n,P} - \mu(P)^{-1}\mathbf{1}_P\|_1 \leq \frac{1}{n}. \tag{20}$$

Let us fix such a sequence $(\mathcal{P}_n)$. Cutting out the space X according to the partition $\mathcal{P}_n$ we get

$$\sum_{k=0}^{n-1} \varphi_{\theta_x}(T^k x) = \sum_{P\in\mathcal{P}_n}\sum_{k=0}^{n-1} \mathbf{1}_P(x)\varphi_{\theta_x}(T^k x).$$

For each element P of $\mathcal{P}_n$ we choose in P a point $x_{n,P}$. If $x \in P$, then the distance $d(\theta_x(T^k x), \theta_{x_{n,P}}(T^k x))$ is bounded by $Cd(x, x_{n,P})^\eta \leq Cn^{-2/\eta}$. We thus have

$$|\sum_{P\in\mathcal{P}_n}\sum_{k=0}^{n-1} \mathbf{1}_P(x)\varphi_{\theta_x}(T^k x) - \sum_{P\in\mathcal{P}_n}\sum_{k=0}^{n-1} \mathbf{1}_P(x)\varphi_{\theta_{x_{n,P}}}(T^k x)| \leq Cnn^{-2}.$$

Let us now study the characteristic function of $\sum_{P\in\mathcal{P}_n}\sum_{k=0}^{n-1}\mathbf{1}_P(x)\varphi_{\theta(x_{n,P})}(T^kx)$:

$$\begin{aligned}
&\mathbb{E}(\exp(i\frac{t}{\sqrt{n}}\sum_{P\in\mathcal{P}_n}\sum_{k=0}^{n-1}\mathbf{1}_P(x)\,\varphi_{\theta(x_{n,P})}(T^kx)))\\
&\quad=\mathbb{E}(\sum_{P\in\mathcal{P}_n}\mathbf{1}_P(x)\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^kx)))\\
&\quad=\sum_{P\in\mathcal{P}_n}\mu(P)\mathbb{E}(\mu(P)^{-1}\mathbf{1}_P(x)\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^kx))).
\end{aligned}$$

By (20) we have

$$|\mathbb{E}(\mu(P)^{-1}\mathbf{1}_P(x)\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^kx)))-\mathbb{E}(\rho_{n,P}\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^kx)))|\le 1/n.$$

We know from Theorem 3.2 that

$$\mathbb{E}(\rho_{n,P}\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^kx)))-\exp(-\frac{1}{2}\,\sigma^2_{\theta(x_{n,P})}t^2)\to 0.$$

The sum $\sum_{P\in\mathcal{P}_n}\mu(P)\exp(-\frac{1}{2}\sigma^2_{\theta(x_{n,P})}t^2)$ is a Riemann sum of the Hölder continuous function $\sigma^2_{\theta_x}$. It converges to

$$\int_X\exp(-\frac{1}{2}\sigma^2_{\theta_x}t^2)\;d\mu(x).$$

Now for the assumption (20), it is known that every smooth manifold has a triangulation (see [Ca35], [Wh57]). The existence of a sequence of partitions $\mathcal{P}_n$ and of functions $\rho_{n,P}$ satisfying the previous conditions are thus satisfied when X is a smooth manifold. So, we have proved the following theorem.

Theorem 3.3 *Let (X,T,μ) be a dynamical system defined on a smooth manifold X for which Property 3.1 holds. Let θ be a map from X to the set of the Hölder continuous maps from X to X such that (19) is satisfied. Then for every centered Hölder continuous function φ on X, we have*

$$\mathbb{E}[\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}(\varphi(\theta_xT^kx)-\overline{\varphi}_{\theta_x}))]\to\int_X\exp(-\frac{1}{2}\sigma^2_{\theta_x}t^2)\;d\mu(x).$$

Now let us give some examples satisfying the hypothesis. The examples given at the end of Section 2 can be treated by this method since Property 3.1 is satisfied for ergodic automorphisms of the torus, ergodic automorphisms on nilmanifolds, diagonal flows on compact quotients of $SL(d,\mathbb{R})$. By the method used in this Section 3 we obtain the same result for Hölder continuous functions under the assumption that the map θ is Hölder continuous. If we identify in these cases the element θ_x of K with the translation by $\theta(x)$ we obtain:

$$\mathbb{E}(\exp(i\frac{t}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(\theta(x)(T^k x))) \to \int_X \exp(-\frac{1}{2}\sigma^2_{\theta(x)}t^2)\ d\mu(x)$$

and, by definition of the image measure of μ by θ on K:

$$\int_X \exp(-\frac{1}{2}\sigma^2_{\theta(x)}t^2)\ d\mu(x) = \int_K \exp(-\frac{1}{2}\sigma^2_k t^2)\ d\mu_\theta(k).$$

By linearity of the Fourier transform, we just have another formulation of the convergence stated in Proposition 2.5. The result is the same. But this second method allows us to consider transformations $\theta(x)$ that are not given by translation by elements of a compact group acting on X. In the case of the automorphisms of the torus, one can for example take for θ_x a regular family of diffeomorphisms of $\mathbb{T}^d$. This method can be also adapted to expanding dynamical systems for which the Perron-Frobenius operator has good spectral properties.

3.3 Convergence of the variance

Proposition 3.4 *Let (X,T,μ) be a dynamical system defined on a manifold X for which Property 3.1 holds. Let θ be a map from X to the set of the Hölder continuous maps from X to X such that (19) is satisfied, and φ a centered Hölder continuous function on X. Then*

$$\mathbb{E}\left(\frac{1}{n}\left(\sum_{k=1}^{n}\varphi(\theta(\cdot)T^k\cdot)\right)^2\right) \to \int_X \sigma^2_{\theta_x}\ d\mu(x).$$

Proof Consider a sequence of partitions $(\mathcal{P}_n)$ with the same properties as in the previous subsection. We have:

$$\begin{aligned}\mathbb{E}\left(\frac{1}{n}\left(\sum_{k=1}^{n}\varphi(\theta(\cdot)T^k\cdot)\right)^2\right) &= \mathbb{E}\left(\sum_{P\in\mathcal{P}_n}\mathbf{1}_P(\cdot)\frac{1}{n}\left(\sum_{k=0}^{n-1}\varphi_{\theta(\cdot)}(T^k\cdot)\right)^2\right)\\ &= \sum_{P\in\mathcal{P}_n}\mu(P)\,\mathbb{E}\left(\frac{\mathbf{1}_P(\cdot)}{\mu(P)}\frac{1}{n}\left(\sum_{k=0}^{n-1}\varphi_{\theta(\cdot)}(T^k\cdot)\right)^2\right).\end{aligned}$$

For every $P\in\mathcal{P}_n$, let $\rho_{n,P}$ be a function with norm $\|\rho_{n,P}\|_\eta$ less than Cn^L such that $\|\rho_{n,P}-\mu(P)^{-1}\mathbf{1}_P\|_1 \le n^{-2}$. We have:

$$|\mathbb{E}\left(\frac{1}{n}\left(\sum_{k=1}^{n}\varphi(\theta(\cdot)T^k\cdot)\right)^2\right) - \sum_{P\in\mathcal{P}_n}\mu(P)\,\mathbb{E}\left(\rho_{n,P}\frac{1}{n}\left(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)\right)^2\right)| \le 1/n.$$

Let us study $\mathbb{E}\left(\rho_{n,P}\frac{1}{n}\left(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)\right)^2\right)$. We fix $\alpha\in(0,1/2)$.

$$\left|\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)-\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=n^\alpha}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)\right|$$
$$=\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=0}^{n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)+\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)-(\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)+\sum_{k=n}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)$$
$$=\frac{1}{n}\mathbb{E}(\rho_{n,P}[(\sum_{k=0}^{n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2+(\sum_{k=n}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)]$$
$$+2\mathbb{E}(\rho_{n,P}(\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))(\sum_{k=0}^{n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)-\sum_{k=n}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot)))$$
$$\leq n^{2\alpha-1}\|\varphi\|_\infty^2+\frac{4}{n}[\mathbb{E}(\rho_{n,P}(\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)]^{1/2}n^\alpha\|\varphi\|_\infty$$

We have

$$\mathbb{E}(\rho_{n,P}(\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)$$
$$\leq\mathbb{E}(\rho_{n,P})\,\mathbb{E}((\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)+|Cov(\rho_{n,P},(\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)|.$$

Because of Property 3.1 the covariance tends to 0 faster than $1/n$. On the other hand $\mathbb{E}((\sum_{k=n^\alpha}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)$ is bounded by Cn for some constant $C>0$. Putting this together one obtains:

$$|\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)-\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=n^\alpha}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)|$$
$$\leq n^{2\alpha-1}\|\varphi\|_\infty^2+4Cn^{\alpha-1}\|\varphi\|_\infty\|\varphi\|_\eta n^{1/2}\leq C\|\varphi\|_\infty\|\varphi\|_\eta n^{\alpha-1/2}.$$

Again because of Property 3.1, we have

$$|\mathbb{E}(\rho_{n,P}\frac{1}{n}(\sum_{k=n^\alpha}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)-\mathbb{E}(\frac{1}{n}(\sum_{k=n^\alpha}^{n+n^\alpha-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)|\leq C\|\varphi\|_\infty\|\varphi\|_\eta n^{-1}$$

Thus we have shown that

$$\left|\mathbb{E}(\frac{1}{n}(\sum_{k=1}^{n}\varphi(\theta(\cdot)T^k\cdot))^2)-\sum_{P\in\mathcal{P}_n}\mu(P)\,\mathbb{E}(\frac{1}{n}(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)\right|\leq C\|\varphi\|_\infty\|\varphi\|_\eta n^{\alpha-1/2}.$$

Because of the mixing property of T we have

$$\left|\mathbb{E}(\frac{1}{n}(\sum_{k=0}^{n-1}\varphi_{\theta(x_{n,P})}(T^k\cdot))^2)-\sigma^2_{\theta(x_{n,P})}\right|\leq C\|\varphi\|_\infty\|\varphi\|_\eta n^{-1},$$

so that

$$\left|\mathbb{E}(\frac{1}{n}(\sum_{k=1}^{n}\varphi(\theta(\cdot)T^k\cdot))^2) - \sum_{P\in\mathcal{P}_n}\mu(P)\,\sigma^2_{\theta(x_{n,P})}\right| \leq C\|\varphi\|_\infty\|\varphi\|_\eta n^{\alpha-1/2}.$$

Since the last sum is a Riemann sum for the function $x \to \sigma^2_{\theta_x}$, this concludes the proof. □

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