Quenched central limit theorem for random walks with a spectral gap

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Abstract

Let G be a discrete group acting on a probability space (X, m) such that the action $(g, x) \to (g.x)$ is ergodic and preserves the measure m. For a probability μ on G we denote by P_μ the contraction of $L^p(m)$ defined by $P_\mu f(x) = \sum_g f(g.x)\mu(g)$. Let $(g_k(\omega))_{k\geq 1}$ be the sequence of independent random variables with distribution μ on G, i.e. the coordinate maps on the space $\Omega := G^{\mathbb{N}^*}$ endowed with the probability product $\mu^{\otimes \mathbb{N}^*}$.

For f a function on X and for a given $\omega \in \Omega$, we consider the ergodic sums $S_n f(\omega, x) = \sum_{k=1}^n f(g_k(\omega)...g_1(\omega)x)$.

Assume that $P_\mu^\otimes$ has a spectral gap property for the diagonal action on $L^2(X \times X, m \times m)$. There are several examples of such a situation. We will present the following "quenched" central limit theorem:

For f in $L^\infty(m)$ with $\|f\| \neq 0$, there is $\sigma(f) > 0$, such that for a.e. $\omega \in \Omega$,

$$\lim_n m\{x : \frac{1}{\sigma(f)\sqrt{n}} S_n f(\omega, x) < a\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}t^2} \, dt.$$

We will also present examples of a quenched CLT under the weaker assumption of stationarity.

This is a joint work with Stéphane Le Borgne (University of Rennes 1).

Contents

1 Annealed and quenched central limit theorems

Framework for a quenched CLT with the formalism of extensions of dynamical systems.

Here we consider the framework of dynamical systems, but there are several situations where the question of a quenched CLT can be formulated. In statistical mechanics or for Markov chains. In some cases it can be considered as the study of non stationary processes depending on a parameter, such that stationarity arises after integration with respect to the parameter.

Let be given a measure preserving dynamical system $(\Omega, \mathbb{P}, \theta)$ and a metric space $(X, \mathcal{B}, m)$ endowed with its Borel σ-algebra $\mathcal{B}$ and a probability m. Denote by $\mathcal{T}$ a semigroup of measurable maps from X to itself which preserves m.

Let $T : \omega \to T(\omega)$ be a measurable map from Ω to $\mathcal{T}$. The map T can be viewed as a cocycle over the dynamical system with values in $\mathcal{T}$.

We obtain a dynamical system defined on $\Omega \times X$ with invariant measure $\mathbb{P} \times m$ by setting

$$\theta_T(\omega, x) = (\theta\omega, T(\omega)x).$$

The iterates of θ_T read

$$\theta_T^k(\omega, x) = (\theta^k\omega, T_1^k(\omega)x), \text{ with } T_1^n(\omega) = T(\theta^{n-1}\omega) \circ ... \circ T(\theta\omega) \circ T(\omega), n \geq 1. \tag{1}$$

If the transformations $T(\omega)$ satisfy a property of hyperbolicity, one can expect that the system defined by θ_T has good statistical properties at least for the action on functions defined on X.

We denote the ergodic sums of $\varphi : x \to \varphi(x)$ on X by $S_n\varphi(\omega, x) = \sum_{k=0}^{n-1} \varphi(T_1^k(\omega)x)$.

The CLT holds for a space $\mathcal{H}$ of real functions in $L_0^2(X, m)$ if, for every $\varphi \in \mathcal{H}$,
- the variance exists: $\sigma^2 = \sigma^2(\varphi) := \lim \frac{1}{n}\|S_n\varphi\|_{2,\mathbb{P}\times m}^2$ and if, $\sigma^2(\varphi) > 0$, there is convergence in distribution

$$\mathbb{P} \times m\{(\omega, x) : \frac{1}{\sigma\sqrt{n}} S_n\varphi(\omega, x) \leq a\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}t^2}\, dt. \tag{2}$$

This is the "annealed" CLT on the product space according to the product measure $\mathbb{P} \times m$, for the iterates of θ_T.

Now the quenched CLT consists in fixing ω in Ω and looking at the distribution of the ergodic sums according to the measure m on the space X.

We say that the **quenched CLT** holds for a space $\mathcal{H}$ of real functions in $L_0^2(X, m)$ if, for every $\varphi \in \mathcal{H}$, for a.e. $\omega \in \Omega$,
- the variance exists and does not depend on ω: $\lim \frac{1}{n}\|S_n\varphi(\omega, .)\|_{2,m}^2 = \sigma^2(\varphi)$.
- If $\sigma^2(\varphi) > 0$, there is convergence in distribution

$$m\{x : \frac{1}{\sigma\sqrt{n}} S_n\varphi(\omega, x) \leq a\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}t^2}\, dt. \tag{3}$$

Remark that this type of quenched CLT differs from the quenched CLT where x is fixed and the the convergence is with respect to the probability $\mathbb{P}_x$ in ω starting from x, as considered by several authors, beginning with M. Gordin and B. Lifshits.

An example: sequence of toral automorphisms

Take for (X, m) the d-dimensional torus with the Lebesgue measure and for $\mathcal{T}$ the semigroup of surjective endomorphisms of $\mathbb{T}^d$. Restricting to the case when the set of values of $T(\omega)$ is finite, we consider a finite set $\mathcal{A}$ of matrices in $\mathrm{GL}(d, \mathbb{Z})$ and a measurable map $T : \Omega \to \mathcal{A}$.

When $\mathcal{A}$ reduces to a single matrix $A \in SL(d, \mathbb{Z})$ without eigenvalue root of unity, then $T(\theta^{k-1}\omega)...T(\theta\omega)\,T(\omega) = A^k$. If φ is Hölder on $\mathbb{T}^d$, the CLT holds for the ergodic sums $S_n\varphi(.) = \sum_{k=0}^{n-1} \varphi(A^k.)$.

Now let $\mathcal{A} = \{A_i, i = 1, ..., r\}$ be a set of hyperbolic matrices in $SL(d, \mathbb{Z})$ and consider a map $T : \omega \to A(\omega)$ taking values in $\mathcal{A}$. We obtain a product of endomorphisms of the torus generated by the stationary sequence $(A(\theta^n\omega))$. We can ask about the CLT for a.e. ω for the sums

$$S_n\varphi(.) = \sum_{k=0}^{n-1} \varphi(T(\theta^{k-1}\omega)...T(\theta\omega)\,T(\omega).).$$

In a joint work with S. Le Borgne and M. Roger, we have applied the method of "multiplicative systems" of Komlòs in some special cases to give a positive answer. We will discuss this later, as well as recent results.

Now we consider under a spectral gap property, but for more general actions, the independent case.

2 Spectral gap and limit theorems

Let $(X, \mathcal{B}, m)$ be as above and let G be a group of Borel invertible maps of X into itself which preserve m and acts ergodically. Let μ be a probability measure on G with support Γ, such that *G is the closed group generated by* Γ.

Let $(\Omega, \mathbb{P}) = (\Gamma^{\mathbb{N}}, \mu^N)$, $\omega = (\omega_1, \omega_2, ...)$. We denote by $T_k(\omega) := \omega_k$ the coordinate maps. With the previous notation, we have $T_1^n(\omega) = \omega_n ... \omega_1$.

These data, denoted by (X, m, Γ, μ), define a *random walk* on X with Markov operator $P = P_\mu$ (and stationary measure m) given by

$$P_\mu \varphi(x) \ = \ \sum_{a \in \Gamma} \varphi(ax)\, \mu(a). \tag{4}$$

The operator corresponding to the diagonal G-action on $(X \times X, m \times m)$ is $P_\mu^{\otimes} \varphi(x, y) := \sum_{a \in \Gamma} \varphi(ax, ay)\, \mu(a)$.

P_μ is a contraction of $L^p(X, m)$, $\forall p \geq 1$, and it preserves the subspace $L_0^2(X, m)$ of functions φ in $L^2(X, m)$ such that $m(\varphi) = 0$. Ergodicity of P_μ is equivalent to ergodicity of the action of G on the measure space $(X, \mathcal{B}, m)$. We will use a strong reinforcement of the ergodicity, the spectral gap property for the operator P when it holds.

Let $P_{0,\mu}$ be the restriction of P_μ to $L_0^2(X, m)$. We say that P_μ satisfies the **spectral gap property** if $\|P_{0,\mu}\| < 1$.

Recall that, with the previous notation,

$$S_n\varphi(\omega, x) := \sum_{k=1}^{n} \varphi(T_1^k(\omega)x).$$

The existence of a spectral gap implies "quenched" properties for the random walk, as shown by A. Furman and Ye. Shalom in 99.

By Kakutani's theorem, for all $\varphi \in L^1(X, m)$, for $\mathbb{P}$-a.e ω, $\lim_n \frac{1}{n} S_n\varphi(\omega, x) = \int \varphi \, dm$, for m-a.e. x. With a spectral gap for P_μ on $L_0^2(X, m)$, there is a.s. a rate of convergence.

Theorem 2.1. *Suppose* $\|P_{0,\mu}\| < 1$ *and let* $\varphi \in L_0^2(X, m)$, *then*

$$\|\frac{1}{n}S_n\varphi(\omega, .)\| = o(\frac{\log^{1/2+\varepsilon} n}{\sqrt{n}}), \ \frac{1}{n}S_n\varphi(\omega, x) = o(\frac{\log^{3/2+\varepsilon} n}{\sqrt{n}}), \text{ for } \mathbb{P} - \text{a.e. } \omega.$$

If $\|P_{0,\mu}^{\otimes}\| < 1$, *there is a.s. an exponential rate of mixing along the random walk generated by* μ.

(CLT) If $\varphi \in L_0^p(X, m)$, $p > 2$, $f \neq 0$, *let* $\sigma_n = \|S_n\varphi\|_2$ *(as a r.v. on* $(\Omega \times X, \mathbb{P} \times m)$*). Then* $\sigma = \lim_n \frac{\sigma_n}{\sqrt{n}}$ *exists, is* $\neq 0$ *and*

$$\sup_{a \in [0,1]} |\mathbb{P} \times m(\{n^{-1/2}S_n\varphi < a\}) - \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{a} e^{-t^2/2\sigma^2} \, dt| = O(\frac{\log^{p/2} n}{n^\delta}), \text{ with } \delta = \min(\frac{1}{2}, \frac{p-1}{2}).$$

The last statement is an "annealed" CLT. Now we consider the "quenched" version of the CLT.

3 Spectral gap and quenched CLT (Independent case)

In 2007, Ayyer, Liverani and Stenlund looked at the case of $\mathrm{SL}(2,\mathbb{Z})$ acting on the torus. More precisely, they considered the model described above, when the set $\mathcal{A}$ consists of positive matrices in $\mathrm{SL}(2,\mathbb{Z})$ and they proved a quenched CLT for regular functions. They use an inequality of large deviations deduced from the theory of perturbation of operators along Nagaev method.

This perturbation method does not seem to apply (immediately) in our situation. But we can adapt their method using the spectral gap property of the operator P_μ on L^2_0. The main steps are the two propositions below.

It can be shown that the variance computed for a.e. fixed ω is the same as the global variance:

Proposition 3.1. *If $\|P^{\otimes}_{0,\mu}\| < 1$, then, for every φ in $L^2_0(X,m)$, for a.e. ω, we have*

$$\lim \frac{1}{n}\int_X S_n\varphi^2 \, dm = \sigma^2.$$

Proposition 3.2. *If $\|P_{0,\mu}\| < 1$, then for every $R > 0$ and $\alpha > 0$, there exists $C > 0$ such that for φ in L_0^{2R}*

$$\mathbb{E}(|S_n\varphi|^{2R}) \leq C n^{(1+\alpha)R}.$$

Theorem 3.3. *(C., Le Borgne) Let G be a group acting ergodically on a probability space (X,m) and μ be a probability measure on G with support Γ as above such that $\|P^{\otimes}_{0,\mu}\| < 1$. Let φ be a real function in $L^p_0(X,m)$, $p > 2$. Then, for a.e. $\omega \in \Omega$, for every $a \in \mathbb{R}$,*

$$\lim_{n\to\infty} m(x : \frac{1}{\sigma\sqrt{n}} S_n\varphi(\omega,x) < a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \exp(-t^2/2)\, dt.$$

4 Spectral gap: examples

There are algebraic examples with the spectral gap property, like group of automorphisms on tori and nilmanifolds, action on homogeneous spaces of Lie groups. It never occurs for G amenable.

There are results of A. Furman, Ye. Shalom, J. Bourgain, A. Gamburd, Y. Guivarc'h, among others.

Tori: For action by automorphism on the torus, the necessary and sufficient condition for the existence of a spectral gap for P_μ is that there does not exist an invariant rational subtorus S such that the action of the group generated by the support of μ on $\mathbb{T}^d/S$ is the action of an abelian group (up to a finite index).

For example, if the group generated by the support of μ has no abelian subgroup of finite index and acts irreducibly on $\mathbb{R}^d$, there is a spectral gap.

For automorphisms of a compact abelian group, the conditions $\|P_{0,\mu}\| < 1$ and $\|P_{0,\mu}^{\otimes}\| < 1$ are equivalent.

Nilmanifolds: Let N be a simply connected nilpotent group, D a lattice in N, $X = N/D$ the corresponding nilmanifold, and $T = N/N'.D$ the maximal torus factor. The following spectral gap property holds for groups of automorphisms or affine transformations of nilmanifolds. Let Γ be a countable group of affine transformations of N/D. Let μ be a probability measure with support Γ. Recently B. Bekka and Y. Guivarc'h have shown that the existence of a spectral gap for the action of Γ on T implies the existence of a spectral gap for the action of Γ on N/D.

5 Another method, without spectral gap

In the case of the torus a quite different method can be used for Hölder functions. It is based on separation of frequencies and a CLT for "multiplicative systems" introduced by Komlòs. This method is adapted to the case of non independent stationary products. But it can also be used for the independent case. It provides a small rate of convergence in the quenched CLT.

The idea is that there is a fast growth of the norms of vectors in $\mathbb{Z}^d$ for the dual action of the endomorphisms, so that the "frequencies" of trigonometric polynomials φ are separated. The CLT is obtained by making gaps in the ergodic sums $S_n\varphi$.

For each ω, $T(\omega) = {}^tA(\omega)$ is an endomorphism of the torus. We use the following conditions, with the notation $A_1^n(\omega) = A(\omega)...A(\theta^{n-1}\omega)$.

Condition 5.1. *There is $c > 0$ such that for a.e. ω, for every $p, q \in \mathbb{Z}^d \setminus \{0\}$ with $\|p\|, \|q\| \leq D$,*

$$A_1^r(\theta^\ell\omega)p \neq q, \ \forall r > c\log D, \ \forall \ell \geq 0. \tag{5}$$

Condition 5.2. *For a.e. ω, there exist $\gamma > 1$, c and $C > 0$ such that for every $p \in \mathbb{Z}^d \setminus \{0\}$*

$$\|A_1^{\ell+r}(\omega)p\| \ \geq C\gamma^{r-c\log\|p\|}\|A_1^\ell(\omega)\|, \ \forall r > c\log\|p\|, \ \forall \ell \geq 1. \tag{6}$$

There is a "uniform" variant of 5.1 valid for matrices in $\mathrm{SL}(2, \mathbb{Z}^+)$.

Condition 5.3. *There is $\delta > 0$, $\gamma > 1$, c and $C > 0$ such that for every $p \in \mathbb{Z}^d \setminus \{0\}$,*

$$\forall A_1, ..., A_r \in \mathcal{A}, \ \|A_1...A_rp\| \geq C\|p\|^{-\delta}\gamma^r, \ \forall r > c\log\|p\|. \tag{7}$$

5.1 Products of independent matrices in $SL(d,\mathbb{Z})$

As an application of the method of separation of frequencies, we obtain in the independent case the following result. Let $\mathcal{A}$ be a finite set of matrices in $SL(d,\mathbb{Z})$. Assume it is
- *proximal*: the semigroup generated by $\mathcal{A}$ contains a contracting sequence (it is satisfied for example when a finite product of elements of $\mathcal{A}$ has a dominant simple eigenvalue),
- *totally irreducible*: for every r, the action of $\mathcal{A}$ on the exterior product of $\bigwedge_r \mathbb{R}^d$ has no invariant finite union of non trivial sub-spaces.

Let μ be a probability on $\mathcal{A}$ such that $\mu(\{A\}) > 0$ for every $A \in \mathcal{A}$. Let $\Omega := \mathcal{A}^{\mathbf{N}} = \{\omega = (\omega_n), \omega_n \in \mathcal{A}, \forall n \in \mathbb{N}\}$ endowed with the product measure $\mathbb{P} = \mu^{\otimes \mathbf{N}}$. For ω in Ω, $A_k(\omega) = A_0(\theta^k\omega)$ is its k-th coordinate. Let $S_n\varphi(\omega, x) := \sum_{k=1}^n \varphi(A_k(\omega) \dots A_1(\omega)x)$. For the action of the product $A_n(\omega) \dots A_1(\omega)$ on the torus, there is for a.e. ω a Central Limit Theorem with a small rate.

Theorem 5.4. *(C., S. Le Borgne, M. Roger) Let φ be a centered Hölder function on $\mathbb{T}^d$ or a centered characteristic function of a regular set. Then, if $\varphi \not\equiv 0$, for $\mathbb{P}$-almost every ω, $\sigma(\varphi) = \lim_n \frac{1}{\sqrt{n}}\|S_n\varphi(\omega, .)\|_2$ exists, is $\neq 0$, independent from ω, and on the space $(\mathbb{T}^d, m)$*

$$(\frac{1}{\sigma(\varphi)\sqrt{n}} S_n\varphi(\omega, .)_{n\geq 1}$$

converges in distribution to the normal law $\mathcal{N}(0,1)$ with a positive rate of convergence.

5.2 Stationary non independent examples

Coming back to the general framework of the beginning, consider stationary (but not necessarily independent) products of automorphisms of $\mathbb{T}^d$. In this direction, we have a partial result, if we restrict the choice of the matrices. Here two examples in dimension 2.

1) **Endomorphisms of $\mathbb{T}^2$ with > 0 coefficients**

Theorem 5.5. *Let φ be a Hölder continuous function or the characteristic function of a regular set. Let the sequence $(T_1^n(\omega))$ be generated by a dynamical system (Ω, θ, μ). If the map $T : \Omega \to \mathrm{Aut}(\mathbb{T}^2)$ takes values in a finite set $\mathcal{A}$ of matrices $\in SL(2, \mathbb{Z})$ with > 0 coefficients, then either for μ-almost $\omega \in \Omega$, $(\|S_n\varphi(\omega, .)\|_2)$ is bounded or, for μ-almost $\omega \in \Omega$, the sequence $(n^{-\frac{1}{2}}\|S_n\varphi(\omega, .)\|_2)$ has a limit $\sigma(\varphi) > 0$ not depending on ω. In the latter case, a non degenerated quenched CLT holds with a rate of convergence.*

For instance, if the sequence $(T_1^n(\omega))$ is generated by an ergodic rotation on the circle, with $T(\omega) = A(\omega) = A$ on an interval and $= B$ on the complementary, where A, B are two matrices in $SL(2, \mathbb{Z}^+)$, then the CLT holds for every such sequence.

For a trigonometric polynomial $\varphi \not\equiv 0$, it can be shown that the variance $\sigma(\varphi)$ is > 0.

2) Kicked systems

Let H be a hyperbolic matrix in $\mathrm{SL}(2,\mathbb{Z})$ and (B_n) be a sequence in $\mathrm{SL}(2,\mathbb{Z})$ such that the sequence $(\mathrm{trace}(B_n))$ is bounded. Let $s \geq 1$ be a fixed integer. Let us consider the sequence M_n of automorphisms of the torus:

$$M_n = B_1 H^s ... B_{n-1} H^s B_n H^s. \tag{8}$$

L. Polterovich and Z. Rudnick called such a sequence (8) a "kicked" system. They proved the following "mixing stability" property: if H is not conjugate to its inverse, for every constant $K > 0$,there exists s_0 such that, for every sequence of "kicks" (B_k) with trace bounde by K, the sequence defined by (8) is mixing for every $s \geq s_0$, i.e. the decorrelation property holds: if φ and ψ are Hölder functions, there exist constants C and $0 < \kappa < 1$ such that

$$|\int_{\mathbb{T}^d} \varphi({}^t M_n x)\, \overline{\psi}(x)\, dm(x)| \leq C\kappa^n, \ \forall n \geq 1.$$

A way to generate stationary kicked systems is the following. Let $\mathcal{A}$ be a set of matrices of the form $\mathcal{A} = \{B_j H^s, j \in \mathbb{N}\}$, where $\{B_j\}$ is a family of matrices in $\mathrm{SL}(2,\mathbb{Z})$ with bounded trace.

Let $\omega \to T(\omega) = B(\omega) H^s$ be a measurable map from Ω to $\mathcal{A}$. Consider the skew product θ_T defined on $\Omega \times \mathbb{T}^2$ as before by $\theta_T: \ (\omega, x) \mapsto (\theta\omega, T(\omega)x)$.

By iterating the map θ_T we obtain a kicked system depending on ω of the form:

$$M_n(\omega) = B(\theta^{n-1}\,\omega) H^s \,...\, B(\theta\omega) H^s\, B_1(\omega) H^s. \tag{9}$$

A quasi-morphism on a group G is a function $r : G \to \mathbb{R}$ such that the function $dr : G \times G \to \mathbb{R}$ defined by $dr(g_1, g_2) = r(g_1 g_2) - r(g_1) - r(g_2)$ is bounded. A homogeneous quasi-morphism also satisfies $r(g^n) = nr(g)$ for all $n \in \mathbb{Z}$.

We use the following results of Polterovich and Rudnick. Let r be a homogeneous quasi-morphism of $\mathrm{SL}(2, Z)$ which vanishes on all parabolic elements. Then there is $c > 0$ such that for every non zero vector $v \in \mathbb{Z}^2$ and every $A \in \mathrm{SL}(2, \mathbb{Z})$,

$$\|Av\| \geq e^{c|r(A)|} \|v\|^{-1}.$$

If H is note conjugate to its inverse, there is an homogeneous quasi-morphism with

$$r(H) = 1 \text{ and } r(B_n H^s B_{n-1} H^s ... B_1 H^s) = \sum_{i=1}^{n} [r(H^s) + r(B_i)] + O(1) = ns + O(n).$$

This implies a uniform lower bound for the Lyapunov exponent. Using this result and Oseledets ergodic multiplicative theorem, we can prove a weak form of the "separation of frequencies" property and deduce from it:

Theorem 5.6. *The quenched CLT is satisfied by a kicked stationary system.*