# Recurrence, ergodicity and invariant measures for cocycles over a rotation

Jean-Pierre Conze

IRMAR, UMR CNRS 6625
Université de Rennes I,
Campus de Beaulieu,
35042 Rennes Cedex, France

**Abstract**

Let $(X, \mathcal{A}, \mu, \tau)$ be an ergodic dynamical system and $\varphi$ be a measurable map from $X$ to a locally compact group $G$. We consider the skew product $\tau_\varphi$ defined on $X \times G$ by $\tau_\varphi : (x, g) \to (\tau x, \varphi(x)g)$.

When the system is an ergodic rotation and $G = \mathbb{R}^d$, we give examples of functions $\varphi$ with bounded variation which are non-regular (i.e. such that the ergodic decomposition of $\mu \times dg$ for $\tau_\varphi$ is based on non-finite $\tau$-invariant measures on the base). We give as well examples for which, on the contrary, the measure $\mu \times dg$ is ergodic and one can identify all ergodic $\tau_\varphi$-invariant locally finite measures. The noncommutative case is also considered and the recurrence of a class of cocycles in the group of triangular matrices $N_3$ is shown.

## Introduction

Let $(X, \mathcal{A})$ be a standard Borel space with an invertible measurable map $\tau : X \to X$ and a probability (or a $\sigma$-finite) measure $\mu$ which is invariant (or quasi-invariant) and *ergodic* for $\tau$. Let $\varphi$ be a measurable function on $X$ with values in a locally compact group $G$. We can associate to $\varphi$ a *cocycle*, noted $(\varphi_n)_{n\in\mathbb{Z}}$ or $(\varphi, \tau)$, defined for $n \geq 1$ by

$$\varphi_n(x) = \varphi(\tau^{n-1}x) \dots \varphi(x),$$

and a map $\tau_\varphi$ (called *skew-product* or *cylindrical* transformation) acting on $X \times G$ by

$$\tau_\varphi : (x, y) \to (\tau x, \varphi(x)y). \tag{1}$$

We denote by $m_G(dg)$, or simply $dg$, a left Haar measure on $G$. The map $\tau_\varphi$ leaves invariant the measure $\mu \times m_G$ denoted by $\lambda_0$ and more generally the measures $\lambda_\chi$ based on a conformal measures $\mu_\chi$ (see below).

If $G$ is abelian, $\varphi_n$, noted additively, coincides with the ergodic sum $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ \tau^k$, for $n > 0$.

The cocycle $(\varphi_n)$ can be viewed as a *stationary walk* in $G$ over the dynamical system $(X, \mu, \tau)$. When it is *recurrent* (i.e. when $(\varphi_n(x))$ returns for a.e. $x$ infinitely often in any neighborhood of the neutral element in $G$), the transformation $\tau_\varphi$ is conservative for the invariant measure $\mu \times m_G$.

Among the questions that arise about cocycles, we can mention: construction of recurrent cocycles $(\varphi, \tau)$ in a group $G$; ergodicity of the measures $\lambda_0$ and $\lambda_\chi$ and expression of their ergodic decomposition if they are not ergodic for $\tau_\varphi$; determination of all ergodic $\tau_\varphi$-invariant *locally finite* measures. Let us briefly comment these different points.

1) When $\varphi$ has its values in $\mathbb{R}$ and is integrable, $(\varphi_n)_{n\in\mathbb{Z}}$ is recurrent if and only if $\mu(\varphi) = 0$ (cf. [Kr75] or [Sc77], see also Lemma 2.5). In higher dimension the recurrence of a cocycle requires stronger assumptions. For any amenable group $G$, over any ergodic system $(X, \mu, \tau)$, there exist measurable recurrent ergodic cocycles $(\varphi, \tau)$ with values in $G$ (cf. [He79], [GoSi85]); but the problem remains of the construction of recurrent cocycles generated by regular functions over specified dynamical systems. When $G$ is $\mathbb{R}^d$ or some nilpotent groups ([Gr05]), an estimate of the growth of the ergodic sums can be used for rotations on the circle and functions $\varphi$ with bounded variation. We will give an example on a nilpotent group.

2) The ergodicity of the measure $\lambda_0$ for the map $\tau_\varphi$, when the base is a one-dimensional rotation, was the subject of many papers since the seventies (some of them are cited in the bibliography) and the present paper is partly in this perspective. Circle extensions of $\mathbb{Z}^d$-actions given by rotations on multidimensional tori are considered by K. Frączek in [Fr00]. It must be emphasized that, over multidimensional rotations, the ergodicity of very simple noncompact skew products seems to be unknown.

The ergodic decomposition of $\lambda_0$ was considered by K. Schmidt, who introduced the notion of regularity of a cocycle ([Sc77]). In a recent paper, A. Raugi ([Ra06]) could express, via a cohomological reduction of the cocycle, the $\tau_\varphi$-ergodic invariant measures, for a class of $\sigma$-finite measures. This makes it possible to give the form of the ergodic decomposition of the measures $\lambda_0, \lambda_\chi$, when they are not ergodic for $\tau_\varphi$ ([CoRa07]). For a *non-regular* cocycle the decomposition introduces a family of non-locally finite measures on $X \times G$ based on $\sigma$-finite non-finite measures on the space $X$. An example of non-regular cocycle was given in [Le95]. We construct here other examples of non-regular cocycles.

3) For a transformation $\tau$ on a space equipped with a quasi-invariant measure $\mu$, there are an "enormous diversity" of $\sigma$-finite, non-locally finite, invariant measures related to $\mu$ (cf. [Sc79]). Therefore the determination of $\tau_\varphi$-invariant measures on $X \times G$ has a meaning only within some restricted classes of measures. The determination of all locally finite ergodic $\tau_\varphi$-invariant measures was recently studied (cf. [ANSS02], [Sa04]). The idea, for a skew product over a rotation, is that these measures should be "Maharam-type" measures. We will extend here some of the results of [ANSS02].

The aim of this paper is to illustrate different aspects of the theory by examples of cocycles over rotations. The plan is the following. After the first section, where we recall some general results on the invariant measures for a cocycle and the notions of regularity and essential values introduced in [Sc77], [FeMo77], we take a rotation as the base for the skew-product. Section 2 is devoted to the recurrence for some BV cocycles with values in a nilpotent group. In section 3 we prove that the measures $\lambda_0$ and $\lambda_\chi$ are ergodic for some class of $\mathbb{R}$-cocycles defined by step functions $\varphi$ and are the only locally finite invariant measures for the map $\tau_\varphi$. In section 4 we construct a simple non-regular step function and we give also a construction of a non-regular cocycle which is valid over any rotation. In the appendix, a diophantine property used in the proof of ergodicity is proved.

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# Contents

# 1 Invariant measures, regularity of a cocycle

## 1.1 Invariant measures

As in the introduction, we consider $(X, \mathcal{A})$ a standard Borel space with an invertible measurable map $\tau : X \to X$ and a $\sigma$-finite measure $\mu$ which is quasi-invariant and ergodic for $\tau$. If $f$ is a function defined on $X$, we write $\tau f$ for $f \circ \tau$.

Let $G$ be a locally compact group with neutral element $e$. Recall that two $G$-valued cocycles $(\varphi, \tau)$ and $(\psi, \tau)$ over the dynamical system $(X, \mu, \tau)$ are *$\mu$-cohomologous* with *transfer function* $u$, if there is a measurable map $u : X \to G$ such that

$$\varphi(x) = u(\tau x)\, \psi(x)\, (u(x))^{-1} \text{ for } \mu - a.e.\ x. \tag{2}$$

A cocycle $(\varphi, \tau)$ is a *$\mu$-coboundary*, if there is a measurable $u$ such that $\varphi(x) = u(\tau x)\, (u(x))^{-1}$, for $\mu$-a.e. $x$.

As it is shown in [Sc79], there is abundance of $\sigma$-finite invariant ergodic measures for a transformation. Let us recall how to get such measures for the map $\tau_\varphi$. Let $\mu$ be a non-atomic measure on $X$ which is quasi-invariant and ergodic for $\tau$ and $\varphi$ a real function such that $\int \varphi \, d\mu = 0$. According to Theorem 10.5 of [Sc79], there is an uncountable family $(\nu_\iota, \iota \in J)$ of infinite non-atomic $\tau$-invariant ergodic mutually singular measures, such that $\varphi$ is a coboundary, $\varphi = \tau u_\iota \, u_\iota^{-1}$, for each system $(X, \nu_\iota, \tau)$ and, if $B$ is a measurable set such that $\nu_\iota(B) = 0$, $\forall \iota$, then $\mu(B) = 0$.

One obtains $\tau_\varphi$-invariant measures by taking the image of $\nu_\iota \times \delta_{y_0}$, $\iota \in J, y_0 \in G$ by the map $(x, y) \to (x, u_\iota(x) y)$.

If we try to explicit the measures which are invariant and ergodic for $\tau_\varphi$, we have therefore to put some restrictions on the class of measures.

We say that a measure $\lambda$ on $X \times G$ satisfies the property (P) if it can be written $\lambda(dx, dg) = \mu(dx) N(x, dg)$, where $\mu$ is a probability measure on $X$ and $N$ a positive Radon kernel (i.e. such that for every $x \in X$, $N(x, dg)$ is a positive Radon measure on the Borel sets of $G$ and the map $x \to N(x, B)$ is measurable, for every Borel set $B$ of $G$).

**Theorem 1.1** *(A. Raugi [Ra06]) Let $\lambda$ be a $\tau_\varphi$-invariant ergodic measure which satisfies (P). Then $\varphi$ is $\mu$-cohomologous to a cocycle $\psi$ with values in a closed subgroup $H$ of $G$: $\varphi = \tau u \, \psi \, u^{-1}$; the image $\tilde\lambda$ of $\lambda$ by the map $(x, g) \to (x, (u(x))^{-1} g)$ is a $\tau_\psi$-invariant ergodic measure with support $X \times H$ and has the form:*

$$\tilde\lambda(dx, dh) = \tilde\mu(dx) \chi(h)\, dh, \tag{3}$$

*where $dh$ is a left Haar measure on $H$, $\chi$ is an exponential on $H$ and $\tilde\mu$ is a $\sigma$-finite measure, equivalent to $\mu$ such that*

$$\tau \tilde\mu(dx) = \chi(\psi(\tau^{-1} x))\, \tilde\mu(dx). \tag{4}$$

*If $H = G$, then we have $u(y) \equiv e$, $\lambda(dy, dg) = \tilde{\mu}(dy)\ \chi(g)\,dg$ and $\tilde{\mu}$ a quasi-invariant measure on $X$ such that, for an exponential $\chi$ on $G$,*

$$\tau\tilde{\mu}(dy) = \chi(\varphi(\tau^{-1}y))\,\tilde{\mu}(dy).$$

For $\gamma \in G$, let $R_\gamma$ be the translation in the second coordinate in $X \times G$, $(x, g) \to R_\gamma(x, g) = (x, g\gamma)$. If $\lambda$ is a $\tau_\varphi$-invariant ergodic measure, its image by $R_\gamma$ is an invariant ergodic measure, which is either singular with respect to $\lambda$ or equal to $\lambda$ up to a constant factor. Let $H(\lambda) := \{\gamma : R_\gamma\lambda \sim \lambda\}$. This group is the closed subgroup $H$ of $G$ which appears in Theorem 1.1.

In specific examples, one could expect that $H = G$, thus characterizing the locally finite invariant measures $\lambda$.

## 1.2 Construction of invariant measures

Theorem 1.1 highlights the role of conformal measures, i.e. quasi-invariant measures satisfying (4). We recall how to construct such measures and the corresponding locally finite $\tau_\varphi$-invariant measures in the framework of a compact metric space $X$ and a Borel map $\varphi$ from $X$ to $\mathbb{R}^d$. For a positive measure $\lambda$ on $X \times G$, "locally finite" means here $\lambda(X \times K) < \infty$, for all compact subsets $K$ in $G$.

Let $\zeta$ be a continuous real valued positive function on $X$. A sufficient condition for the existence of a quasi-invariant probability measure $\nu$ such that $\tau\nu = \zeta\nu$ is $\inf_{\mu \in E(\tau)} |\int \log\zeta\ d\mu| = 0$, where $E(\tau)$ is the set of all ergodic $\tau$-invariant probability measures. For an ergodic rotation, this condition reduces to $\int_X \log\zeta\ dx = 0$. (See also [Sc79]).

Remark that we may get an atomic measure on the orbit of a point $x \in X$ such that $\sum_{k\in\mathbb{Z}} \zeta_k(x) < \infty$, where $\zeta_k$ is the multiplicative cocycle defined by $\zeta$. For instance, for every ergodic rotation one can construct a continuous function $\zeta$ such that, for every rational $x$, $\sum_{k\in\mathbb{Z}} \zeta_k(x) < +\infty$ (cf. [Br02]). For such a function and any rational, the measure concentrated on the orbit of $x$ defined by $\nu(\{T^n x\}) = \zeta_n(x)$, $\forall n \in \mathbb{Z}$, satisfies $\tau\nu = \zeta\nu$.

If $\tau$ is an ergodic rotation on the circle and $\log\zeta$ has bounded variation, this does not happen: in this case Inequality (13) (section 3.1) implies $\sum_{k\in\mathbb{Z}} \zeta_k(x) = \infty, \forall x \in X$. Moreover, the measure $\nu$ satisfying the quasi-invariance equation for a given $\zeta$ is unique (cf. [MeSt93], [CoGu00]). As a consequence, we have:

*If $\varphi$ is a function with values in $\mathbb{R}^d$ with bounded variation and null integral, for every ergodic rotation $\tau$ on the circle, for every exponential $\chi$ on $\mathbb{R}^d$, there exists a unique non-atomic probability measure $\mu_\chi$ (called $\chi \circ \varphi$-conformal) such that*

$$\tau\mu_\chi = \chi \circ \varphi \circ \tau^{-1}\ \mu_\chi. \tag{5}$$

*The measure (sometimes called measure of Maharam)*

$$\lambda_\chi(dx, dy) := \mu_\chi(dx) \times \chi(y)dy. \tag{6}$$

*is a $\tau_\varphi$-invariant locally finite measure.*

The $\tau_\varphi$-invariant ergodic locally finite measures have been described for some cylindrical transformations (cf. in particular O. Sarig ([Sa04]), J. Aaronson, H. Nakada, O. Sarig, R. Solomyak ([ANSS02])). It is proved in [ANSS02] that for $\varphi = 1_{[0,\beta]} - \beta$, if $\tau$ is a rotation by $\alpha$, assuming an arithmetical condition on $(\alpha, \beta)$, then the $\tau_\varphi$-invariant ergodic measures on $X \times \mathbb{R}$ are given by (6). We will extend this result in section 3.

## 1.3 Regularity, essential values

In the following definitions we consider an invariant probability measure $\mu$.

**Definition 1.2** A cocycle $(\varphi, \tau)$ is *$\mu$-regular*, if there exist a closed subgroup $H$ of $G$ and a measurable map $u : X \to G$ such that $\psi := (u \circ \tau)^{-1} \varphi u$ takes $\mu$-a.e. its values in $H$ and $\tau_\psi : (x, h) \to (\tau x, \psi(x)h)$ is ergodic for the product measure $\mu \otimes m_H$ on $X \times H$.

In the regular case, any $\tau_\varphi$-invariant function can be written $F(u(x)^{-1} g)$, for some function $F$ which is invariant for the left translations by elements of $H$. The regular case corresponds to a "good" ergodic decomposition of $\mu \times dg$.

If the cocycle is non-regular, then the measures $\mu_x$ of the ergodic decomposition of $\mu \otimes m_G$ are singular with respect to the measure $\mu$, there are uncountably many of them pairwise mutually singular and (at least for $G$ abelian) they are infinite (cf. [CoRa07]).

Examples of non-regular cocycle over rotations were given by Lemańczyk in [Le95]. In section 4, we give an example of a non-regular cocycle over a rotation given by a step function.

We recall now the notion of essential value of a cocycle (K. Schmidt [Sc77], J. Feldman, C.C. Moore [FeMo77]).

**Definitions 1.3** An element $a \in \overline{G} = G \cup \{\infty\}$ is an *essential value* of the cocycle $(\varphi, \tau)$ if, for every neighborhood $V$ of $a$, for every measurable subset $B$ such that $\mu(B) > 0$, there is $n \in \mathbb{Z}$ such that

$$\mu(B \cap \tau^{-n} B \cap \{x : \varphi_n(x) \in V\}) > 0.$$

We denote by $\overline{\mathcal{E}}(\varphi)$ the set of essential values of the cocycle $(\varphi, \tau)$, $\mathcal{E}(\varphi) = \overline{\mathcal{E}}(\varphi) \cap G$ the set of finite essential values and $\mathcal{P}(\varphi)$ the group of left period of the measurable $\tau_\varphi$-invariant functions $f$ on $X \times G$.

It can be shown that $\mathcal{E}(\varphi)$ and $\mathcal{P}(\varphi)$ are closed subgroups of $G$ and they are equal. In the abelian case, two cohomologous cocycles have the same set of essential values and we have $\overline{\mathcal{E}}(\varphi) = \{0\}$ if and only if $\varphi$ is a coboundary.

In the abelian case, if $\varphi$ is cohomologous to $\varphi_1$ and to $\varphi_2$, two functions with values respectively in closed subgroups with an intersection reduced to $\{e\}$, then $\mathcal{E}(\varphi) = \{e\}$.

The following simple lemma is a consequence of the preceding remark (we will see how to apply this lemma in the proof of Thm. 4.2).

**Lemma 1.4** *If $\varphi$ is a $\mathbb{Z}$-valued cocycle such that there exists $s \notin \mathbb{Q}$ for which the multiplicative equation $e^{2\pi i s\varphi} = \psi^{-1}\ \psi \circ \tau$ has a measurable solution $\psi$, then $\mathcal{E}(\varphi) = \{0\}$. If $\varphi$ is not a coboundary, then $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$ and the cocycle $\varphi$ is non-regular.*

# 2 Growth of the ergodic sums over a rotation, application to recurrence

**Notations 2.1** Given an irrational number $\alpha$, we denote by $\tau_\alpha$, or simply $\tau$, the rotation $x \to x + \alpha \bmod 1$ on $X = \mathbb{R}/\mathbb{Z}$, $\mu_0$ the Lebesgue measure. If $f$ is a function defined on $X$, we write $\tau f$ for $f \circ \tau$ and $f_k$ for the ergodic sums $f + ... + \tau^{k-1} f, k \geq 1$.

## 2.1 Growth of the ergodic sums

**• Continued fraction**

First, we briefly recall some well known results on the diophantine properties of an irrational number $\alpha \in ]0, 1[$ and its development in continued fraction $\alpha = [0; a_1, ..., a_n, ...]$.

We denote by $(p_n/q_n)_{n\geq 0}$ the sequence of its convergents. An integer $q$ belonging to the sequence $(q_n)$ will be called, to simplify, a *denominator* of $\alpha$. Recall that we have $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and for $n \geq 1$:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2}, \\ (-1)^n &= p_{n-1} q_n - p_n q_{n-1}. \end{aligned}$$

Noting, for a real $u$, $\{u\}$ its fractional part, $((u)) := \inf(\{u\}, \{1 - u\}) = \inf_{n\in\mathbb{Z}} |u - n|$, we have for $n \geq 0$:

$$((q_n\alpha)) = (-1)^n (q_n\alpha - p_n), \tag{7}$$

$$1 = q_n((q_{n+1}\alpha)) + q_{n+1}((q_n\alpha)). \tag{8}$$

The relation (8) implies, for every $n \geq 0$:

$$((q_n\alpha)) + ((q_{n+1}\alpha)) \leq \inf(\frac{1}{q_n}, \frac{2}{q_{n+1}}), \tag{9}$$

$$\frac{1}{q_{n+1} + q_n} \leq ((q_n\alpha)) \leq \frac{1}{q_{n+1}}. \tag{10}$$

An irrational number $\alpha = [0; a_1, ..., a_n, ...]$ has *bounded partial quotients* (abbreviated as *bpq*) if the sequence $(a_n)$ is bounded.

The type of a number $\alpha$ is the real number $\eta > 0$ such that

$$\inf_k [k^{\eta-\varepsilon}((k\alpha))] = 0, \quad \inf_k [k^{\eta+\varepsilon}((k\alpha))] > 0, \ \forall \varepsilon > 0. \tag{11}$$

The inequality (10) shows that any irrational number is of type $\geq 1$. It is well known that almost every $\alpha$ (for the Lebesgue measure on the circle) is of type 1.

**Lemma 2.2** *If $\alpha$ is of type $\eta$, we have $\sum_{k\geq 1} \frac{1}{k^{\eta+\delta}} \frac{1}{((k\alpha))} < \infty$ for any $\delta > 0$.*

<u>Proof</u> For $i \geq 1$, let $S_i$ be the strictly increasing sequence $(k_{i,j})_{j\geq 1}$ such that $((k_{i,j}\alpha)) \in [2^{-(i+1)}, 2^{-i}[$. Set $K_i = \min_j \{k_{i,1}, k_{i,j+1} - k_{i,j}\}$.
Let $0 < \varepsilon < \delta$. By (11) there is a constant $C(\varepsilon)$ such that

$$\begin{aligned} \frac{C(\varepsilon)}{k_{i,1}^{\eta+\varepsilon}} &\leq ((k_{i,1}\alpha)) < 2^{-i}, \\ \frac{C(\varepsilon)}{(k_{i,j+1} - k_{i,j})^{\eta+\varepsilon}} &\leq (((k_{i,j+1} - k_{i,j})\alpha)) < 2^{-(i-1)} \end{aligned}$$

and therefore

$$K_i \geq (C(\varepsilon) 2^{i-1})^{\frac{1}{\eta+\varepsilon}}.$$

We have $k_{i,j} \geq j K_i, \forall j \geq 1$, which implies, for a constant $D(\varepsilon)$

$$\begin{aligned} \sum_{k\in S_i} \frac{1}{k^{\eta+\delta}} \frac{1}{((k\alpha))} &= \sum_{j\geq 1} \frac{1}{k_{i,j}^{\eta+\delta}} \frac{1}{((k_{i,j}\alpha))} \leq \sum_{j\geq 1} \frac{2^{i+1}}{(jK_i)^{\eta+\delta}} \\ &= \frac{2^{i+1}}{K_i^{\eta+\delta}} \sum_{j\geq 1} \frac{1}{j^{\eta+\delta}} \leq D(\varepsilon) 2^{i(1-\frac{\eta+\delta}{\eta+\varepsilon})}; \end{aligned}$$

hence the convergence of the series $\sum_{k\geq 1} \frac{1}{k^{\eta+\delta}} \frac{1}{((k\alpha))}$. $\square$

*Inequality of Denjoy-Koksma*

If $f$ is a function on $X = \mathbb{R}/\mathbb{Z}$ with bounded variation (BV) denoted by $V(f)$, the ergodic sums $(f_n)$ satisfies the Denjoy-Koksma inequality: let $p$ and $q$ be relatively primes integers such that $|q\alpha - p| < 1/q$, then we have

$$|\sum_{0}^{q-1} f(x + \ell\alpha) - q \int_X f \, d\mu_0| \leq V(f), \forall x \in X. \tag{12}$$

Applied to the denominators $(q_n)$ of $\alpha$, this implies

$$|\sum_0^{q_n-1} f(x+\ell\alpha) - q_n \int_X f \, d\mu_0| \leq V(f), \forall x \in X. \tag{13}$$

Recall that if $f$ is a BV function, its Fourier coefficients satisfy $c_k(f) = O(\frac{1}{k})$. This will be used implicitly. We can also consider the class of functions in $L^2$ for which this last condition holds and get a $L^2$-bound for the sum in (13) in order to obtain recurrence for a broader class of cocycles. We recall the following result of [ALMN92].

**Proposition 2.3** *If $f$ is a function on the circle whose Fourier coefficients are in $O(\frac{1}{k})$, there is a constant $C$ such that, for any irrational number $\alpha$ and for any denominator $q$ of $\alpha$: $\| \sum_{k=0}^{q-1} f(.+k\alpha) - q\mu_0(f)\|_2 \leq C$.*

• **Growth of the ergodic sums**

One can bound the growth of the ergodic sums of a BV function $f$ over $\tau_\alpha$ by using the Denjoy-Koksma inequality and the "development in base $(q_n)$" where $(q_n)$ are the denominators of $\alpha$. We assume that $\mu_0(f) = 0$.

Any integer $n \geq 1$ can be written: $n = \sum_1^{\ell_n-1} b_j q_j$, with $0 \leq b_j \leq a_{j+1}$, $j = 1, ..., \ell_n - 1$, $b_{\ell_n-1} \geq 1$, where $\ell_n$ is determined by the condition

$$q_{\ell_n-1} \leq n < q_{\ell_n}. \tag{14}$$

The growth of the sequence $(q_n)$ is at least exponential and thus $\ell_n = O(\log n)$.

We set $\theta(n) := \sum_1^{\ell_n-1} a_{j+1}$. If $f$ has its Fourier coefficients in $O(\frac{1}{k})$, it follows from Proposition 2.3:

$$\|f_n\|_2 \leq C\,\theta(n), \ \forall n \geq 1.$$

If $f$ is BV, (13) implies

$$\|f_n\|_\infty \leq V(f)\,\theta(n), \ \forall n \geq 1.$$

We have now to bound the growth of $\theta(n)$. If $\alpha$ has bpq, then clearly $\theta(n) = O(\log n)$.

If $\alpha$ is of type 1, (11) implies that, for every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that $a_n \leq C(\varepsilon) q_{n-1}^\varepsilon$, $\forall n \geq 1$, hence: $n < (a_{\ell_n} + 1) q_{\ell_n-1} < (C(\varepsilon) q_{\ell_n-1}^\varepsilon + 1) q_{\ell_n-1}$, which implies:

$$C_1(\varepsilon) n^{1-\varepsilon} \leq C_1(\varepsilon) n^{\frac{1}{1+\varepsilon}} \leq q_{\ell_n-1}. \tag{15}$$

**Lemma 2.4** *If $\alpha$ is of type 1, for every $\varepsilon > 0$, $\theta(n) = O(n^\varepsilon)$.*

Proof Using (14) we have

$$\theta(n) \leq \ell_n \sup_{k \leq \ell_n} a_k \leq C(\varepsilon) q_{\ell_{n-1}}^\varepsilon \log n \leq C(\varepsilon) n^\varepsilon \log n.$$

□

## 2.2 A lemma on recurrence

**Lemma 2.5** *Let $(E, \mathcal{A}, \mu, \tau)$ be a dynamical system, with $\mu$ an invariant probability measure, and $\varphi$ a measurable function with values in a locally compact group $G$ with left Haar measure $m_G$. Assume that there are an increasing sequence of sets $(D_n)$ in $G$ and a strictly increasing sequence $(k_n)$ such that*

$$\lim_n \mu(x : \varphi_{k_n}(x) \notin D_n) = 0 \text{ and } \lim_n \frac{m_G(D_n.K)}{n} = 0 \tag{16}$$

*for any compact set $K \subset G$. Then the cocycle $(\varphi, \tau)$ is recurrent.*

Proof Let $\lambda$ be the product measure $\mu \times m_G$. We have to show that if $B$ is a wandering set for $\tau_\varphi$ (i.e. with pairwise disjoint images $(\tau_\varphi^\ell B, \ell \geq 0)$), then $\lambda(B) = 0$. Assume that $\lambda(B) > 0$. We can also assume that $B$ is included in $E \times K$ for a compact set $K$ in $G$.

Let $E_n = \{x : \varphi_{k_n}(x) \in D_n\}$ and $B_n = (E_n \times K) \bigcap B$. If we choose $n_0$ such that $\lambda(B_j) \geq \frac{1}{2}\lambda(B)$, for $j \geq n_0$, then $\lambda(\cup_{j=n_0}^n \tau_\varphi^{k_j} B_j) \geq \frac{1}{2}(n - n_0)\lambda(B)$.

On the other hand, this union is included in $E \times (D_n.K)$. Therefore we have

$$\lambda(\cup_{j=n_0}^n \tau_\varphi^{k_j} B_j) \leq m_G(D_n.K).$$

This implies $\liminf_n \frac{m_G(D_n.K)}{n} \geq \frac{1}{2}\lambda(B) > 0$, contrary to the assumption. $\square$

## 2.3 Application to stationary walks in the nilpotent group $N_3$

When the cocycle is defined by a BV function with values in $\mathbb{R}^d$ over a one dimensional rotation, the Denjoy-Koksma inequality implies easily recurrence. The situation is different for cocycles taking their values in a nilpotent group since the coordinates of the ergodic sums are polynomial expressions. The simplest non-commutative example is the group $G = N_3$ of matrices of the form

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in \mathbb{R}^3.$$

The group $G$ can be identified with $\mathbb{R}^3$ endowed with the composition law:

$$(a, b, c).(a', b', c') = (a + a', b + b', c + c' + ab').$$

An example of a recurrent and ergodic cocycle with values in the triangular group $N_3$ was recently given by G. Greschonig ([Gr05]). We present here a more general class of recurrent cocycles in $N_3$.

Let $\Phi = (f, g, h)$ be a map from $X$ into $\mathbb{R}^3$. We assume that $f, g, h$ are real valued BV and centered functions: $\mu_0(f) = \mu_0(g) = \mu_0(h) = 0$. A supplementary condition on the pair $(f, g)$ will appear below.

The transformation $\tau_\Phi$ is defined on $X \times N_3$ by

$$\tau_\Phi(x, g) = (x + \alpha, \Phi(x).g) = (x + \alpha, a + f(x), b + g(x), c + h(x) + bf(x)).$$

For $k \geq 2$, we have by iteration:

$$\tau_\Phi^k(x, a, b, c) = (x + k\alpha, a + f_k(x), b + g_k(x), c + h_k(x) + bf_k(x) + \sum_{j=1}^{k-1} f(\tau^j x) g_j(x)). \quad (17)$$

When $k = q_n$, a denominator of $\alpha$, the ergodic sums $f_k, g_k, h_k$ are bounded by $V(f)$. The recurrence of the cocycle defined by $\Phi$ in $N_3$, is related to an estimate of the sums $\sum_{j=1}^n \tau^j f \, g_j$.

After a formal calculation on Fourier series it appears that there is a "linear drift" of the form $n\gamma(f, g)$. If $\alpha$ is of type 1, a necessary and sufficient condition of recurrence is that the constant $\gamma(f, g)$ is null.

Let $f = \sum_{p \neq 0} c_p e_p$, $g = \sum_{p \neq 0} d_p e_p$ be the Fourier series of $f$ and $g$, with $e_p = e^{2\pi i p .}, p \in \mathbb{Z}$. We have formally the following Fourier development:

$$\begin{aligned}
\sum_{j=1}^n \tau^j f \, g_j &= \sum_{p,q \neq 0} c_p d_q e_{p+q} \left( \sum_{k=1}^{n-1} e^{2\pi i p k \alpha} (\sum_{j=0}^{k-1} e^{2\pi i q j \alpha}) \right) \\
&= n \sum_{p \neq 0} c_p d_{-p} \, (e^{-2\pi i p \alpha} - 1)^{-1} - \sum_{p \neq 0} c_p d_{-p} \, (e^{-2\pi i p \alpha} - 1)^{-1} (\frac{e^{2\pi i p n \alpha} - 1}{e^{2\pi i p \alpha} - 1}) \\
&+ \sum_{p,q \neq 0, p+q \neq 0} c_p d_q e_{p+q} \, (e^{2\pi i q \alpha} - 1)^{-1} (\frac{e^{2\pi i (p+q) n \alpha} - 1}{e^{2\pi i (p+q) \alpha} - 1} - \frac{e^{2\pi i p n \alpha} - 1}{e^{2\pi i p \alpha} - 1}).
\end{aligned}$$

The coefficient $\gamma(f, g)$ of the term in $n$ is

$$\gamma(f, g) = \sum_{p \neq 0} \frac{c_p \, d_{-p}}{1 - e^{-2\pi i p \alpha}} = \sum_{p \geq 1} [\Re(c_p \overline{d_p}) + \Im(c_p \overline{d_p}) \frac{\cos \pi p \alpha}{\sin \pi p \alpha}]. \quad (18)$$

In this expression the series $\sum_{p \geq 1} \Re(c_p \overline{d_p})$ converges and is equal to $\frac{1}{2}\mu_0(fg)$, but the convergence of the series $\sum_p \Im(c_p \overline{d_p}) \frac{\cos \pi p \alpha}{\sin \pi p \alpha}$ requires diophantine conditions for $\alpha$.

From Lemma 2.2, the series is absolutely convergent if $\alpha$ is of type $\eta < 2$. If $f = g$, the coefficient $\gamma(f, g)$ reduces to $\sum_{p \geq 1} \Re(|c_p|^2) = \frac{1}{2}\mu_0(f^2)$.

**Theorem 2.6** *Let $\gamma(f, g)$ be given by (18). If $\alpha$ is of type 1, the stationary walk defined over the rotation $x \to x + \alpha$ by a map $\Phi$ from $\mathbb{T}^1$ to $N_3$ with centered BV components $(f, g, h)$ is recurrent if and only if $\gamma(f, g) = 0$ .*

Proof a) For every function $g$ on $X$, for every $q \geq 1$, we have

$$g = \frac{1}{q}\sum_{k=1}^{q} g_k - \tau(\frac{1}{q}\sum_{k=1}^{q} g_k) + \frac{1}{q}\tau g_q.$$

Therefore we can write

$$g = \psi^{(q)} - \tau\psi^{(q)} + \zeta^{(q)}, \tag{19}$$

with

$$\psi^{(q)} = \frac{1}{q}\sum_{k=1}^{q} g_k,\ \zeta^{(q)} = \frac{1}{q}\tau g_q.$$

b) *Estimation of* $\int \psi^{(q)} f\ d\mu_0$

We have

$$\int \psi^{(q)} f\ d\mu_0 = -\sum_{p\neq 0} \frac{c_p\, d_{-p}}{1 - e^{-2\pi i p\alpha}}(1 + \varepsilon_{p,q}),$$

with $\varepsilon_{p,q} = -\frac{1}{q}\frac{e^{-2\pi i q p\alpha}-1}{e^{-2\pi i p\alpha}-1} e^{-2\pi i p\alpha}$ and

$$|\varepsilon_{p,q}| = \frac{1}{q}|\frac{e^{2\pi i q p\alpha} - 1}{e^{2\pi i p\alpha} - 1}| \leq \inf(1, \frac{1}{q}\frac{2}{((p\alpha))}).$$

If $\sum_{p\neq 0} |\frac{c_p\, d_{-p}}{1-e^{-2\pi i p\alpha}}| < \infty$, this implies that

$$\lim_q \int \psi^{(q)} f\ dx = -\sum_p \frac{c_p\, d_{-p}}{1 - e^{-2\pi i p\alpha}} = -\gamma(f, g).$$

Under the assumption that $\alpha$ is of type 1, to estimate the rate of convergence, we have to bound the sum of the series $S_q$ defined by

$$S_q := \sum_{p\geq 1} \frac{1}{p^2}\frac{1}{((p\alpha))} \inf(1, \frac{((qp\alpha))}{q((p\alpha))}).$$

We have

$$S_q \leq \frac{1}{q^2}\sum_{p=1}^{q-1} \frac{1}{p}\frac{1}{((p\alpha))^2} + \frac{1}{q^{\frac{1}{2}}}\sum_{p\geq q} \frac{1}{p^{\frac{3}{2}}}\frac{1}{((p\alpha))} = (1) + (2).$$

For the term (2), Lemma 2.2 implies the convergence $\sum_{p\geq 1} \frac{1}{p^{\frac{3}{2}}}\frac{1}{((p\alpha))} < \infty$.

The term (1) is bounded by $\frac{1}{q^2}(\sum_{p=1}^{q-1} \frac{1}{p^2}\frac{1}{((p\alpha))})^{\frac{1}{2}}(\sum_{p=1}^{q-1} \frac{1}{((p\alpha))^3})^{\frac{1}{2}}$. The factor $\sum_{p=1}^{q-1} \frac{1}{p^2}\frac{1}{((p\alpha))}$ is bounded (Lemma 2.2). Moreover, using the fact that there is only one value of $r\alpha$ mod 1, for $1 \leq r < q$, in each interval $[\frac{k}{q}, \frac{k+1}{q}[$, $k = 1, ..., q-1$, we can bound $\sum_{p=1}^{q-1} \frac{1}{((p\alpha))^3})^{\frac{1}{2}}$ by $(\sum_{p=1}^{q-1} \frac{q^3}{p^3})^{\frac{1}{2}} \leq Cq^{\frac{3}{2}}$.

For $\alpha$ of type 1, it follows that $S_q = 0(q^{-\frac{1}{2}})$ and then

$$|\int f\psi^{(q)}\, d\mu_0 - \gamma(f,g)| = 0(q^{-\frac{1}{2}}). \tag{20}$$

c) If we take for $q$ a denominator of $\alpha$, $\psi^{(q)}$ and $\zeta^{(q)}$ satisfy the following inequalities (below we denote by $\| \;\|$ the uniform norm)

$$\begin{aligned} \|\psi^{(q)}\| &\leq \theta(q)V(g), && (21)\\ V(\psi^{(q)}) &\leq \frac{1}{2}qV(g), && (22)\\ \|\zeta^{(q)}\| &\leq \frac{V(g)}{q}. && (23) \end{aligned}$$

This implies

$$\begin{aligned} \sum_{k=1}^{n}\tau^k f\, g_k &= \sum_{k=1}^{n}(\psi^{(q)}\tau^k f - \tau^k(\psi^{(q)}f)) + \sum_{k=1}^{n}\zeta_k^{(q)}\tau^k f\\ &= \sum_{k=1}^{n}(\psi^{(q)}\tau^k f - \tau^k(\psi^{(q)}f)) + \sum_{k=1}^{n}\tau^{k-1}\zeta^{(q)}\ \tau^k f_{n-k+1}, \end{aligned}$$

hence using (20)

$$\begin{aligned} &\|\sum_{k=1}^{n}\tau^k f\, g_k - n\gamma(f,g)\|\\ \leq\ & \|\psi^{(q)}\|\,\|f_n\| + \|(\psi^{(q)}f - \int \psi^{(q)}f)_n\| + \|\zeta^{(q)}\|\sum_{k=1}^{n}\|\tau^k f_{n-k+1}\| + n|\int \psi^{(q)}f d\mu_0 + \gamma(f,g)|\\ \leq\ & \|\psi^{(q)}\|\, V(f)\theta(n) + V(\psi^{(q)}f)\theta(n) + \|\zeta^{(q)}\|\sum_{k=1}^{n}\|f_{n-k+1}\| + \frac{Cn}{\sqrt{q}}. \end{aligned}$$

From (22) and (23), it follows

$$\begin{aligned} &\|\sum_{k=1}^{n}\tau^k f\, g_k - n\gamma(f,g)\| && (24)\\ \leq\ & (2\theta(q)V(g)V(f) + \frac{1}{2}qV(g)\|f\|)\theta(n) + \frac{V(g)}{q}V(f)n\theta(n) + \frac{Cn}{\sqrt{q}} && (25)\\ \leq\ & C(q+\frac{n}{q})\theta(n) + \frac{Cn}{\sqrt{q}}. && (26) \end{aligned}$$

For $q = q_{r_n - 1}$ such that $q_{r_n-1} \leq \sqrt{n} < q_{r_n}$, we have (cf. (15)): $q_{r_n-1} \geq C(\sqrt{n})^{1-\varepsilon}$ ; hence

$$\frac{n}{q_{r_n-1}} \leq C\frac{n}{(\sqrt{n})^{1-\varepsilon}} = Cn^{\frac{1}{2}+\frac{\varepsilon}{2}} \text{ and } \frac{n}{\sqrt{q_{r_n-1}}} \leq Cn^{1-\frac{1}{4}(1-\varepsilon)}.$$

Lemma 2.4 and (26) imply, for a constant $C_1$ depending on $\varepsilon$,

$$\|\sum_{k=0}^{n-1} \tau^k f\, g_k - n\gamma(f,g)\| \leq C_1\sqrt{n}(1+n^{\frac{\varepsilon}{2}})\theta(n) + C\frac{n}{\sqrt{q_{r_n-1}}} \leq 2C_1 n^{\frac{1}{2}+\frac{3\varepsilon}{2}} + Cn^{\frac{3}{4}+\frac{\varepsilon}{4}}. \quad (27)$$

If $\gamma(f,g)=0$, we have $\sum_{k=0}^{n-1} \tau^k f\, g_k = O(n^\delta)$, for any $\delta > \frac{3}{4}$.

The two other components have a growth with the rate $O(n^\varepsilon)$, for any $\varepsilon > 0$. We can apply Lemma 2.5 above with $k_n = n$ and $D_n$ defined, for a small $\varepsilon > 0$ and a constant $C$ big enough, as follows:

$$D_n := \{(x,y,z), 0 \leq |x| \leq Cn^\varepsilon, 0 \leq |y| \leq Cn^\varepsilon, 0 \leq |z| \leq Cn^{(\frac{3}{4}+\varepsilon)}\}.$$

One easily check using (17) that the sequence $(D_n)$ satisfies Condition (16) of Lemma 2.5. This gives the recurrence of the stationary walk in the group $N_3$ if $\gamma(f,g)=0$.

Conversely for a rotation by $\alpha$ of type 1, if $\gamma(f,g) \neq 0$, according to (27) the cocycle $(\Phi_n)$ behaves like $n\gamma(f,g)$ and the walk is transient. □

**Remark**: The coefficient $\gamma(f,g)$ defined by (18) vanishes if the following condition is satisfied

$$c_k d_{-k} = 0, \forall k \in \mathbb{Z}. \quad (28)$$

This condition (28) is satisfied if $f(x+\frac{1}{2}) \equiv f(x)$ and $g(x+\frac{1}{2}) \equiv -g(x)$, which is the assumption used in [Gr05] where ergodicity is also proved for a particular class of cocycle.

## 2.4 Recurrent cocycles in higher dimension

For a nilpotent group of order 2, we have to study expressions of the form

$$\sum_{k=1}^{n} \tau^k f\left(\sum_{j=1}^{k} \tau^j g(\sum_{\ell=1}^{j} \tau^\ell h)\right). \quad (29)$$

The method used for $N_3$ is particular to the dimension. For higher dimensions a way to construct recurrent cocycles is to consider stronger conditions, like the following:

Let $p$ be an integer $\geq 2$. If the pair $(f,g)$ satisfies the condition

$$\sum_{l=0}^{p-1} f(y+\frac{l}{p})\, g(x+\frac{l}{p}) = 0, \forall x, y, \quad (30)$$

then the expressions (29) can be bounded and a recurrence property can be proved.

This condition (30) is equivalent to $c_k(f)\, c_{k'}(g) = 0$, $\forall (k,k')$ such that $p$ divides $k+k'$. It is satisfied in particular if $f$ is invariant under the map $x \to x+\frac{1}{p}$ and $g$ satisfies $\sum_{k=0}^{p-1} g(x+\frac{k}{p}) = 0$.

For $p=2$, it is equivalent to ($f(x) \equiv f(x+\frac{1}{2})$ et $g(x) \equiv -g(x+\frac{1}{2})$) or ($f(x) \equiv -f(x+\frac{1}{2})$ and $g(x) \equiv g(x+\frac{1}{2})$).

# 3 Examples of ergodic BV $\mathbb{R}^d$-cocycles

## 3.1 Quasi-periods for a class of BV functions $\varphi$

The concept of quasi-period given in the following definition makes it possible to obtain essential values for classes of functions with bounded variation and discontinuities. We suppose that $G$ is a locally compact abelian group and we use additive notation.

The method of quasi-periods will be applied to rotations $x \to x + \alpha \bmod 1$ on the circle with the uniform measure $\mu_0$ or a conformal measure $\mu_\chi$. We recall that $\mu_\chi$ is the conformal measure specified in section 1 which corresponds to an exponential $\chi$ on $\mathbb{R}$.

**Definition 3.1** Let $(X, \mathcal{A}, \mu, \tau)$ be a dynamical system, where $\mu$ is quasi-invariant and $\varphi$ a measurable function on $X$ with values in $G$. We say that an element $c \in G$ is a *quasi-period* of a recurrent cocycle $(\varphi, \tau)$ if there exist $\delta > 0$ and sequences $(\ell_n)_{n\geq 1}$, $(\varepsilon_n > 0)_{n\geq 1}$, with $\lim_n \varepsilon_n = 0$, such that

$$\lim_n \tau^{\ell_n} x = x, \text{ for } \mu\text{−a.e. } x \in X \text{ and } \mu(A_n) \geq \delta, \forall n \geq 1,$$

where $A_n = \{x : d(\varphi_{\ell_n}(x), c) < \varepsilon_n\}$.

**Lemma 3.2** *Any quasi-period of $(\varphi, \tau)$ is a period of the $\tau_\varphi$-invariant functions (relatively to the measure $\lambda_0$) and thus an essential value of the cocycle.*

Proof 1) We use the notations of the previous definition. Let $f(x, t)$ be a $\tau_\varphi$-invariant function on $X \times G$, $f(\tau x, \varphi(x) + t) = f(x, t)$. We can suppose $f$ bounded (since $\tau_\varphi$ is conservative) and it is possible to regularize it by convolution with respect to the variable $t$, by replacing $f$ by the function $(x, t) \to \int_G f(x, s)\, h(t - s)\, ds$, where $h$ is a continuous function on $G$ with compact support. Therefore we can suppose $f$ to be continuous in $t$.

Let $c$ be a quasi-period. The set $\{x : f(x, c + t) = f(x, t), \text{ for almost every } t\}$ is $\tau$-invariant. The transformation $\tau$ being ergodic, to show the relation $f(x, c + t) = f(x, t)$, for $\mu$-almost every $x$, it is enough to show that the measure of this set is $\neq 0$.

Remark that the condition $\lim_n \tau^{\ell_n} x = x$, $\forall x \in X$, implies for $\zeta$ integrable on $X$:

$$\lim_n \int_X |\zeta(\tau^{\ell_n} x) - \zeta(x)|\, d\mu(x) = 0. \tag{31}$$

Indeed, this property is satisfied by the continuous functions on $X$ and can be extended to the class of integrable functions, thanks to the invariance of the measure $\mu$.

2) Let $u$ be a continuous $> 0$ integrable function on $\mathbb{R}$. Applying (31), we get

$$\lim_n \int_{\mathbb{R}} \int_X |f(\tau^{\ell_n} x, t) - f(x, t)|\, u(t)\, d\mu(x)\, dt = 0.$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_X 1_{A_n}(x)|f(x,t)-f(x,t-c)|\ u(t)\ d\mu(x)\,dt \\
\leq\ &\int_{\mathbb{R}}(\int_X |f(x,t)-f(\tau^{\ell_n}x,t)|\ d\mu(x))\ u(t)\,dt \\
+\ &\int_{\mathbb{R}}\int_X 1_{A_n}(x)|f(x,t-\varphi_{\ell_n}(x))-f(x,t-c)|\ u(t)\ d\mu(x)\,dt.
\end{aligned}$$

The two terms at right tend to 0, so that

$$\lim_n \int_X 1_{A_n}(x)\ (\int_{\mathbb{R}} |f(x,t)-f(x,t-c)|\ u(t)\,dt)\ d\mu(x) = 0.$$

The condition $\mu(A_n) \geq \delta > 0$ implies that the function

$$x \to \int_{\mathbb{R}} |f(x,t)-f(x,t-c)|\ u(t)\,dt$$

is null on a set of measure $\geq \delta$. $\square$

**Remarks** 1) There is a similar result for the proper functions: if $f$ is a $\tau_\varphi$-proper function and $c$ a quasi-period, there exists $\theta_c$ of modulus 1 such that: $f(x, c+t) = \theta(c) f(x,t)$.

2) If $\tau$ is an irrational rotation $x \to x+\alpha \bmod 1$ on the circle, $\varphi$ a BV function and the sequence $(\ell_n)$ is extracted from the sequence $(q_n)$ associated to $\alpha$, then the result of Lemma 3.2 is also valid when $\mu$ is the $\chi\circ\varphi$-conformal measure $\mu_\chi$ and the invariance of the functions is $\lambda_\chi$-a.e.. Indeed we can apply (13) which makes it possible to obtain (31), since it implies for a constant $C > 0$:

$$C^{-1} \leq \frac{d(\tau^{q_n}\mu_\chi)}{d\mu_\chi} \leq C.$$

By a method close to that of Lemma 3.2, we have the following result in which $\varphi$ is any BV function with values in $\mathbb{R}$.

**Theorem 3.3** *Let $\mu = \mu_0$ (the Lebesgue measure) or $\mu = \mu_\chi$ and $\lambda$ be the corresponding invariant measure on $X \times \mathbb{R}$. For every BV function $\varphi$, if $(\varphi_{q_n})$ does not tend to 0 in probability in the sense of $\mu$, the $\tau_\varphi$-invariant functions (relatively to the measure $\lambda$) are periodic and have a common period.*

Proof Let $f(x,y)$ such that $f(\tau x, y+\varphi(x)) = f(x,y)$, for $\mu-$a.e. $x$. As in the proof of Lemma 3.2, we can assume that $f$ is continuous with respect to $y$. Let $u$ be a continuous nonnegative integrable function on $\mathbb{R}$. As in Lemma 3.2, we have

$$\int\int |f(x,y)-f(\tau^{q_n}x,y)|\ u(y)\,dy\ d\mu(x) \to 0$$

and thus, since $f(\tau^{q_n} x, y) = f(x, y - \varphi_{q_n}(x))$

$$\int\int |f(x,y) - f(x, y - \varphi_{q_n}(x))| \; u(y)\, dy \; d\mu(x) \to 0.$$

From this convergence in $\| \, \|_1$ norm, we deduce the existence of a subsequence $(q_{n_k})$ for which the pointwise convergence for $\mu$-almost every $x$ holds:

$$\int |f(x,y) - f(x, y - \varphi_{q_{n_k}}(x))| \; u(y)\, dy \to 0. \tag{32}$$

This subsequence can be chosen such as $(\varphi_{q_{n_k}})$ does not tend to 0 in probability. We can then can build a sequence of measurable functions $(t_j(x))_{j\geq 1}$ with values in $\mathbb{N}$, tending to $+\infty$, extracted from the sequence $(q_{n_k})$, such that

$$\lim_k \varphi_{t_j(x)}(x) = c(x),$$

where $c$ is a measurable not almost everywhere null nonnegative function.

By passing to the limit, we get

$$\int |f(x,y) - f(x, y - c(x))| \; u(y)\, dy = 0.$$

For a set of values of $x$ of measure $> 0$, the functions $y \to f(x,y)$ are therefore periodic. If they are not constant, using the invariance by $\tau_\varphi$ and the ergodicity of $\tau$, we obtain that, for a.e. $x$, the period does not depend on $x$. If we consider now a countable dense family of $\tau_\varphi$-invariant functions, the construction (i.e the extraction of a subsequence $(q_{n_k})$ in (32)) can be made simultaneously for the functions of this family.

This implies that there is a common period for all invariant functions. $\square$

**Remark** An analogous proof was sketched in [Co80], Proposition 1, but must be modified there: the subsequences should be extracted first from $(q_n)$ or a multiple $(\ell_n q_n)$ so that convergence as in (32) still holds. Using Lemma 3.9 this can be done in the context of [Co80] with a modification of "lemme 3", in order to correct the proof of ergodicity which is given there.

## 3.2 Ergodic measures for $\varphi = \sum_i c_i 1_{I_i} - \beta$

We take now $G = \mathbb{R}$ and use the additive notation. We consider cocycles defined by functions $\varphi$ of the form

$$\varphi(x) = \sum_i c_i 1_{I_i}(x \bmod 1) - \beta, \tag{33}$$

where $(I_i)$ is a finite family of intervals of $[0, 1]$, the coefficients $c_i$ are integers and $\beta$ is such that $\mu(\varphi) = 0$.

**Notations 3.4** Let $\mathcal{V}$ be the set of limit values of the sequence $(\{q_n\beta\})_{n\geq 1}$ and $\mathcal{F}$ be the interval of integers:

$$\mathcal{F} = \{l \in \mathbb{Z} : |\ell| \leq V(\varphi) + 1\}.$$

The cocycle $(\varphi_n)$ can be written: $\varphi_n(x) = u_n(x) - \{n\beta\},\ n \geq 1$, where the functions $u_n$ have integral values. From (13), it follows $|\varphi_{q_n}(x)| \leq V(\varphi)$ and therefore the values of $u_{q_n}(x)$ belong to $\mathcal{F}$.

**Lemma 3.5** *For every $\gamma \in \mathcal{V}$, for every $\tau_\varphi$-invariant ergodic locally finite measure $\lambda$, there exists an integer $u \in \mathcal{F}$ such that $u - \gamma$ satisfies $R_{u-\gamma}\lambda \sim \lambda$.*

Proof Let $\gamma$ be a limit point of the sequence $(\{q_n\beta\}_{n\geq 1})$ and $(t_k)$ be a sequence such that $\gamma_k := \{q_{t_k}\beta\} \to \gamma$.

Let $F$ be a nonnegative continuous function with compact support on $X \times \mathbb{R}$. According to the invariance of the measure $\lambda$, we have, for every $k \geq 0$,

$$\begin{aligned}
\int\int F(x,y)\, d\lambda(x,y) &= \int\int F(x + q_{t_k}\alpha, y + \varphi_{q_{t_k}}(x))\, d\lambda \\
&= \sum_{u\in\mathcal{F}} \int\int_{u_{q_{t_k}}=u} F(x + q_{t_k}\alpha, y + u - \gamma_k)\, d\lambda \\
&\leq \sum_{u\in\mathcal{F}} \int\int F(x + q_{t_k}\alpha, y + u - \gamma_k)\, d\lambda\ ;
\end{aligned}$$

we obtain in the limit

$$\int\int F(x,y)\, d\lambda \ \leq\ \sum_{u\in\mathcal{F}} \int\int F(x, y + u - \gamma)\, d\lambda, \tag{34}$$

which implies that there is, for at least one value of $u \in \mathcal{F}$, a measure $R_{u-\gamma}\lambda$ not singular with respect to the measure $\lambda$. $\square$

**Lemma 3.6** *For every $\gamma \in \mathcal{V}$, there exists $u \in \mathcal{F}$ such that $u - \gamma$ is a quasi-period (for $\mu = \mu_0$ or $\mu = \mu_\chi$).*

Proof With the notations of the preceding lemma, there exists $u \in \mathcal{F}$ such that, for infinitely many indices in the sequence $(t_k)$: $\mu\{x : u_{q_{t_k}}(x) = u\} \geq Card(\mathcal{F})^{-1}$.

Since $q_{t_k}\alpha \bmod 1 \to 0$, the quasi-period property (definition 3.1) is satisfied by $u - \gamma$. $\square$

If $\mathcal{V} \neq \{0, 1\}$, not every $u - \gamma$, for $u$ integer in $\mathcal{F}$ and $\gamma \in \mathcal{V}$, is zero. There is thus at least a real $\neq 0$ of this form satisfying Lemma 3.5 (resp. Lemma 3.6).

• **Ergodic measures when** $\mathrm{Card}(\mathcal{V}) = \infty$

The following theorem extends results of [ANSS02].

**Theorem 3.7** *If the set of limit values of the sequence* $(((q_n\beta))_{n\geq 1})$ *is infinite, the measure* $\lambda_0$*, and more generally the measures* $\lambda_\chi$*, are ergodic. They are the only* $\tau_\varphi$*-invariant ergodic locally finite measures.*

<u>Proof</u> Under the assumption of the theorem there are, according to Lemma 3.6, infinitely many quasi-periods $\in [-(V(\varphi)+1), V(\varphi)+1]$ and Lemma 3.2 implies that the group of periods of the bounded invariant functions, which is a closed subgroup of $\mathbb{R}$, coincides with $\mathbb{R}$. These functions are thus constant and the measures $\lambda_0$ and $\lambda_\chi$ are ergodic.

Let $\lambda$ be a $\tau_\varphi$-invariant ergodic locally finite measures on $X \times \mathbb{R}$. By Lemma 3.5, the closed subgroup $H := \{\gamma : R_\gamma\lambda \sim \lambda\}$ is $\mathbb{R}$. Therefore $R_\gamma\lambda = \chi(\gamma)\lambda, \forall\gamma \in \mathbb{R}$, where $\chi$ is an exponential on $\mathbb{R}$. Since the measure $\lambda$ is locally finite, this implies that $\lambda = \lambda_\chi$, with $\lambda_\chi(dx, dt) = \mu_\chi(dx)\chi(t)dt$ as in (6), with $\mu_\chi$ the unique quasi-invariant probability measure on $X$ satisfying (5). $\square$

**Remarks** 1) The method of Lemma 3.2 also implies that, under the assumptions of the theorem, the proper functions, necessarily bounded, have the form: $g(x)e^{2\pi i\theta y}$ where $\theta$ is a real and $g$ satisfies for some $\gamma$ of modulus 1 the functional equation: $g \circ \tau = \gamma e^{-2\pi i\theta\varphi} g$.

2) The conclusions of Theorem 3.7 hold if $\alpha$ has bpq and $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$. Indeed, if $\alpha$ has bpq and if the sequence $(((q_n\beta)))$ has a finite number of limit points, there exists a non-zero integer $p$ such that the limit points $\gamma_i$ of $(((q_np\beta)))_{n\geq 1}$ satisfy $0 \leq \gamma_i \leq \frac{1}{4}q_n((q_n\alpha))$ for every $n \geq 0$. This implies then $p\beta \in \mathbb{Z}\alpha + \mathbb{Z} \bmod 1$ (Corollary 5.2, section 5, cf. also [KrLi91], Theorem 2).

• **Ergodicity of** $\lambda_0$

The question of ergodicity and the determination of ergodic measures become more difficult if one gives up the assumption of Theorem 3.7. In what follows, we will restrict ourselves to the measure $\lambda_0$ and show the ergodicity of $\lambda_0$ by distinguishing different situations for the set of limit values of $(((q_n\beta))_{n\geq 1})$.

**Proposition 3.8** *If the sequence* $(((q_n\beta))_{n\geq 1})$ *has an irrational value among its limit points, the measure* $\lambda_0$ *is ergodic.*

<u>Proof</u> The cocycle $\varphi_n$ can be written $\varphi_n = u_n - \{n\beta\}$. Let $\gamma \in \mathcal{V}$ be a limit point of $(\{q_n\beta\}) \in ]0, 1[$ and $(t_j)$ a subsequence such that $\lim_j \{q_{t_j}\beta\} = \gamma$. Let $A_{j,k} := \{u_{q_{t_j}} = k\}$. Using the fact that $\int \varphi_n \, dx = 0$, we have

$$\sum_k k\mu(A_{j,k}) = \int_0^1 u_{q_{t_j}}(x)dx = \{q_{t_j}\beta\} \to \gamma. \tag{35}$$

There exists a sequence of indices $\mathcal{S}$, two distinct integers $k_1$ and $k_2$ and $\delta > 0$ such that $\mu(A_{j,k_i}) \geq \delta$, for $j \in \mathcal{S}$, $i = 1, 2$. Indeed, in the opposite case, one could construct an infinite sequence $\mathcal{S}'$ and for every $j \in \mathcal{S}'$ an integer $k_j$ such that $\lim_{j\to\infty, j\in\mathcal{S}'} \mu(A_{j,k_j}) = 1$ and, according to (35), $\gamma$ would be a limit of a sequence of integers.

One obtains thus two quasi-periods of the form $k_i - \gamma$, of which the difference is a non-zero integer. If there is an irrational value in $\mathcal{V}$, the closed subgroup generated by the quasi-periods is equal to $\mathbb{R}$, which implies the assertion. $\square$

*Case where the limit points of $(((q_n\beta)))_{n\geq 1}$ are rational and in finite number*

There exists in this case an integer $\beta'$ multiple of $\beta$ for which $\lim_n((q_n\beta')) = 0$. Replacing $\beta$ by $\beta'$ and the cocycle by an iterate, we are reduced to the situation where $\lim_n((q_n\beta)) = 0$

This case can arise if the partial quotients of $\alpha$ are not bounded, since in this case the set $E(\alpha) = \{\beta : ((q_n\beta)) \to 0\}$ is uncountable (cf. Remark 5.3). Lemma 3.6 is not enough in this case. We will use Corollary 5.2 given in appendix and the following lemma.

**Lemma 3.9** *Let $\varphi$ be a step function with $D$ points of discontinuity. For $\ell$ and $q$ integers $\geq 1$, let*

$$A_{q,\ell} = \bigcap_{1\leq s\leq \ell} \{x : \varphi_q(x) = \varphi_q(x + sq\alpha)\}.$$

*We have $\mu(A_{q,\ell}) > 1 - 2Dq\epsilon$, with $\epsilon = \ell((q\alpha))$.*

Proof Let $\Delta$ be the set of discontinuities of $\varphi$. If $x \notin A_{q,\ell}$, we can find $s$, $1 \leq s < \ell$, and $j$, $0 \leq j < q$, such that $\varphi(x + j\alpha) \neq \varphi(x + j\alpha + sq\alpha)$. This implies that $\varphi$ has a discontinuity $\delta$ on the circle between $x + j\alpha$ and $x + j\alpha + sq\alpha$, and therefore $x$ belongs to the interval $B(\delta - j\alpha, \epsilon)$ centered at $\delta - j\alpha$, of length $2\epsilon$.

According to the inequality $\sup_{1\leq s\leq \ell}((sq\alpha)) \leq \ell((q\alpha))$, the complementary $A^c_{q,\ell}$ is contained in the set (whose measure is less then $2Dq\epsilon$)

$$\bigcup_{0\leq j<q,\, \delta\in\Delta} B(\delta - j\alpha, \epsilon).$$

$\square$

**Proposition 3.10** *: If $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ and $((q_n\beta)) \to 0$, the measure $\lambda_0$ is ergodic.*

Proof We use the notations of Lemma 3.9. For $x \in A_{q,\ell}$, we have $\varphi_{\ell q} = \ell\varphi_q$. Let $(L_k)$ be the sequence of integers defined by the condition

$$L_k = [\eta/((q_{n_k}\beta))],\ k \geq 1,$$

where $\eta$ is such that $\eta < \frac{1}{16D}$, the sequence $(n_k)$ being chosen so that

$$((q_{n_k}\beta)) > \frac{1}{4} q_{n_k}((q_{n_k}\alpha)).$$

The existence of such a sequence follows from Corollary 5.2.

Denote by $B_k$ the sets $A_{q_{n_k},L_k}$. By Lemma 3.9, we have

$$\mu(B_k) > 1 - 2Dq_{n_k}L_k((q_{n_k}\alpha)) \geq 1 - 8D\eta \geq \frac{1}{2}.$$

For $x \in B_k$ and $\ell \leq L_k$, we have

$$\varphi_{\ell q_{n_k}}(x) = \ell\varphi_{q_{n_k}}(x) = \ell u_{q_{n_k}}(x) - \ell((q_{n_k}\beta)).$$

On the other hand, for $\ell \leq L_k$, we have

$$\begin{aligned} ((\ell q_{n_k}\alpha)) &\leq L_k((q_{n_k}\alpha)) \\ &\leq \eta\frac{((q_{n_k}\alpha))}{((q_{n_k}\beta))} \leq 4\frac{\eta}{q_{n_k}} \to 0. \end{aligned}$$

If $\mu(\{x : u_{q_n}(x) = 0\}) \to 1$, this shows that any value between 0 and $\eta$ is a quasi-period of the cocycle, and therefore is a period of the $\tau_\varphi$-invariant functions. In that case, the invariant functions are constant and the measure $\lambda_0$ is ergodic for $\tau_\varphi$.

In the opposite case, it is clear that we have found a quasi-period which is a non-zero integer $c$. If there is a non-constant invariant function, Lemma 3.11 below implies the existence of an integer $p$ and of a measurable function $g$ of modulus 1 satisfying the equation $g \circ \tau = e^{2\pi i\frac{p}{c}\varphi}g$.

Raising both sides of this equation to the power $c$, we get, according to the form of $\varphi$: $(g \circ \tau)^c = e^{2\pi i p\varphi}g^c = e^{-2\pi i p\beta}g^c$. Therefore $p\beta$ is a proper value of the rotation $\tau_\alpha$, which implies $\beta \in \mathbb{Q}\alpha + \mathbb{Q}$, contrary to the assumption. $\square$

**Lemma 3.11** *Let $(\varphi, \tau)$ be a $\mathbb{R}$-cocycle. If $\tau_\varphi$ is conservative, there exists a $\tau_\varphi$-invariant periodic non-constant function $f$, with period $c > 0$ ($f(x, y + c) = f(x, y)$, for $\lambda_0$-a.e. $(x, y)$), if and only if there are an integer $p \neq 0$ and a measurable function $g$ with modulus 1 such that*

$$g(\tau x) = e^{2\pi i\frac{p}{c}\varphi(x)}g(x), \ \ \mu\text{-a.e.}. \tag{36}$$

<u>Proof</u> As $\tau_\varphi$ is conservative, we can assume $f$ bounded. The Fourier series in $y$ of the periodic function $f$ has the form: $f(x, y) = \sum_{p\in\mathbb{Z}} u_p(x)e^{-2\pi i\frac{p}{c}y}$, where the measurable functions $u_p$ satisfy an equation of the form (36).

If $f$ is not constant, we get for some $p \neq 0$, the existence of a solution of modulus 1 for the equation (36) and $\varphi$ is cohomologous with a function with values in $\frac{c}{p}\mathbb{Z}$.

Conversely, if $g$ is solution of (36), the function $g(x)e^{-2\pi i\frac{p}{c}y}$ is invariant and $c$-periodic. $\square$

From the two preceding propositions it follows:

**Theorem 3.12** *For every irrational $\alpha$, if $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$, the measure $\lambda_0 = \mu \times dy$ on $\mathbb{T}^1 \times \mathbb{R}$ is ergodic for the cylindrical transformation $\tau_\varphi$, where $\varphi$ is given by (33).*

## 3.3 Extension to $\mathbb{R}^d$, $d > 1$

We consider now a stationary walk over a rotation $x \to x+\alpha \bmod 1$, defined by a function $\Phi$ with values in $\mathbb{R}^d$.

If the components of $\Phi$ are BV and of null integral, the Denjoy-Koksma inequality ensures recurrence, for any dimension $d$. To obtain ergodicity, the methods presented above can be used to get a statement like Theorem 3.13 below. Of course it would be necessary to make a finer arithmetic study to have results similar to those of dimension 1 given in the preceding theorems.

Let us consider $\Phi = (\varphi_j)$ from $\mathbb{R}/\mathbb{Z}$ into $\mathbb{R}^d$ with components $\varphi_j = \sum_i c_{i,j} 1_{I_{i,j}} - \beta_j$, where the coefficients $c_{i,j}$ are integers and $\beta_j = \sum_j c_{i,j}$ length$(I_{i,j})$.

**Theorem 3.13** *For almost every choice of $(\beta_j) \in [0,1]^d$, the measures $\lambda_0$, $\lambda_\chi$ (where $\chi$ is an exponential on $\mathbb{R}^d$) are ergodic for the transformation $\tau_\Phi$ defined in $\mathbb{T} \times \mathbb{R}^d$ over the rotation by an irrational $\alpha$ by the cocycle $\Phi$. There are no other locally finite $\tau_\Phi$-invariant ergodic measures.*

Proof For almost every choice of $(\beta_j, j = 1, ..., d) \in [0,1]^d$, the sequence $((q_n\beta_1, ..., q_n\beta_d) \bmod 1)_{n\geq 1}$ is equidistributed in $\mathbb{R}^d/\mathbb{Z}^d$. One then can apply the method of proof of Theorem 3.7. $\square$

**Rotations in higher dimension**

Another generalization consists in replacing at the basis of the skew product the rotation by a rotation of higher dimension. For questions related to multidimensional rotations let us mention in particular papers by Fráczek ([Fr00]), Rittaud ([Ri00]).

If the function $\varphi$ has its values in $\mathbb{R}$, one still has the property of recurrence, but the absence of the Denjoy-Koksma inequality (cf. Yoccoz ([Yo80])) makes the question of ergodicity much more difficult.

In this direction, one of the simplest example is

$$\tau_\phi(x_1, x_2, y) = (x_1 + \alpha_1, x_2 + \alpha_2, y + 1_{[0,\frac{1}{2}[}(x_1) 1_{[0,\frac{1}{2}[}(x_2) - \frac{1}{4}).$$

For a product of cylindrical transformations of the form $(x + \alpha_i, y + \varphi_i(x))$, by applying Lemma 2.4, one obtains easily the recurrence if the $\alpha_i$'s are of type 1 and the functions $\varphi_i$ are BV and of null integral.

About the recurrence of $\mathbb{R}^d$-valued cocycles over a 2-dimensional rotation, see also [ChCo08].

# 4 Examples of non-regular cocycles

## 4.1 A non-regular BV cocycle

Let us still consider a step function on the circle $\varphi = \sum_i c_i 1_{I_i} - \beta$, but with $\beta = u/v \in \mathbb{Q}$. If the $c_i$ are integers and $\beta \neq 0$, it can be shown that the cocycle is regular for the Lebesgue measure $\mu_0$ or the measures $\mu_\chi$. Indeed a common period for all $\tau_\varphi$-invariant functions can be found as in Lemma 3.6.

The situation is different if $\beta = 0$. We are going to show that in that case the cocycle, which has its values in $\mathbb{Z}$, can be non-regular and therefore non-ergodic, but with the group of periods reduced to $\{0\}$. More precisely, if $\alpha$ is not with bpq, one can find $\beta$ and $r$ such that for $\varphi = 1_{[0,\beta]} - 1_{[0,\beta]}(. + r)$ , one has $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$. Here the essential values are defined with respect to the Lebesgue measure $\mu_0$ on the circle. This implies that the transformation $\tau_\varphi$ is not ergodic for the product measure $\mu_0 \times dy$, but without periodic invariant functions. The ergodic decomposition of $\mu_0 \times dy$ is given by a family of $\sigma$-finite measures based on $\sigma$-finite invariant measures for the rotation on the circle.

For the construction we consider the function defined on the circle by $\varphi_{\beta,s} := e^{2\pi i s 1_{[0,\beta]}}$.

K. Merril ([Me85]) extended results of Veech and Stewart by giving a necessary condition and a sufficient condition for the function $\varphi_{\beta,s}$ to be a quasi-coboundary. Recently M. Guenais and F. Parreau obtained necessary and sufficient conditions for it ([GuPa06]). For the construction that we have in mind, it will be enough to use the following statement:

**Theorem 4.1** *([Me85]) 1) Let $\beta$ be a real of the form $\beta = \sum_1^\infty b_n q_n \alpha \bmod 1$, with $b_n \in \mathbb{Z}$. If*

$$\sum_1^\infty |b_n| q_n ((q_n \alpha)) < \infty, \ \sum ((b_n s)) < \infty,$$

*then $\varphi_{\beta,s} := e^{2\pi i s 1_{[0,\beta]}}$ is a multiplicative quasi-coboundary.*

*2) Conversely, if $\varphi_{\beta,s}$ is a multiplicative quasi-coboundary, then $\beta$ can be written: $\beta = \sum_1^\infty b_n q_n \alpha \bmod 1$, with $b_n \in \mathbb{Z}$, $\lim_n b_n q_n((q_n\alpha)) = 0$ and $\lim_n ((b_n s)) = 0$.*

*3) In particular, if $\alpha$ has bpq and $\beta \notin \mathbb{Z}\alpha$, $\varphi_{\beta,s}$ is not a multiplicative quasi-coboundary, for every $s \neq 0$.*

For $r \in \mathbb{R}$, we denote by $\rho_r$ the translation $x \to x + r \bmod 1$, by $\mathcal{U}(X)$ the group of measurable functions from $X$ to the group $\mathcal{U}$ of complex numbers of modulus 1.

**Theorem 4.2** *Let $\tau$ be the rotation by $\alpha$ on the circle. If $\alpha$ is not with bpq, there exist $\beta$ and $r$ such that $\varphi = 1_{[0,\beta]} - 1_{[0,\beta]} \circ \rho_r$ defines a non-regular cocycle $(\varphi, \tau)$ in $\mathbb{Z}$.*

Proof By Theorem 4.1, if $\alpha$ is not with bpq, there is an uncountable set of values of $\beta$ such that, for an uncountable set of $s$, $\varphi_{\beta,s}$ is a multiplicative coboundary. There are thus $\beta \notin \alpha\mathbb{Z} + \mathbb{Z}$, $s \notin \mathbb{Q}$, a number $\gamma \in \mathcal{U}$ and $\psi \in \mathcal{U}(X)$ such that

$$e^{2\pi i s 1_{[0,\beta]}} = \gamma \frac{\psi \circ \tau}{\psi}.$$

For this choice of $\beta$ and of $s$, $e^{2\pi i s(1_{[0,\beta]}-1_{[0,\beta]}\circ\rho_r)}$ is a multiplicative coboundary for every $r$. On the other hand, if the function $1_{[0,\beta]} - 1_{[0,\beta]} \circ \rho_r$ is an additive coboundary for every $r$, then by Proposition 4.3 below this implies that $1_{[0,\beta]} - \beta$ is an additive coboundary up to some additive constant $c$ (and necessarily $c = 0$, since the integral of $1_{[0,\beta]} - \beta$ is zero). But this would imply that $e^{2\pi i\beta}$ is an eigenvalue of the rotation by $\alpha$.

Therefore there is $r \in \mathbb{R}$ such that $1_{[0,\beta]} - 1_{[0,\beta]} \circ \rho_r$ is not an additive coboundary. For such a value of $r$, Lemma 1.4 shows that the function $\varphi = 1_{[0,\beta]} - 1_{[0,\beta]} \circ \rho_r$ satisfies: $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$. □

**Proposition 4.3** *Let $\varphi$ be a real-valued measurable function on a compact abelian group $K$. Let $\tau$ be an ergodic rotation on $K$. If for every $g \in K$ there exists a measurable function $\psi_g$ such that*

$$\varphi - g\varphi = \psi_g \circ \tau - \psi_g, \tag{37}$$

*then $\varphi$ is an additive quasi-coboundary: $\varphi = c + h \circ \tau - h$, for a measurable function $h$ and a constant $c$.*

Proof From (37) we have, for every $s \in \mathbb{R}$ :

$$e^{2\pi i s\varphi(x)}/e^{2\pi i s\varphi(g.x)} = e^{2\pi i s\psi_g(\tau x)}/e^{2\pi i s\psi_g(x)}. \tag{38}$$

According to Proposition 3 in [Le93], for every $s$ there exist $\zeta_s$ in $\mathcal{U}(X)$ and $\lambda_s$ in $\mathcal{U}$ such that:

$$e^{2\pi i s\varphi} = \lambda_s \ \tau\zeta_s/\zeta_s. \tag{39}$$

By a theorem of [MoSc80], (39) implies the result. □

**Remarks 4.4** 1) It remains the question of the existence of a BV function such that, for a rotation by $\alpha$ with bpq, one has $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$. In the following subsection a construction of a non-regular cocycle is given, without assumption on $\alpha$.

2) Remark that the following corollary (where $\gamma_r(x)$ is not assumed to be measurable with respect to the two variables $(r, x)$), follows from the previous proposition:

**Corollary 4.5** *: Let $(\gamma_r)_{r\in\mathbb{R}}$ be a cocycle for the action of the group of the circle by translation onto itself (i.e for each $r \in \mathbb{R}$, $\gamma_r$ is 1-periodic measurable and $\gamma_{r+r'} = \gamma_r + \gamma_{r'}(. + r)$). Then there are a measurable function $u$ and constants $(c_r, r \in \mathbb{R})$ such that $\gamma_r(.) = c_r + u(.) - u(. + r)$.*

*Proof*: Let $\alpha \notin \mathbb{Q}$. We have for every $r \in \mathbb{R}$: $\gamma_\alpha(. + r) - \gamma_\alpha(.) = \gamma_r(. + \alpha) - \gamma_r(.)$. We conclude by Proposition 4.3. □

## 4.2 Construction of a non-regular cocycle, for any irrational $\alpha$

Let $\theta_n$ be defined by

$$\theta_n(x) = a_n\, 1_{[0,\frac{1}{2}]}(r_n x \bmod 1). \tag{40}$$

The sequences $(a_n)$ and $(r_n)$ are chosen in the following manner:

- The sequence $(a_n)$ is a sequence of integers $\geq 1$ (for instance $a_n = 3^n$) such that $a_n$ divides $a_{n+1}$ and

$$\sum_1^{n-1} a_k \leq \frac{1}{2} a_n. \tag{41}$$

- For $(r_n)$ we take a subsequence $(r_n) = (q_{k_n})$ of the sequence of denominators of $\alpha$, where $(k_n)$ is such that, for all $n \geq 1$,

$$q_{k_n} \geq n a_n, \tag{42}$$

$$\sqrt{q_{k_{n+1}}} \geq 2 a_n q_{k_n+1}, \tag{43}$$

According to (10), the construction is possible for any irrational $\alpha$. We have then

$$a_{n+1}((r_{n+1}\alpha))^{\frac{1}{2}} < \frac{1}{2} q_{k_n+1}^{-1}, \forall n \geq 1. \tag{44}$$

According to the invariance of the Lebesgue measure by the transformations $x \to r_k x \bmod 1$, for $r_k \in \mathbb{N}^*$, we have

$$\|\theta_n - \theta_n(. + \alpha)\|_2 = \sqrt{2}\, a_n\, ((r_n\alpha))^{\frac{1}{2}}.$$

Let $\varphi$ be defined by

$$\varphi(x) = \sum_1^\infty [\theta_n(x) - \theta_n(x + \alpha)]. \tag{45}$$

Condition (44) implies that $\varphi \in L^2$ and

$$\sum_{p \geq n+1} \|\theta_p - \theta_p(. + \alpha)\|_2 = \sqrt{2} \sum_{p \geq n+1} a_p ((r_p\alpha))^{\frac{1}{2}} < q_{k_n+1}^{-1}. \tag{46}$$

We have according to (41)

$$\|S_j(\sum_{k=0}^{n-1}(\theta_k - \theta_k \circ \tau))\|_\infty = \|\sum_{k=0}^{n-1}(\theta_k - \theta_k \circ \tau^j)\|_\infty \leq \sum_{k=0}^{n-1} a_k \leq \frac{1}{2} a_n. \tag{47}$$

**Lemma 4.6** *: $\varphi$ is not a coboundary.*

Proof Suppose that $\varphi$ is a coboundary. There exists then a real number $M$ and for every $j$ a set $B_j$ of measure $> \frac{1}{2}$ such that $|S_j(\varphi)| \leq M$ on $B_j$.

On $B_j$ we have according to (47): $|S_j(\varphi - \sum_0^{n-1}(\theta_k - \theta_k \circ \tau))| \leq M + \frac{1}{2}a_n$.
We have also:

$$\begin{aligned}\|S_j(\theta_n - \theta_n \circ \tau)\|_2^2 &= \int |\theta_n(r_n x) - \theta_n(r_n x + r_n j\alpha)|^2 \, d\mu_0 \\ &= \int |\theta_n(x) - \theta_n(x + r_n j\alpha)|^2 \, d\mu = 2((j r_n \alpha)),\end{aligned}$$

Therefore, for $j = j_n := [\frac{1}{2}q_{k_n+1}]$, we get, according to (10)

$$\|S_{j_n}(\theta_n - \theta_n \circ \tau)\|_2^2 = 2a_n^2\,((j_n q_{k_n}\alpha)) = 2a_n^2[\frac{1}{2}q_{k_n+1}]q_{k_n+1}^{-1} \simeq a_n^2.$$

Hence:

$$\int_{B_{j_n}} |S_{j_n}(\theta_n - \theta_n \circ \tau)|^2 d\mu_0 \geq \int_0^1 |S_{j_n}(\theta_n - \theta_n \circ \tau)|^2 d\mu_0 - \mu(B_n^c)a_n^2 \geq \frac{1}{2}a_n^2. \tag{48}$$

For the rest $R_n := \sum_{p=n+1}^{\infty}(\theta_p - \theta_p \circ \tau)$, (46) implies:

$$\begin{aligned}\|S_{j_n}(R_n)\|_2 &\leq j_n(\sum_{p=n+1}^{\infty} \|\theta_p - \theta_p \circ \tau)\|_2 \\ &\leq j_n \sum_{p=n+1}^{\infty} a_p((r_p\alpha))^{\frac{1}{2}} \leq 1.\end{aligned}$$

Finally, according to (48) and to the previous inequality, we get for every $n \geq 1$:

$$\begin{aligned}\frac{1}{2}(M + \frac{1}{2}a_n) &\geq (\int_{B_{j_n}} |S_{j_n}(\varphi - \sum_0^{n-1}(\theta_k - \theta_k \circ \tau))|^2 d\mu)^{\frac{1}{2}} \\ &= (\int_{B_{j_n}} |S_{j_n}(\theta_n - \theta_n \circ \tau) + S_{j_n}(R_n)|^2 d\mu)^{\frac{1}{2}} \\ &\geq \frac{1}{\sqrt{2}}a_n - \|S_{j_n}(R_n)\|_2 \geq \frac{1}{\sqrt{2}}a_n - 1,\end{aligned}$$

which is impossible. □

**Proposition 4.7** *The function $\varphi$ given by (45) satisfies $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$ and defines a non-regular cocycle.*

Proof If $\overline{\mathcal{E}}(\varphi) = \{0\}$, then (cf. [Sc77]) $\varphi$ is a coboundary contrary to the preceding lemma. It suffices to show that $\varphi$, which has integer values, has no integer $\neq 0$ as essential value.

The set $\overline{\mathcal{E}}(\varphi)$ coincides with the set essential values of the cocycle generated by the function $\varphi - \sum_1^n [\theta_k - \theta_k \circ \tau]$ which differs from $\varphi$ by a coboundary.

This function is equal to $\sum_{n+1}^\infty [\theta_k - \theta_k \circ \tau]$ and the values of the corresponding cocycle are 0 or multiple of $3^{n+1}$. Therefore, the integers $\ell$ such that $0 < |\ell| < 3^{n+1}$ are not essential values of $\varphi$. As $n$ is arbitrary, $\mathcal{E}(\varphi) = \{0\}$. $\square$

# 5 Appendix : A diophantine property for $(\alpha, \beta)$

We prove here a diophantine property for $(\alpha, \beta)$ used in Proposition 3.10. In what follows, $(q_n)$ is the sequence of denominators of a given irrational $\alpha$. For a real $\beta$ we study the sequence $(((q_n\beta))_{n\geq 1})$. (cf. [Co80], [KrLi91]).

Let $(M_n)$ be a sequence (defined for $n \geq n_0$, for some $n_0 \geq 1$). For $\ell \geq n_0$, let $\Lambda_\ell$ be the set of real numbers $\beta \in [0, 1]$ such that:

$$\Lambda_\ell = \{\beta : ((q_n\beta)) \leq M_n((q_n\alpha)), \forall n \geq \ell\}. \tag{49}$$

We call interval of rank $n$, for $k = 0, ..., q_n - 1$, an interval $I_n^k$ of the form

$$I_n^k = [\frac{k}{q_n} - \frac{M_n}{q_n}((q_n\alpha)), \frac{k}{q_n} + \frac{M_n}{q_n}((q_n\alpha))].$$

We assume that the sequence $(M_n)$ is increasing and such that

$$M_n((q_n\alpha)) < \frac{1}{2}, \forall n \geq n_0. \tag{50}$$

This implies that, for $n \geq n_0$, the intervals $I_n^k$, $k = 0, ..., q_n - 1$, are pairwise disjoint. For every $\beta \in \Lambda_{n_0}$, there is a sequence $(k_n(\beta))_{n\geq n_0}$ such that:

$$\{\beta\} = \cap_{n\geq n_0} I_n^{k_n(\beta)}.$$

If the sequence $(M_n)_{n\geq n_0}$ satisfies

$$2M_{n+1}\frac{((q_{n+1}\alpha))}{q_{n+1}} + 2M_n\frac{((q_n\alpha))}{q_n} < \frac{1}{q_{n+1}}, \ \forall n \geq n_0, \tag{51}$$

an interval of rank $n$, for $n \geq n_0$, intersects at most one interval of rank $n + 1$.

**Lemma 5.1** *: Let $(M_n)$ be an increasing sequence satisfying (50) and (51). If $\beta$ is such that $((q_n\beta)) \leq M_n((q_n\alpha))$ for $n \geq n_0$, for some $n_0$, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$.*

Proof We are going to show that, for every $k$ such that $I^k_{n_0+1}$ intersects an interval of order $n_0$, the set $\Lambda_{n_0+1} \cap I^k_{n_0+1}$ contains an element of the form $r\alpha - s$, for integers $r$ and $s$. As this set contains at most one point, it is reduced to this element.

This will imply the result since we have for every $\beta \in \Lambda_{n_0}$:

$$\{\beta\} = \Lambda_{n_0+1} \cap I^{k_{n_0+1}(\beta)}_{n_0+1}$$

and $\beta$ belongs to the intersection $I^{k_{n_0+1}(\beta)}_{n_0+1} \bigcap I^{k_{n_0}(\beta)}_{n_0}$.

Let $I^k_{n_0+1}$, $0 \leq k < q_{n_0+1}$, and $I^{k'}_{n_0}$, $0 \leq k' < q_{n_0}$, be two intervals of order respectively $n_0+1$ and $n_0$. Let $\lambda = k'((q_{n_0+1}\alpha)) + k((q_{n_0}\alpha))$.

(8) and (7) imply $0 < \lambda < 1$ and $\lambda$ is of the form $r\alpha - s$, wher $r$ and $s$ are the integers:

$$\begin{aligned} r &= (-1)^{n_0}(k'q_{n_0+1} - kq_{n_0}), \\ s &= (-1)^{n_0}(k'p_{n_0+1} - kp_{n_0}). \end{aligned}$$

Let us show that if $I^k_{n_0+1}$ and $I^{k'}_{n_0}$ intersect, then $\lambda$ belongs to $\Lambda_{n_0+1} \cap I^k_{n_0+1}$. We have:

$$|\frac{k'}{q_{n_0}} - \frac{k}{q_{n_0+1}}| \leq M_{n_0}\tfrac{((q_{n_0}\alpha))}{q_{n_0}} + M_{n_0+1}\tfrac{((q_{n_0+1}\alpha))}{q_{n_0+1}};$$

hence:

$$|k'q_{n_0+1} - kq_{n_0}| = |r| \leq M_{n_0}q_{n_0+1}((q_{n_0}\alpha)) + M_{n_0+1}q_{n_0}((q_{n_0+1}\alpha)).$$

According to (8), the right hand term is the barycenter of $M_{n_0}$ and $M_{n_0+1}$. As $M_{n_0+1} \geq M_{n_0}$, this implies: $|r| < M_{n_0+1}$.

We have therefore, for $n \geq n_0+1$, since $(M_n)$ is an increasing sequence,

$$((q_n\lambda)) = ((q_n r\alpha)) \leq |r|((q_n\alpha)) \leq M_{n_0+1}((q_n\alpha)) \leq M_n((q_n\alpha)),$$

which implies $\lambda \in \Lambda_{n_0+1}$.

Let us show that $\lambda \in I^k_{n_0+1}$ interval of order $n_0+1$. Still using (8), we have:

$$\begin{aligned} |\frac{k}{q_{n_0+1}} - \lambda| &= |\frac{k}{q_{n_0+1}}(1 - q_{n_0+1}((q_{n_0}\alpha))) - k'((q_{n_0+1}\alpha))| \\ &= |(kq_{n_0} - k'q_{n_0+1})|\frac{((q_{n_0+1}\alpha))}{q_{n_0+1}} \leq M_{n_0+1}\frac{((q_{n_0+1}\alpha))}{q_{n_0+1}}, \end{aligned}$$

which implies: $\lambda \in I^k_{n_0+1}$. $\square$

By taking $M_n = \frac{1}{4}q_n$, we recover a result of C. Kraaikamp and P. Liardet [KrLi91]:

**Corollary 5.2** : *If there exists $n_0$ such that $((q_n\beta)) \leq \frac{1}{4}q_n((q_n\alpha))$, for $n \geq n_0$, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ mod 1.*

Proof The sequence $(M_n)$ defined by $M_n = \frac{1}{4}q_n$, $n \geq n_0$, is increasing, satisfies (50) and the condition (51) since, according to (9), we have for every $n \geq n_0$:

$$M_{n+1}\,\frac{((q_{n+1}\alpha))}{q_{n+1}} + M_n\,\frac{((q_n\alpha))}{q_n} = \frac{1}{4}[((q_{n_0}\alpha)) + ((q_{n_0+1}\alpha))] < \frac{1}{2q_{n+1}}.$$

Therefore we can apply Lemma 5.1. □

**Remarks 5.3** 1) If we take for $(M_n)$ a constant sequence, $M_n = (q_{n_0} - 1)/2$, $n \geq n_0$, then (8) implies (51). The previous argument becomes simpler: at rank $n_0$ there is, at most, $q_{n_0}$ intervals of rank $n_0$ providing a non-empty intersection. As the elements of the form $p\alpha$ mod 1, for $|p| < (q_{n_0} - 1)/2$ belong to the set $\Lambda_{n_0}$, they are exactly all elements of $\Lambda_{n_0}$. It shows the following result (cf. [Co80]), weaker than the corollary, since $M_n$ here is constant, but slightly more precise on the form of $\beta$:

*Let $n_0$ be such that $q_{n_0}$ is odd. If $((q_n\beta)) \leq \frac{q_{n_0}-1}{2}((q_n\alpha))$, for $n \geq n_0$ then $\beta = p\alpha$ mod 1, with $|p| \leq \frac{q_{n_0}-1}{2}$.*

2) When $\alpha$ is pbq, Corollary 5.2 gives a result of G. Larcher ([La88]): *Let $\alpha$ be a real number with bpq. If $\beta \in \mathbb{R}$ is such that $\lim_n((q_n\beta)) = 0$, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$.*

On the other hand, if $\alpha$ is not with bpq, then the set $E(\alpha) = \{\beta : \lim_n((q_n\beta)) = 0\}$ is an uncountable additive subgroup of $\mathbb{R}$ (cf. [KrLi91]).

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Jean-Pierre Conze
e-mail: conze@univ-rennes1.fr