Spectral Theory of Dynamical Systems and Related Topics

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Role of r-mixing in martingale methods for limit theorems in algebraic N^d-actions

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For N^d-actions, a martingale method has been applied recently by D. Volný to ergodic sums over rectangles. For more general sets, we will discuss the needed estimates in order to use this method and their link with r-mixing and solutions of "S-unit equations" in the case of some actions by algebraic endomorphisms.

Introduction

In the study of the stochastic behavior of multidimensional models, there are difficulties that do not occur in the one-dimensional models obtained by iteration of a single transformation, for instance the lack of mixing of all orders which can occur for 2-mixing actions.

To make the problem precise, it is suitable to choose an explicit model and to see what methods are available for these multidimensional models.

An example is the model provided in the algebraic framework by the action of automorphisms or endomorphisms of compact abelian groups.

For connected groups mixing of all orders holds. In this talk, we consider also a family of non connected groups for which we have to deal with non mixing configurations due to the existence of solutions of "S-unit type" equations.

Examples:
Tori: connected case, where there is mixing of all order,
Shift-invariant subgroups: non connected case, non mixing of all order, hence a more difficult case.

General question: Let us consider:
- a compact abelian group G endowed with its Haar measure μ,
- $T_1, ..., T_d$, d commuting algebraic automorphisms or surjective endomorphisms,
- the action of $\mathbb{Z}^d$ or $\mathbb{N}^d$ on G by $T^{\underline{\ell}} := T_1^{\ell_1} ... T_d^{\ell_d}$ (underlined letters represent vectors).

This action is assumed to be totally ergodic (that is: ergodic for every $\underline{\ell} \in \mathbb{Z}^d \backslash \{\underline{0}\}$.

We denote by $\widehat{G}$ the dual group of characters on G, by χ_0 the trivial character.

Total ergodicity (here) is equivalent to: $T^{\underline{\ell}}\chi \neq \chi$ for $\underline{\ell} \neq \underline{0}$ and any character $\chi \neq \chi_0$, to the Lebesgue spectrum property, to 2-mixing).

A function f on G is called "regular" if it has an absolutely convergent Fourier series. If its integral is 0, this implies that f has (for the $(T^{\underline{\ell}})$ action) a spectral density φ_f which is continuous on $\mathbb{T}^d$. Recall that the Fourier coefficients of φ_f (on $\mathbb{T}^d$) satisfy

$$\int_{\mathbb{T}^d} e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle} \varphi_f(\underline{t}) \, d\underline{t} = \langle T^{\underline{\ell}} f, f \rangle, \; \forall \underline{\ell}.$$

If f is function , $T^{\underline{\ell}} f$ stands for $f \circ T^{\underline{\ell}}$.

In the talk, the distribution is taken with respect to the Haar measure μ on G. For instance, if we act by $\times 2$, $\times 3$ mod 1 on the circle, the distribution is taken with respect to the Lebesgue measure. All functions are assumed to be centered ($\mu(f) = 0$).

If $f : G \to \mathbb{R}$ is regular on G, one can investigate the statistical behavior of the random field $(T^{\underline{\ell}} f)_{\underline{\ell} \in \mathbb{N}^d}$, in particular the following limits (in distribution w.r. to μ):

a) (ergodic sums) for a sequence (D_n) of sets in $\mathbb{N}^d$

$$\lim_n |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} T^{\underline{\ell}} f$$

b) (ergodic sums along a random walk) if $Z_n = Y_0 + ... + Y_{n-1}$ is a r.w. on $\mathbb{Z}^d$ or $\mathbb{N}^d$,

$$\lim_n a_n^{-1} \sum_{0 \leq k < n} T^{Z_k(\omega)} f, \text{ for a.e fixed } \omega,$$

where (a_n) is a normalizing sequence (which may depend on ω).

The case when G is a connected is easier, because mixing of all orders holds. For non connected groups. in particular shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ (characteristic p, where $p \geq 2$ is a prime integer) and some commutative actions by endomorphisms or automorphisms on such groups, mixing of all orders is not satisfied.

Nevertheless, it is possible to show that, for the model that we consider, non-mixing configurations are sparse in some sense. This allows us to apply the cumulant method, as we did for the connected case, to prove limit theorems.

Is it possible to use a martingale method? Yes, as we will see, in the case of exact endomorphisms. For a summation method like ergodic sums on sets which are not rectangles, the method could required results on S-units linked to r-mixing. We will discuss this point.

Furthermore, it requires the reduction trick (Gordin's method) which does not work for all summation methods.

1. Martingale method

A toy model: action by $Sx = 2x \mod 1$, $Tx = 3x \mod 1$ on $\mathbb{T}^1$

Let $(X, \mathcal{A}, \mu)$ be the circle with its Borel σ-algebra and the Lebesgue measure μ. The maps T, S are two commuting measure preserving transformations of $(X, \mathcal{A}, \mu)$.

Let (R_n) be a sequence of rectangles in $\mathbb{N}^2$ with length of sides tending to ∞ and f a regular function on the circle, for instance a trigonometric polynomial such that $\mu(f) = 0$ and $\|f\|_2 = 1$. Suppose we want to prove a CLT for

$$|R_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in R_n} f(3^{\ell_1} 2^{\ell_2} x).$$

To simplify, let us take squares $R_n = [0, n-1] \times [0, n-1]$. We consider $\sum_{0 \leq j,i < n} f(3^i 2^j x)$.

First it is possible to reduce f (up to a coboundary) to the case when it has no frequencies multiple of 2 or 3, which implies $\int f(.)\, g(2.)\, d\mu = 0$, $\int f(.)\, g(3.)\, d\mu = 0$, for every $g \in L^2(\mu)$.

We suppose the reduction is made. Hence, f satisfies the double (reverse) martingale difference condition:

$$\mathbb{E}(f|T^{-1}\mathcal{A}) = 0, \ \mathbb{E}(f|S^{-1}\mathcal{A}) = 0. \tag{1}$$

For simplicity let us assume moreover that f is a real trigonometric polynomial $f(x) = \sum_\ell c_\ell e^{2\pi i \ell x}$, with $c_0 = 0$ and $\|f\|_2 = 1$. Let us prove the CLT, which reads here:

$$n^{-1} \sum_{0 \leq j, i < n} f(3^i 2^j x) \ \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0,1). \tag{2}$$

The proof is based on McLeish theorem recalled below. To check the hypotheses of McLeish theorem, we will give two proofs:
- one of the proofs follows a recent paper of Dalibor Volný,
- the second one is based on a computation related to mixing.

The advantage of the first proof is is generality (it can be applied to totally commuting random fields), but it is adapted to the case of Birkhoff sums over rectangles.The advantage of the second one is that it applies to more general methods of summations, in particular Birkhoff sums over domains which are not rectangles.

Recall the central limit theorem of McLeish for arrays of martingale differences:

Theorem 1. *(McLeish) If* $(X_{n,i},\, i = 1, ..., k_n)$ *is an array of martingale (or reverse martingale) differences such that* $\sup_n \max_{i=1}^{k_n} \mathbb{E}X_{n,i}^2 < \infty$ *and*
(i) $\max_{i=1}^{k_n} |X_{n,i}| \to 0$ *in probability,*
(ii) $\sum_{i=1}^{k_n} X_{n,i}^2 \to 1$ *in probability,*
then $\sum_{i=1}^{k_n} X_{n,i}$ *converges to* $\mathcal{N}(0,1)$ *in distribution.*

We apply Theorem 1 with: $X_{n,i} = T^i[\frac{1}{n} \sum_{j=0}^{n-1} S^j f], i = 0, ..., n-1$, which is an array of reverse martingale differences. Let us check conditions (*i*) and (*ii*). We will focus on (*ii*) and give two proofs.

1) *First proof* (D. Volnỳ): We have to prove (ii) for $X_{n,i}$. Actually we will show L_1-convergence, that is:

$$\|\frac{1}{n} \sum_{i=0}^{n-1} T^i \,(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} S^j f)^2 - 1\|_1 \to 0. \tag{3}$$

It is enough to show: given $\varepsilon > 0$, $\exists\ m$ and L such that, for $n \geq L$,

$$\|\frac{1}{m}\sum_{i=0}^{m-1} T^i \,(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} S^j f)^2 - 1\|_1 \leq \varepsilon. \tag{4}$$

Let us fix a positive integer m and for constants $a_1, ..., a_m$ consider $(\sum_{i=1}^{m} a_i T^i S^j f)$, $j \geq 1$. They are reverse martingale differences by (1) and by the central limit theorem of Billingsley and Ibragimov,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\sum_{i=1}^{m} a_i T^i S^j f \ \ \underset{n\to\infty}{\overset{distrib}{\longrightarrow}}\ \ \mathcal{N}(0, \sum_{i=1}^{m} a_i^2).$$

Let $F_{i,n} := \frac{1}{\sqrt{n}}\sum_{j=1}^{n} T^i S^j f$. From this it follows that the sequence of random vectors $(F_{1,n}, ..., F_{m,n})$ converges in distribution to a vector $(W_1, ..., W_m)$ of mutually independent and $\mathcal{N}(0,1)$-distributed random variables. For a given $\varepsilon > 0$, if $m = m(\varepsilon)$ is sufficiently big, then we have $\|\frac{1}{m}\sum_{u=1}^{m} W_u^2 - 1\|_1 < \varepsilon/2$.

By a truncation argument and uniform integrability of $(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} S^j f)^2)_{n\geq 1}$ we deduce, from the convergence in distribution of $(F_{1,n}, ..., F_{m,n})$ to $(W_1, ..., W_m)$, that for $m = m(\varepsilon)$ sufficiently big, $\|\frac{1}{m}\sum_{i=1}^{m} F_{i,n}^2 - 1\|_1 \leq \varepsilon$.

2) *Second proof.*

Suppose we have a trigonometric polynomial as above. Let $(w_n(i,j))_{n\geq 1}$ be a sequence of positive weights such that $\sum_{i,j} w_n(i,j)^2 = 1$. An example of weights is $w_n(i,j) = |D_n|^{-\frac{1}{2}} 1_{D_n}(i,j)$, where D_n is an increasing sequence of sets, for instance rectangles as above.

So we consider the normalized sums: $\sum_{i,j} w_n(i,j) T^i S^j f$. They satisfies

$$\|\sum_{i,j} w_n(i,j) T^i S^j f\|_2^2 = \sum_i \int (\sum_j w_n(i,j) S^j f)^2 d\mu = \sum_{i,j} w_n(i,j)^2 \int f^2 d\mu = 1$$

and we want to prove their convergence in distribution toward $\mathcal{N}(0,1)$.

Notation: $\psi_{n,i} = \sum_j w_n(i,j) S^j f$, $X_{n,i} = T^i \psi_{n,i}$.

To obtain the result via McLeish theorem, we have to check conditions (*i*) and (*ii*). Let us show (*ii*). Actually, we will prove $\|\sum_i X_{n,i}^2 - 1\|_2 = \|\sum_i T^i (\sum_j w_n(i,j) S^j f)^2 - 1\|_2 \to 0$, which is:

$$\|\sum_i T^i (\psi_{n,i}^2 - \int \psi_{n,i}^2 \, d\mu)\|_2 \to 0.$$

We have to evaluate:

$$\sum_j w_n(i,j)w_n(i,j)w_n(i,j')w_n(i',k')w_n(i',k') \int T^iS^jf\, T^iS^{j'}f\, T^{i'}S^kf\, T^{i'}S^{k'}f\, d\mu$$

The expansion of $\psi_{n,i}^2 - \int \psi_{n,i}^2$ gives:

$$\sum_j w_n(i,j)w_n(i,j)w_n(i,j')w_n(i',k')w_n(i',k')$$

$$\sum_{\ell_1,\ell_2,\ell_3,\ell_4} c_{\ell_1}\, c_{\ell_2}\, c_{\ell_3}\ c_{\ell_4} e^{2\pi i[3^i(2^j\ell_1+2^{i'}\ell_2)+3^{i'}(2^k\ell_3+2^{k'}\ell_4)]x}.$$

Since the integrals are substracted, remark that

$$\sum_{2^i\ell_1+2^{i'}\ell_2=0} c_{\ell_1}\, c_{\ell_2} = 0, \quad \sum_{2^i\ell_3+2^{i'}\ell_4=0} c_{\ell_3}\, c_{\ell_4} = 0.$$

After integration, all terms such that $3^i(2^j\ell_1 + 2^{i'}\ell_2) + 3^{i'}(2^k\ell_3 + 2^{k'}\ell_4) \neq 0$ disappear. Take $w_n(i,j) = |D_n|^{-\frac{1}{2}}\, 1_{D_n}(i,j)$. It suffices to bound the number of terms for (i,j), (i,j'), $i',k')$, (i',k') in D_n, such that:

$$3^i(2^j\ell_1 + 2^{i'}\ell_2) + 3^{i'}(2^k\ell_3 + 2^{k'}\ell_4) = 0, \tag{5}$$

For it, we use a result on S-units that we will recall later: if in (5) no proper subsum vanishes and if the ℓ_i's have no common factor, then there is a finite number of solutions. Taking into account a previous remark, the only possible vanishing subsums are (simultaneously):
$3^i 2^j \ell_1 + 3^{i'} 2^k \ell_3 = 0$, $3^i 2^{j'} \ell_2 + 3^{i'} 2^{k'} \ell_4 = 0$, or
$3^i 2^j \ell_1 + 3^{i'} 2^{k'} \ell_4 = 0$, $3^i 2^{j'} \ell_2 + 3^{i'} 2^k \ell_3 = 0$.

The ℓ_i's are prime to 2 and 3. Therefore $i = i'$, $j = k$, $j' = k'$ or $i = i'$, $j = k'$, $j' = k$.

If $D_n = [0, a_n - 1] \times [0, b_n - 1]$, with $a_n, b_n \to \infty$, the number of choices is $a_n b_n^2$. The normalization of the ergodic sums is by $|D_n|^{-\frac{1}{2}}$; hence we divide by $|D_n|^2 = a_n^2 b_n^2$. This implies the result.

This shows (*ii*). An analogous proof can be given for (*i*).

The CLT for (2) follows.

Complete commutation

Now we consider in general γ_1, γ_2, two commuting surjective endomorphisms of G such that Ker (γ_1) is finite. One easily checks that the following conditions are equivalent for the adjoint operators:

$$\begin{aligned} T_{\gamma_2}\Pi_{\gamma_1} &= \Pi_{\gamma_1}T_{\gamma_2}, & (6)\\ \text{Ker}\,(\gamma_1) &\cap \text{Ker}\,(\gamma_2) = \{0\}. & (7)\end{aligned}$$

(6) is what M. Gordin called complete commutation.

The complete commutation allows the use of the martingale method for ergodic means on rectangles, as we have seen for the example of the 2-dimensional action generated by $2x \bmod 1$, $3x \bmod 1$ on $\mathbb{T}^1$. But the method is not available for other summation methods like the summation along a random walk (more precisely, the proof of condition ii) does not work).

Another possibility is the method of moments. We will recall it briefly, but after mentionning the question of r-mixing for a multi-dimensional action.

2. r-mixing and the method of moment or cumulants

The proof of the CLT given by Leonov in 1960 for a single ergodic endomorphism of a compact abelian group G is based on the computation of the moments of the ergodic sums of trigonometric polynomials and uses mixing of all orders. For $\mathbb{Z}^d$-actions by automorphisms on compact abelian groups, mixing of all orders is satisfied for actions on connected compact abelian groups, but may fail (cf. Ledrappier (1978)) for a non connected group. Nevertheless, as we will see, when the non mixing configurations are sparse enough the moment method can be applied.

Moments, cumulants

Let $(X_1, ..., X_r)$ be a random vector. For any subset $I = \{i_1, ..., i_p\} \subset \{1, ..., r\}$, we put $m(I) := \mathbb{E}(X_{i_1} \cdots X_{i_p})$. The cumulant of order r is

$$C(X_1, ..., X_r) = \sum_{\pi \in \mathcal{P}} (-1)^{p-1}(p-1)!\; m(I_1) \cdots m(I_p), \qquad (8)$$

where $\pi = \{I_1, I_2, ..., I_p\}$ runs through the set $\mathcal{P}$ of partitions of $\{1, ..., r\}$ into $p \leq r$ nonempty subsets, for $p = 1, ..., r$.

Theorem 2. *Let* $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$ *be a random process (with* X_k *centered in* L^2*) and* $(w_n)_{n\geq 1}$ *a sequence of positive weights on* $\mathbb{Z}^d$*. Let* $Y^{(n)} = \sum_{\underline{\ell}} w_n(\underline{\ell})\, X_{\underline{\ell}}$, $n \geq 1$, *such that* $\|Y^{(n)}\|_2 \neq 0$*. Then the condition*

$$\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\in(\mathbb{Z}^d)^r} w_n(\underline{\ell}_1)...w_n(\underline{\ell}_r)\, C(X_{\underline{\ell}_1}, ..., X_{\underline{\ell}_r}) = o(\|Y^{(n)}\|_2^r), \forall r \geq 3.$$

$$\textit{implies} \quad \frac{Y^{(n)}}{\|Y^{(n)}\|_2} \xrightarrow[n\to\infty]{distrib} \mathcal{N}(0,1).$$

Non-mixing r-tuples

Let $f = \sum_{j\in J} c_j\chi_j$ be a trigonometric polynomial and $\Phi = (\chi_j, j \in J)$. We take $X_{\underline{k}} = T^{\underline{k}}f$ and consider the cumulants: $C(T^{\underline{a}_1}f, ..., T^{\underline{a}_r}f)$.

The set of "*non-mixing*" r-tuples for $\Phi = (\chi_j, j \in J)$ ("bad configurations") is defined as

$$\mathcal{N}(\Phi, r) := \{(\underline{a}_1, ..., \underline{a}_r) : \exists \chi_{i_1}, ..., \chi_{i_r} \in \Phi : C(T^{\underline{a}_1}\chi_{i_1}, ..., T^{\underline{a}_r}\chi_{i_r}) \neq 0\}.$$

If $(\underline{a}_1, ..., \underline{a}_r) \notin \mathcal{N}(\Phi, r)$, then, by expansion, $C(T^{\underline{a}_1}f, ..., T^{\underline{a}_r}f) = 0$.

If $(\underline{a}_1, ..., \underline{a}_r) \in \mathcal{N}(\Phi, r)$, i.e., $(\underline{a}_1, ..., \underline{a}_r)$ is a non-mixing r-tuple for Φ, then it satisfies $T^{\underline{a}_1}\chi_{j_1}...T^{\underline{a}_r}\chi_{j_r} = \chi_0$, for some $(j_1, ..., j_r) \in J^r$.

Let $(w_n)_{n\geq 1}$ be a sequence of positive weights on $\mathbb{Z}^d$ and f a regular real function on G with spectral density φ_f, such that $\varphi_f(0) \neq 0$, since otherwise the limiting distribution is δ_0. Suppose the weights are such that

$$\sigma_n^2(f) := \| \sum_{\underline{\ell}} w_n(\underline{\ell})\, T^{\underline{\ell}} f \|_2^2 \sim (\sum_{\underline{\ell}} w_n^2(\underline{\ell}))\, \varphi_f(0).$$

Theorem 2 implies

Theorem 3. *If, for any finite family* Φ *of characters*,

$$\sum_{(\underline{\ell}_1,\ldots,\underline{\ell}_r)\in\mathcal{N}(\Phi,r)} \prod_{j=1}^{r} |w_n(\underline{\ell}_j)| = o\Big((\sum_{\underline{\ell}\in\mathbb{Z}^d} w_n^2(\underline{\ell}))^{\frac{r}{2}}\Big), \forall r \geq 3, \quad (9)$$

then

$$(\sum_{\underline{\ell}\in\mathbb{Z}^d} w_n^2(\underline{\ell}))^{-\frac{1}{2}} \sum_{\underline{\ell}\in\mathbb{Z}^d} w_n(\underline{\ell}) f(T^{\underline{\ell}}.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \varphi_f(0)). \quad (10)$$

Now, the problem is to show that the sets $\mathcal{N}(\Phi, r)$ are small in some sense.

3. Mixing of order r for $\mathbb{N}^d$-action by endomorphisms

For a general $\mathbb{N}^d$-action, $\underline{\ell} \to T^{\underline{\ell}}$, preserving a probability measure μ, the property of mixing of order $r \geq 2$ is that, for any r-tuple of bounded measurable functions $f_1, ..., f_r$ with 0 integral, every $\varepsilon > 0$, there is M such that

$$\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M, \forall j \neq j' \Rightarrow |\int T^{\underline{\ell}_1} f_1 ... T^{\underline{\ell}_r} f_r \, d\mu| < \varepsilon. \tag{11}$$

For the action by algebraic endomorphisms on G, mixing of order r is equivalent to: for every set $\Phi = \{\chi_1, ..., \chi_r\}$ of r characters $\neq \chi_0$, there is $M > 0$ such that $\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M$ for $j \neq j'$ implies $T^{\underline{\ell}_1}\chi_1 ... T^{\underline{\ell}_r}\chi_r \neq \chi_0$ (because for a character χ, either its integral is 0 or χ is the trivial character χ_0).

In other words, r-mixing for an $\mathbb{N}^d$-action by endomorphisms is equivalent to: for every set Φ of r non trivial characters, there is M s.t. $\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M, \forall j \neq j' \Rightarrow (\underline{\ell}_1, ..., \underline{\ell}_r)$ is not solution of

$$T^{\underline{\ell}_1}\chi_1 ... T^{\underline{\ell}_r}\chi_r = \chi_0. \tag{12}$$

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Example: action by $\times 2$*,* $\times 3 \mod 1$ on $\mathbb{T}^1$

A set Φ of non zero characters is given by an r-tuple $\{k_1, ..., k_r\}$ of non zero integers. Equation (12) reads $k_1 2^{a_1} 3^{b_1} + ... + k_r 2^{a_r} 3^{b_r} = 0$. Hence we consider the equation:

$$k_1 2^{a_1} 3^{b_1} + ... + k_r 2^{a_r} 3^{b_r} = 1, \ ((a_1, b_1), ..., (a_r, b_r)) \in (\mathbb{Z}^2)^r. \quad (13)$$

It is known that, for a given set $k_1, ..., k_r$, there is only a finite number of r-tuples $((a_1, b_1), ..., (a_r, b_r))$ solutions of (13), if no proper sub-sum vanishes. It implies that the (invertible extension) 2-dimensional action generated by $\times 2, \times 3$ is mixing of all orders.

The result on solutions of (13) is a special case of a theorem on S-units applied in 1992 by K. Schmidt and T. Ward to prove in general: *Every 2-mixing* $\mathbb{Z}^d$*-action by automorphisms on a compact connected abelian group G is mixing of all orders.*

There is a long history of equations of type (13) (S-unit equations), (van der Poorten, Evertse, Schlickewei, W. Schmidt, ...). In a paper of the last three authors the following statement is proved:

Let K be an algebraically closed field of characteristic 0. Let $(K^*)^r$ be the direct product consisting of r-tuples $x = (x_1, ..., x_r)$ of non zero elements $x_i \in K$. Let Γ be a subgroup of $(K^*)^r$. For $(a_1, ..., a_r) \in (K^*)^r$, let us consider the equations, with $x \in \Gamma$:

$$a_1 x_1 + ... + a_r x_r = 1, \tag{14}$$

If Γ has finite rank d, the number of non-degenerate solutions $x \in \Gamma$ of equation (14), if no sub-sum of the left-hand side of (14) vanishes, is finite (with an explicit bound).

This is what has been used to check McLeish condition previously.

At the opposite, in the non connected case (for example for endomorphisms of shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$), we will see that there are infinitely many bad r-tuples, for $r \geq 3$.

4. $\mathbb{F}_p^{\mathbb{Z}^d}$ and its dual group, shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$

Let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, $p > 1$ a prime integer. Let $G_0 = G_0^{(d)}$ be the compact abelian group $\mathbb{F}_p^{\mathbb{Z}^d}$ identified with the space $\mathcal{S}_d = \mathbb{F}_p[[x_1^{\pm}, ..., x_d^{\pm}]]$ of formal power series in d variables with coefficients in $\mathbb{F}_p$.

A point $\zeta = (\zeta_{\underline{k}}, \underline{k} \in \mathbb{Z}^d)$ in G_0 is represented by the formal power series with coefficients in $\mathbb{F}_p$: $\zeta(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^d} c(\zeta, \underline{k})\, \underline{x}^{-\underline{k}}$.
In this representation of the group G_0, the action of the shifts $\sigma_1, ..., \sigma_d$ on G_0 is: $\sigma_j \zeta(\underline{x}) = x_j \zeta(\underline{x})$.

$\mathcal{P} = \mathcal{P}_d = \mathbb{F}_p[x_1^{\pm}, ..., x_d^{\pm}]$ is the ring of Laurent polynomials in d variables with coefficients in $\mathbb{F}_p$. A Laurent polynomial P in $\mathcal{P}$ reads

$$P(x_1, ..., x_d) = \sum_{\underline{k} \in S(P)} c(P, \underline{k})\, \underline{x}^{\underline{k}},$$

where $S(P)$ is a finite subset of $\mathbb{Z}^d$. For $P \in \mathcal{P}$ and $\zeta \in \mathcal{S}$, the product $P\zeta$ is well defined. It is easy to see that for any character χ on $\mathbb{F}_p^{\mathbb{Z}^d}$ there is a polynomial $P \in \mathcal{P}_d$ such that

$$\chi(\zeta) = \chi_P(\zeta) = e^{\frac{2\pi}{p} i c(P\zeta, \underline{0})}.$$

Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ (cf. K. Schmidt's book (1995))

By what precedes, the dual $\widehat{G}_0$ of G_0 is $\mathcal{P}$ endowed with its additive group structure.

Let $G \subset G_0$ be a shift-invariant closed subgroup of $\mathbb{F}_p^{\mathbb{Z}^d}$.

The annulator $G^\perp$ of G in $\widehat{G}_0$ is $\{P : \chi_P(\zeta) = 1, \ \forall \zeta \in G\}$, i.e., $\{P : c(P\zeta, \underline{0}) = 0, \ \forall \zeta \in G\}$. Since G is shift-invariant, the relation is satisfied for $\underline{x}^{\underline{k}}\zeta(\underline{x})$, for every $\underline{k} \in \mathbb{Z}^d$. Hence $P\zeta = 0, \ \forall \zeta \in G$.

Therefore $G^\perp$ can be identified with the ideal

$$\mathcal{J} = \{P \in \mathcal{P}_d : P\zeta = 0, \ \forall \zeta \in G\}.$$

By duality in $\mathbb{F}_p^{\mathbb{Z}^d}$, we have $G = (G^\perp)^\perp$. This shows that

$$G = \{\zeta : P\zeta = 0, \ \forall P \in \mathcal{J}\}. \tag{15}$$

The dual of G is identified with the quotient $\widehat{G}_0/G^\perp$, i.e., $\widehat{G} = \mathcal{P}/\mathcal{J}$.

Conversely, every ideal $\mathcal{J}$ defines a shift-invariant subgroup of G_0 by (15).

5. Counting non mixing r-tuples

Now we consider the action on $\mathbb{F}_p^{\mathbb{Z}}$ given by polynomials in one variable. Let $\mathcal{R} = (R_1, ..., R_d)$ be a family of d prime polynomials in one variable. We assume $R_1(x) = x$. Their natural extension is the shift-actions on the shift-invariant group $G_{\mathcal{J}} \subset \mathbb{F}_p^{\mathbb{Z}^d}$ defined by the ideal $\mathcal{J} = \operatorname{Ker} h$, with $h(Q)(x) = Q(x, R_2(x), ..., R_d(x))$.

Ledrappier's celebrated example is a special case of this construction (for $R(x) = 1 + x$). Its invertible extension given by the $\mathbb{Z}^2$-shift action on the shift-invariant group $G_{\mathcal{J}}$, where $\mathcal{J}$ is the ideal generated by the polynomial $1 + x_1 + x_2$. The group $G_{\mathcal{J}}$ is the set of configurations ζ in $\mathbb{F}_2^{\mathbb{Z}^2}$ such that $\zeta(n, m) + \zeta(n+1, m) + \zeta(n, m+1) = 0 \bmod 2$, $\forall (n, m) \in \mathbb{Z}^2$.

A set of characters is given by a finite family of polynomials $P_1, ..., P_r$. A "bad configurations" of the $\mathbb{N}^d$-action defined by $\mathcal{R}$ for this set, is an r-tuple $(\underline{a}_1, ..., \underline{a}_r) \in \mathbb{N}^r$ such that in $\mathbb{F}_p[x]$

$$P_1(x) \prod_{i=1}^{d} R_i(x)^{a_{1,i}} + ... + P_r(x) \prod_{j=1}^{d} R_r(x)^{a_{r,i}} = 0. \tag{16}$$

(16) is analogous to an S-unit equation (like for the 2 and 3).

It can be reduced to the case where the P_j's are scalars.

To count bad configurations we introduce the following definition: A polynomial Γ in d-variables, $\Gamma(\underline{x}) = \sum_{\underline{a} \in \mathbb{N}^d} c(\underline{a}) \prod_{i=1}^{d} x_i^{a_i}$ is a special $\mathcal{R}$-polynomial if

$$\Gamma(R_1(x), R_2(x), ..., R_d(x)) = \sum_{\underline{a} \in \mathbb{N}^d} c(\underline{a}) \prod_{i=1}^{d} R_i^{a_i} = 0. \tag{17}$$

Let $\mathcal{D}$ be any family of prime polynomials containing the polynomial $R_1(x) = x$. The polynomials $x_\rho - \rho(x), \rho \in \mathcal{D}$, will be called *basic special $\mathcal{D}$-polynomials* (abbreviated in "bs $\mathcal{D}$-polynomial"). We say that a polynomial Γ is *shifted* from another polynomial Γ_0 if $\Gamma(\underline{x}) = \underline{x}^{\underline{a}}\,\Gamma_0(\underline{x})$ for some monomial $\underline{x}^{\underline{a}}$.

A polynomial Λ is called *generalized basic special $\mathcal{D}$*-polynomial (abbreviated in "gbs $\mathcal{D}$-polynomial"), if it is obtained from a basic special $\mathcal{D}$-polynomial Δ by shift and dilation (exponentiation with a power of p as exponent).

Therefore, Λ is a gbs $\mathcal{D}$-polynomial, if there are $\underline{a} \in \mathbb{Z}^d$, $t \geq 0$ and a bs $\mathcal{D}$-polynomial Δ such that: $\Lambda(\underline{x}) = \underline{x}^{\underline{a}}\,(\Delta(\underline{x}))^{p^t}$.

Extending a result shown by Arenas-Carmona, Bergelson, Berend (2008) for Ledrappier's example, we have:

Theorem 4. *Let* r *be an integer* ≥ 2. *For every family* $\mathcal{R} = (R_j, j = 1, ..., d)$ *of* $d \geq 1$ *polynomials, there is a finite constant* $t(r, \mathcal{R})$ *and a finite family* $\mathcal{E}$ *of polynomials in one variable containing* $\mathcal{R}$ *such that every special* $\mathcal{R}$*-polynomial of length* $\leq r$ *is a sum of at most* $t(r, \mathcal{R})$ *gbs* $\mathcal{E}$*-polynomials. Moreover* $t(r) = O(r^\delta)$, *for some constant* δ.

Let D be a domain in $\mathbb{Z}^d$. A corollary of the previous theorem is:

Theorem 5. *The number $\theta(D,r)$ of special $\mathcal{R}$-polynomials Γ with r terms, supported in a domain D, satisfies for a constant $\theta(r)$*

$$\theta(D,r) = O(|D|^{r/3}(\log \operatorname{diam} D)^{\theta(r)}). \tag{18}$$

Denote by A an r-uple $(\underline{a}_1, ..., \underline{a}_r)$, $\widetilde{P} = (P_1, ..., P_r)$ a set of polynomials. The corresponding cumulant is

$$c_{\widetilde{P}}(A) = c(R^{\underline{a}_1}P_1, ..., R^{\underline{a}_r}P_r).$$

The problem, in the method of cumulants for limit theorems, is to get a bound for $\#\{A \in D^r : c_{\widetilde{P}}(A) \neq 0\}$. This is done in the theorem:

Theorem 6. *For each $r \geq 3$, there exists $\theta_1(r) \geq 1$ such that*

$$\#\{A \in D^r : c_{\widetilde{P}}(A) \neq 0\} = O(|D|^{\frac{r}{2}-\frac{1}{2}}(\log \operatorname{diam} D)^{\theta_1(r)}). \tag{19}$$

6. Example of application to limit theorems

The case when G is a connected compact abelian group was considered in a previous paper in collaboration with G. Cohen. Here we consider limit theorems in the framework of shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$.

Let us consider a family $\mathcal{R} = (R_1(x) = x, R(_2(x), ..., R_d(x))$ of endomorphisms of $\mathbb{F}_p^{\mathbb{Z}}$, extended to automorphisms of $\mathbb{F}_p^{\mathbb{Z}^d}$ or extended to the $\mathbb{Z}^d$-shift action on the shift invariant group G defined by the constraints associated to the ideal $\mathcal{J} = \text{Ker}\,(h_{\mathcal{R}})$. The (R_j)'s are chosen to be algebraically independent. Therefore we have a totally ergodic $\mathbb{Z}^d$-action on G.

Let us give an example of the results (for $d = 2$).

Theorem 7. *Let (Z_k) be a centered random walk with a finite moment of order 2. Let $\underline{\ell} \to T^{\underline{\ell}}$ be the standard $\mathbb{Z}^2$-action by automorphisms (the shifts) on $G_{\mathcal{J}}$. If f is a regular function with spectral density φ_f then, there exists a constant C,for a.e. ω,*

$$(Cn\text{Log}n)^{-\frac{1}{2}} \sum_{k=0}^{n-1} T^{Z_k(\omega)} f(.) \ \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \ \mathcal{N}(0,1).$$

Thank you for your attention!