Remarks on affine stationary walks on $\mathbb{R}$

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Let $(E, \mathcal{A}, \mu, \theta)$ be an invertible measure preserving *ergodic* dynamical system and $(a(x), b(x))$ be two real-valued functions on E with $a(x) > 0$. We consider the "stationary walk" over (E, μ, θ) defined on the affine group by

$$\tilde{\theta}(x, \begin{pmatrix} y & z \\ 0 & 1 \end{pmatrix}) = (\theta x, \begin{pmatrix} a(x) & b(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & z \\ 0 & 1 \end{pmatrix}).$$

We have $\tilde{\theta}^n(x, \begin{pmatrix} y & z \\ 0 & 1 \end{pmatrix}) = (\theta^n x, \begin{pmatrix} a_n(x)y & a_n(x)z + u_n(x) \\ 0 & 1 \end{pmatrix})$, with

$$\begin{aligned} a_0(x) = 1,\ a_n(x) &= a(\theta^{n-1}x)\, \theta^{n-2}x) \dots a(x), n \geq 1, && (1) \\ u_0(x) = 0,\ u_{n+1}(x) &= a(\theta^n x)\, u_n(x) + b(\theta^n x), n \geq 0. && (2) \end{aligned}$$

The corresponding affine stationary walk generated on $\mathbb{R}$ by the sequence of transformations $z \to a(\theta^n x)z + b(\theta^n x)$ is the process $(u_n(x))_{n \geq 1}$ defined recursively by (2).

We discuss the asymptotic behavior of $(u_n(x))_{n \geq 1}$: is it recurrent (i.e. it returns infinitely often to some compact set) or transient (i.e. $\lim_n |u_n(x)| = \infty$) ? The aim of this note is to gather a few remarks on this question.

Assume $\log a$ to be μ-integrable and let $c := \int \log a \, d\mu$.

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With the notation of (1), we have the following formulas, for $n \geq 1$:

$$\begin{aligned} u_n(x) &= \sum_{k=1}^{n} a_{n-k}(\theta^k x)\ b(\theta^{k-1}x) && (3) \\ &= a_n(x) \sum_{k=1}^{n} (a_k(x))^{-1}\ b(\theta^{k-1}x). && (4) \end{aligned}$$

The process $(u_n(x))$ is related to the following functional equation, where the unknown is the μ-a.e. finite measurable function v:

$$v(\theta x) = a(x)v(x) + b(x),\ \mu - a.e.. \quad (5)$$

Observe that, if an a.e. finite solution v of (5) exists, it is unique (when a is not a multiplicative coboundary) and (u_n) verifies

$$u_n(x) = v(\theta^n x) - a_n(x)v(x),\ n \geq 0. \quad (6)$$

Lemma 1.1 *If $\int \log a \, d\mu \neq 0$ and $\log |b|$ integrable, there exists an a.e. finite solution v of (5).*

Proof We have $\lim_k \frac{1}{k} \log |b(\theta^{-k-1}x)| = 0$, for a.e. x. Assume $\int \log a \, d\mu < 0$. The ergodic theorem implies $\lim_k \frac{1}{k} \sum_{j=1}^{k-1} \log a(\theta^{-j}x)) = c < 0$, for a.e. x. Therefore we have $\lim_k (a_k(\theta^{-k}x) \, |b(\theta^{-k-1}x)|)^{\frac{1}{k}} < 1$, for μ-a.e. $x \in X$; hence the solution is given by the absolutely convergent series

$$v(x) = \sum_{k=0}^{\infty} a_k(\theta^{-k}x) \, b(\theta^{-k-1}x). \tag{7}$$

If $\int \log a \, d\mu > 0$, the same argument shows that the solution is given by the absolutely convergent series

$$v(x) := - \sum_{k=1}^{\infty} (a_k(x))^{-1} \, b(\theta^{k-1}x).$$

□

Proposition 1.2 *i) Assume that $c := \int \log a \, d\mu \neq 0$. Let v be the solution of (5) given by Lemma 1.1.*

a) If $c < 0$, then $\lim_n (u_n(x) - v(\theta^n x)) = 0$, the process $(u_n(x))$ is asymptotically stationary and recurrent for μ-a.e. x.

b) If $c > 0$ and $v(x) \neq 0$, then $\lim_n u_n(x) = +\infty$ or $-\infty$ according to the sign of $v(x)$. If $c > 0$ and $v(x) = 0$ on a set D of measure > 0, then $u_n(x) = v(\theta^n x)$ and $(u_n(x))$ is recurrent for a.e. $x \in D$.

ii) Assume that $\int \log a \, d\mu = 0$.

a) If a is a multiplicative coboundary, $a = h/h \circ \theta$ with hb integrable, then for μ-a.e. x $(u_n(x))$ is recurrent if $\int b \; h \circ \theta \; d\mu = 0$, transient to $+\infty$ (resp. $-\infty$) if $\int b \; h \circ \theta \; d\mu > 0$ (resp. $\int b \; h \circ \theta \; d\mu < 0$).

b) If (5) has an a.e. finite solution, the sequence $(u_n(x))$ is recurrent for μ-a.e. x.

c) If $b(x) \geq 0$, for a.e. x, with $b \not\equiv 0$, then $\lim_n \mu(\{x : u_n(x) > M\}) = 1, \forall M$ and (5) has no solution.

Proof *i*) Statement *a*) follows from (6) since for μ-a.e. x, $(\theta^n x)$ visits infinitely often sets on which v is bounded by some constant.

Statement *b*) follows from (4) which shows that $u_n(x) = a_n(x) w_n(x)$, where $(w_n(x))$ is a sequence tending to $-v(x)$. On the set $D = \{x : v(x) = 0\}$, we use (6).

ii)*a*) If a is a multiplicative coboundary $a = h/h \circ \theta$, we have according to (3):

$$u_n(x) \quad = \quad (h(\theta^n x))^{-1} \sum_{k=1}^{n-1} h(\theta^k x) b(\theta^{k-1}x) + b(\theta^{n-1}x), n \geq 1.$$

The result follows from the conservativeness of the system $(x, y) \to (\theta x, y + h(\theta x) b(x))$, if $\int b\ h \circ \theta\ d\mu = 0$. By conservativeness of the transformation $(x, y) \in X \times \mathbb{R} \to (\theta x, y + h(x) b(\theta^{-1} x))$, the sums $\sum_{k=1}^{n-1} h(\theta^k x) b(\theta^{k-1} x)$ and the factor $h(\theta^n x))^{-1}$ are simultaneously small for an infinite sequence of times $(n_j(x))$.

b) We have $u_n(x) = v(\theta^n x) - a_n(x) v(x)$ (cf. (6)). By conservativeness of the transformation $(x, y) \in X \times \mathbb{R} \to (\theta x, y + \log a(x))$, for a.e. x there is an increasing sequence of times $(n_j(x))$ such that $(v(\theta^{n_j(x)} x))_{k \geq 1}$ is bounded and $a_{n_j(x)}(x)$ is small. This implies recurrence.

c) Let v be a solution of (5). If $b \geq 0$, then we have

$$v(x) \geq \sum_{k=0}^{n} a_k(\theta^{-k} x)\, b(\theta^{-k-1} x), \forall n \geq 1. \tag{8}$$

Assume $b \not\equiv 0$ and let $C := \{x : b(x) \geq c\}$, where c is a constant such that $\mu(C) > 0$. The series in (8) is divergent on C, since by conservativeness of $(x, y) \in X \times \mathbb{R} \to (\theta x, y + \log a(x))$, for a.e. x there is a constant $d > 0$ and an infinite sequence $(n_j(x))$ such that $a_{n_j(x)}(\theta^{-j} x)\, b(\theta^{-n_j(x)-1} x) \geq d$.

By the same argument, since

$$u_n(\theta^{-n} x) \geq \sum_{k=0}^{n-1} a(\theta^{-1} x) ... \, a(\theta^{-k} x) b(\theta^{-k-1} x) \to +\infty, \text{ for a.e. } x,$$

we have, by invariance of the measure, $\lim_n \mu(\{x : u_n(x) > M\}) = 1, \forall M$, i.e. *transience in probability.* $\square$

Remarks 1.3 1) In *ii)c)*, if $\int \log a\ d\mu = 0$ and $b(x) \geq 0$, a quantitative statement consists in finding an increasing sequence (t_n) such that $\lim_n \mu(\{x : u_n(x) > t_n\}) = 1$. This amounts to estimate the growth of the sums $r_n(x) := \sum_{k=0}^{n-1} a(\theta^{-1} x) ... \, a(\theta^{-k} x) b(\theta^{-k-1} x)$ and depends strongly on the system.

For a rotation and a regular function a, the growth is slow (if $\log a$ has bounded variation and if the rotation has bounded partial quotients, the growth is at most polynomial).

At the opposite, consider the independent case and more generally when $\log a$ satisfies a non degenerate central limit theorem. Assuming for simplicity that $\inf_x b(x) > 0$, observe that there are constants $0 < c \leq C < \infty$ such that $c < r_n(\theta x) / r_n(x) < C$. Therefore, the set $F_\lambda := \{x : \limsup_n (r_n(\theta x) e^{-\lambda \sqrt{n}}) > 0\}$ is θ-invariant. By the CLT, for every $\lambda > 0$, $\mu(F_\lambda) > 0$. By ergodicity, this implies $\mu(F_\lambda) = 1$.

2) There no "dichotomy" (as for random walks on groups) in general for the asymptotic behavior of $(u_n(x))_{n \geq 1}$: there can exist a set B with $0 < \mu(B) < 1$ such that the sequence $(u_n(x))$ is recurrent for $x \in B$ and transient for $x \in B^c$. Indeed let a be such that $\int \log a\ d\mu > 0$, $a(x) \neq 1$, for every x and let B such that $0 < \mu(B) < 1$ and $\theta^{-1} B \cap B = \emptyset$, then using (5) to define $b := 1_B \circ \theta - a 1_B$, we get $b(x) \neq 0$, for every x, and a solution $v = 1_B$ of (5) null on the set B. We apply then *i) b)* of the previous Proposition.

3) When the map $x \to (a(x), b(x))$ takes two values, we can reduce it to the case $b \geq 0$: assume that there are a set A of measure $\mu(A) = \frac{1}{2}$ and constants c_1, c_2 and $d > 1$ such that

- for $x \in A$: $a(x) = d$, $b(x) = c_1$;
- for $x \in A^c$: $a(x) = d^{-1}$, $b(x) = c_2$.

By changing the origin, the transformations are reduced to
for $x \in A$: $a(x) = d$, $b(x) = c_1 - dc_2$; for $x \in A^c$: $a(x) = d^{-1}$, $b(x) = 0$.

Without loss of generality, we can assume that $c_1 - dc_2 > 0$, if $c_1 - dc_2 \neq 0$. The set $\mathbb{R}^+$ is then invariant. We know that we have then by *ii c)* transience in probability: $\lim_n \mu(\{x : u_n(x) > M\}) = 1, \forall M$.

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The most interesting case is when $c = 0$, as in ii) of Proposition 1.2. In general it seems difficult to find the asymptotic behavior of (u_n). We discuss below two particular cases in order to show that the *pointwise* behavior depends on the properties of the dynamical system. We consider transformations on (E, μ) the circle endowed with the Lebesgue measure.

Example 1

Take for θ the map $x \to 2x \bmod 1$, $a(x) = 2$ and $b(x) = 1$ if $x \in [0, \frac{1}{2}[$, $a(x) = 2^{-1}$ and $b(x) = 0$, if $x \in [\frac{1}{2}, 1[$. The sequence $(a(\theta^n x), b(\theta^n x))$ is a sequence of iid random variables.

The following proposition results from the reference: *Babillot (M.), Bougerol (Ph.), Elie (L.): The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case, Ann. Probab. 25 (1997), no. 1, p. 478-493.*

Proposition 2.1 *In Example 1, we have $\liminf_n u_n(x) < +\infty$ for a.e. x, therefore the process $(u_n(x)$ is pointwise recurrent for a.e. x.*

The same result holds if we take for $(a(\theta^n x), b(\theta^n x))$ any sequence of iid integrable r.v.s (X_n, Y_n) such that $\mathbb{E}(\log X_0) = 0$.

Let $S_n(x) := \sum_{k=0}^{n-1} \log X_k(x)$. The proof of Proposition 2.1 relies on the introduction of the subprocess obtained by taking the process $(u_n(x), S_n(x))$ at the downward ladder times $(L_n(x))$ of the random walk $(S_n(x))$. There is a unique finite invariant probability measure for the Markov chain $(u_{L_n(x)}(x))$ and the ergodic theorem implies that the subprocess visits infinitely often compact sets on $\mathbb{R}$.

Example 2

We assume now that θ is a rotation $x \to x + \gamma \bmod 1$ where γ is an irrational number. Let $a = e^{\varphi}$ where φ is a function with *bounded variation* on the circle such that $\mu(\varphi) = 0$. Let $b(x)$ be the indicator of an interval J of measure > 0.

Proposition 2.2 *In Example 2, with the preceding assumptions, we have $\lim_n u_n(x) = +\infty$, for every $x \in X$.*

Proof Let (q_j) be the sequence of convergents of γ given by its continued fraction development. By the Denjoy-Koksma inequality we have $a(\theta^{-1}x)...a(\theta^{-q_j+1}x)) \geq e^{-V}$, where V is the variation of φ.

If $n-k = q_j$, for some j, then $a_{n-k}(\theta^k x) \geq e^{-V}$ for every x. By minimality of the rotation, we know that there is $p \geq 1$ such that, for every $y \in X$, there is an integer $\ell(y)$, $0 \leq \ell(y) < p$, such that $\theta^{\ell(y)} y \in J$.

This implies, for $k = n - q_j + \ell(\theta^{n-q_j}x)$,

$$a_{n-k}(\theta^k x) b(\theta^k x) \geq e^{-V} \|a\|_\infty^{-p}.$$

Therefore,

$$u_n(x) = \sum_{k=1}^{n} a_{n-k}(\theta^k x) b(\theta^{k-1}x) \geq \#\{j : q_j < n-p\}\ e^{-V} \|a\|_\infty^{-p},$$

so that $\lim_n u_n(x) = +\infty$. $\square$