Limit theorems for sequential expanding dynamical systems on $[0, 1]$

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ABSTRACT. We consider the asymptotic behaviour of a sequence (θ_n), $\theta_n = \tau_n \circ \tau_{n-1} \cdots \circ \tau_1$, where $(\tau_n)_{n\geq 1}$ are non-singular transformations on a probability space.

After briefly discussing some definitions and problems in this general framework, we consider the case of piecewise expanding transformations of the interval. Exactness and statistical properties (a central limit theorem for BV functions after a moving centering) can be shown for some families of such transformations.

The method relies on an extension of the spectral theory of transfer operators to the case of a sequence of transfer operators.

Introduction

Let (θ_n) be a sequence of non-singular transformations on a probability space $(X, \mathcal{A}, m)$. When the measure is preserved, the extension of notions (ergodicity, mixing,...) from the case of the iterates of a single transformation to a sequence (θ_n) was considered by D. Berend and V. Bergelson ([**BB84**]) and some examples were given of what they called *sequential dynamical systems*.

Recently some authors have given examples of sequential systems of hyperbolic type ([**Ba95**], [**AF04**], [**PR03**]). In the last reference, a property of stable mixing for a sequence of automorphisms of the 2-torus has been discussed by L. Polterovich and Z. Rudnick. Sequential systems have been also considered in [**BBH05**] in the context of the "Bendford law", for transformations composed on $I\!R^+$ before taking mod 1. Another situation where sequential systems appear is that of random sequences of transformations (see for instance [**Ki88**], [**Ke82**], [**BY93**], [**Bu99**], [**Vi97**]).

We consider here the asymptotic behaviour of a sequence (θ_n), $\theta_n = \tau_n \circ \tau_{n-1} \cdots \circ \tau_1$, where $(\tau_n)_{n\geq 1}$ are piecewise expanding transformations of the interval $[0, 1)$, and discuss properties like exactness and limit theorems for such sequential systems. (The measure m is the Lebesgue measure, which is only quasi-invariant.)

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A motivation is the following. Let $x \to \beta x$ mod 1 be a β-transformation for $\beta > 1$. Suppose that at each step of the iteration we make a small error and replace β by β_n tending to β. If $\theta_n(x)$ is defined by : $\theta_0(x) = x$, $\theta_n(x) = \beta_n\, \theta_{n-1}(x)$ mod 1, $n \geq 1$, is it true that for a.e. $x \in [0,1]$ the asymptotic distribution of the sequence $(\theta_n(x))_{n\geq 0}$ is, , the absolutely continuous invariant measure (ACIM) of the transformation $x \to \beta x$ mod 1 ?

A positive answer can be proved for β-transformations and more generally for some classes of piecewise expanding transformations of $[0,1]$. Moreover, statistical properties for such sequences, like a CLT for BV functions (after a moving centering), could be investigated. It is closely related to the question of stochastic stability of expanding transformations. But we will give also global results and obtain exactness for some families of such transformations. The method relies on an extension of the spectral theory of transfer operators to the case of a sequence of transfer operators.

In the first section we consider the general case of a sequential dynamical system where the measure m is only quasi-invariant and briefly discuss some definitions and problems in this general framework. In section 2 general results on products of operators of quasi-compact type are given. In sections 3, 4 and 5 we apply these results to the particular case of sequences of piecewise expanding maps on the interval and prove a "Borel-Cantelli Lemma" and a central limit theorem for regular functions.

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Contents

1. Sequential dynamical systems

We consider a probability space $(X, \mathcal{A}, m)$ and a family $\mathcal{C}$ of non-singular transformations on it: for $\tau \in \mathcal{C}$, we assume that $\tau\ m \ll m$.

We will call *sequential dynamical system* any sequence (τ_n) of transformations in $\mathcal{C}$.

We denote by (θ_n) the sequence of composed transformations:

$$\theta_n = \tau_n \circ \tau_{n-1} \cdots \circ \tau_1, n \geq 1.$$

If (τ_n) is a constant sequence (i.e. $\tau_n = \tau$ for a transformation $\tau \in \mathcal{C}$) (θ_n) is simply the sequence of iterated transformations (τ^n).

NOTATIONS 1.1. If τ is a transformation in $\mathcal{C}$, we denote by T the operator of composition by τ. The transfer operator P_τ corresponding to τ is defined in $L^1(m)$ by:

$$\int P_\tau f \; g \; dm = \int f \; g \circ \tau \; dm, \forall f \in L^1, \, g \in L^\infty.$$

When τ_n, $n = 1, 2, ...$, are transformations in $\mathcal{C}$, we write simply T_n, P_n for the operators corresponding to τ_n and

$$\Pi_n = P_n P_{n-1} \cdots P_1.$$

With these notations, we have for $f \in L^1, g \in L^\infty$:

$$\begin{aligned} T_1 \cdots T_n g(x) &= g(\tau_n \cdots \tau_1 x), \\ \int T_1 \cdots T_n g \; f \; dm &= \int g \; P_n \cdots P_1 f \; dm = \int g \; \Pi_n f \; dm, \\ P_n(T_n g \; f) &= g \; P_n f. \end{aligned}$$

• **Invariant measure, wandering sets**

In the case of an unique transformation τ, a classical problem is the existence of a τ-invariant measure which is equivalent to the measure m, or at least absolutely continuous with respect to m. Such a probability measure μ is called an *ACIM* for τ. In that case, we have $\mu = \varphi m$, with $\varphi \geq 0$, $\int \varphi \; dm = 1$ and $P_\tau \varphi = \varphi$.

In general, when the transformations τ_k depend on k, there is no joint invariant measure, but it is convenient to make for the sequence $(\Pi_n 1)$ the following assumption (1.1) which implies its *equi-integrability* since we have $\|\Pi_n 1\|_1 = 1$:

for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that

$$\forall B \in \mathcal{A}, m(B) < \eta(\varepsilon) \Rightarrow m(\theta_n^{-1} B) = \int_B \Pi_n 1 \; dm < \varepsilon. \tag{1.1}$$

In particular (1.1) is satisfied if there is $p \in]1, +\infty]$ such that the sequence $(\Pi_n 1)$ is bounded for the L^p-norm. As we will see, this is the case (for the uniform norm), for some families of piecewise expanding maps of the interval.

Using the Dunford-Pettis compactness criterion, it is easy to deduce, for a single transformation τ, the existence of an ACIM for τ from (1.1) (Theorem of Hajian and Kakutani).

We say that a set A is *mean wandering* if it satisfies:

$$\lim_N \frac{1}{N} \sum_{j=1}^N m(\theta_j^{-1} A) = 0.$$

With property (1.1), one can prove the existence of a mean wandering set A_0 which is "*maximal*" in the sense that : if B is such that $\lim_N \frac{1}{N} \sum_1^N m(\theta_j^{-1} B) = 0$, then $B \subset A_0$ (up to a set of m-measure 0).

To show it, let us take a sequence (C_k) of mean wandering sets such that $\lim_k m(C_k) = \sup_A m(A)$, the supremum being taken on the family of sets A which are mean wandering.

Let $A_0 = \bigcup_k C_k$. This set is mean wandering: let $\varepsilon > 0$, $\eta(\varepsilon) > 0$ given by (1.1) and k such that $m(A_0 - \cup_1^k C_i) \leq \eta$. We have, for N big enough,

$$\frac{1}{N} \sum_{j=1}^N m(\theta_j^{-1} A) \leq \varepsilon + \sum_{i=1}^k \frac{1}{N} \sum_{j=1}^N m(\theta_j^{-1} C_i) \leq 2\varepsilon$$

and the set A_0 is clearly maximal.

We note that $m(A_0^c) > 0$. From the maximality of A_0, it follows that, if $A \in \mathcal{A}$ is such that $m(A \cap A_0^c) > 0$, then

$$\limsup_N \frac{1}{N} \sum_1^N m(\theta_j^{-1} A) > 0. \tag{1.2}$$

• **Ergodicity and mixing**

We say that a sequence (τ_n) is *ergodic in mean* if $\theta_n = \tau_n \circ \tau_{n-1} \cdots \circ \tau_1$ satisfies the equivalent conditions:

$$\lim_N \frac{1}{N} \sum_{k=1}^N [m(B \cap \theta_k^{-1} A) - m(B) m(\theta_k^{-1} A)] \quad = \quad 0 \tag{1.3}$$

$$\forall g \in L^1, f \in L^\infty \ \lim_N \frac{1}{N} \sum_{k=1}^N \langle f \circ \theta_k - \int f \circ \theta_k \, dm, g \rangle \quad = \quad 0. \tag{1.4}$$

We say that the sequence is *mixing* if

$$\forall A, B \in \mathcal{A} \ \lim_n [m(B \cap \theta_n^{-1} A) - m(B)\, m(\theta_n^{-1} A)] = 0. \tag{1.5}$$

Proposition 1.2. *Let A be such that $m(A \cap A_0^c) > 0$. Ergodicity in mean implies that, for almost all x, the sequence $(\theta_n x)$ visits A infinitely often and, if $m(B) > 0$,*

$$\limsup_N \frac{1}{N} \sum_{j=1}^N m(\theta_j^{-1} A \cap B) > 0.$$

Remark 1.3. If the sequence $(\Pi_n 1)$ is uniformly bounded, condition (1.3) is equivalent to the convergence in L^p-norm, for $1 \leq p < \infty$ (cf. [**BB84**]):

$$\lim_N \frac{1}{N} \| \sum_{k=0}^{N-1} [f \circ \theta_k - \int f \circ \theta_k \, dm] \|_p = 0, \forall f \in L^p. \tag{1.6}$$

This is a consequence of the inequality, for $p \geq 1$:

$$\|\frac{1}{n} \sum_{k=0}^{n-1} [1_A \circ \theta_k - m(1_A \circ \theta_k)]\|_p^{2p} \leq \|\frac{1}{n} \sum_{k=0}^{n-1} [1_A \circ \theta_k - m(1_A \circ \theta_k)]\|_1^2$$

and the fact that the set of functions f such that (1.6) holds is closed in $(L^p, \| \, \|_p)$.

REMARK 1.4. For the iterates of a single transformation, ergodicity in mean characterizes the usual ergodicity property of τ-invariant measure and is equivalent to the "pointwise ergodicity". But this consequence of the ergodic theorem is not true in the general case of a sequential dynamical system.

For example, let $(\tau_t, t \in \mathbb{R})$ be an aperiodic measure preserving flow on a probability space $(X, \mathcal{A}, m)$. M. Akcoglu, A. Bellow, A. del Junco and R. Jones ([**ABJJ93**]) have shown that for any increasing sequence of integers (n_k) and any sequence $(t_k \neq 0)$ converging to zero, the following "strong sweeping out property" is true: given any $\varepsilon > 0$, there is a set A with $m(A) < \varepsilon$ and

$$\liminf_N \frac{1}{N} \sum_1^N 1_A(\tau_{n_k+t_k} x) = 0, \ \limsup_N \frac{1}{N} \sum_1^N 1_A(\tau_{n_k+t_k} x) = 1.$$

Therefore, if we sample a dynamical system with a small error tending to 0, we cannot expect to still have a law of large number valid for any bounded measurable observable.

This suggest that, for a sequential dynamical system, "pointwise ergodicity" should be defined with respect to particular families of (regular) functions or sets. The sequence (τ_n) can be called *pointwise ergodic* if, for a.a. x,

$$\forall f \in \mathcal{F}, \lim_n \frac{1}{n} \sum_{k=1}^n [f(\theta_k x) - \int f \circ \theta_k \, dm] = 0, \tag{1.7}$$

where $\mathcal{F}$ is a convenient set of functions or the set of characteristic functions of sets which form an algebra generating the σ-algebra $\mathcal{A}$.

We will give later examples of sequences of piecewise expanding maps of the interval which are pointwise ergodic in the previous meaning.

• **Stochastic and sequential stabilities**

DEFINITIONS 1.5. Let $\mathcal{T}$ be a metric set of parameters and let $\mathcal{C} = \{\tau_t : t \in \mathcal{T}$ be a family of transformations on $(X, \mathcal{A}, m)$. Fix $t_0 \in \mathcal{T}$, and for $\varepsilon > 0$, let U_ε be an ε-neighbourhood of t_0. Let Y be a random variable with values in U_ε and law ν_ε. Consider the operator P_{ν_ε} defined by

$$P_{\nu_\varepsilon} f = \mathbb{E}(P_{\tau_Y} f).$$

Assume that the transformation $\tau = \tau_{t_0}$ has a unique ACIM μ and that there exists a measure μ_ε which is invariant by P_{ν_ε}. We say that the transformation τ is *stochastically stable* (in the family $\mathcal{T}$) if $\lim_{\varepsilon \to 0} \mu_\varepsilon = \mu$.

In particular, if $\mu_\varepsilon = \delta_{\tau_\varepsilon}$, $\tau_\varepsilon \in U_\varepsilon$, "stochastic stability" becomes stability by deterministic perturbations. The property of sequential stability, that we consider now, is close to this property.

Assuming the space X to be a metric space, we say that τ is *sequentially stable* in $\mathcal{T}$ if, for every sequence (t_n) in $\mathcal{T}$ such that $\lim_n t_n = t_0$, writing simply $\tau_n = \tau_{t_n}$, we have for every continuous f:

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(\tau_n \tau_{n-1} ... \tau_1 x) = \mu(f), \text{ for } m\text{-a.a. } x.$$

This means that, if we make a small error at each step, replacing the transformation τ by τ_n, with $\lim_n \tau_n = \tau$, then we have for m-almost all points x the same asymptotic distribution for the sequence $(\theta_n(x) = \tau_n\tau_{n-1}...\tau_1 x)$ as for $(\tau^n x)$, namely the distribution given by the τ-ACIM μ.

• **Asymptotic σ-algebra, exactness**

Fix a sequence$\{\tau_n\} \subset \mathcal{C}$, and for $k \geq 1$ set

$$\mathcal{A}_k = \tau_1^{-1} \cdots \tau_k^{-1}\mathcal{A}.$$

The sequence of σ-algebras $(\mathcal{A}_k)$ is strictly decreasing if the transformations τ_n are non-invertible.

The *asymptotic σ-algebra* is defined as the intersection:

$$\mathcal{A}_\infty = \bigcap_{k\geq 1} \tau_1^{-1} \cdots \tau_k^{-1}\mathcal{A}.$$

Let f be in $L^1(m)$. We use the notations of 1.1. Remark that for $f \in L^\infty$ the quotients $|\Pi_n f|/\Pi_n 1$ are bounded by $\|f\|_\infty$ on $\{\Pi_n 1 > 0\}$ and we have $\Pi_n f(x) = 0$ on the set $\{\Pi_n 1 = 0\}$. We define these quotients as 0 on $\{\Pi_n 1 = 0\}$. The following relations hold:

$$m(T_1 \cdots T_n f) = m(f\Pi_n 1), \tag{1.8}$$

$$\mathbb{E}(f|\,\mathcal{A}_k) = T_1 \cdots T_k(\frac{\Pi_k f}{\Pi_k 1}), \tag{1.9}$$

and, for $0 \leq \ell < k \leq n$:

$$\mathbb{E}(T_1 \cdots T_\ell f\,|\,\mathcal{A}_k) = T_1 \cdots T_k(\frac{P_k \cdots P_{\ell+1}(f\ \Pi_\ell 1)}{\Pi_k 1}). \tag{1.10}$$

By the martingale theorem, for every $f \in L^1(m)$, we have convergence of the sequence of conditional expectations $(\mathbb{E}(f|\mathcal{A}_n))_{n\geq 1}$ to $\mathbb{E}(f|\mathcal{A}_\infty)$ and therefore:

$$\lim_n \|T_1...T_n(\frac{\Pi_n f}{\Pi_n 1}) - \mathbb{E}(f|\mathcal{A}_\infty)\|_1 = 0. \text{ in norm } \|\ \|_1 \text{ and } m\text{-a.e.}$$

Definition 1.6. We say that the sequence (τ_n) is **exact** if its asymptotic σ-algebra $\mathcal{A}_\infty$ is trivial.

Exactness implies mixing and is equivalent to $\lim_n \|\Pi_n f\|_1 = 0, \forall f \in L_0^1(m)$, since we have by (1.9):

$$\|\mathbb{E}(f|\mathcal{A}_n)\|_1 = \|T_1...T_n(\frac{\Pi_n f}{\Pi_n 1})\|_1 = \|\Pi_n f\|_1. \tag{1.11}$$

Examples 1.7.

A) We take first for $\mathcal{C}$ the family of translations on a compact group.

1) It is easy to see that mean ergodicity of a sequence (τ_n) defined by $\tau_n x = x + \alpha_n$ is equivalent to the equidistribution of the sequence (u_n) defined by $u_n = \alpha_1 + \cdots \alpha_n$.

For the torus $\mathbb{T}^d$, the property is satisfied by a sequence (α_n) converging modulo 1 to α, if the translation by α modulo 1 is ergodic: for all continuous functions f on the torus, the sequence $(\frac{1}{N}\sum_{n=0}^{N-1} f(.+\alpha_1+\cdots\alpha_n))$ is equicontinuous and any converging subsequence has a limit which is invariant by translation by α and therefore is equal to $m(f)$.

2) Let (X_n) be a sequence of iid r.v. with values in the torus and $\tau_n(\omega)x = x + X_n(\omega)$. We have (almost surely in ω) ergodicity of $(\tau_n(\omega))$ if and only if the law of X_0 is not supported by a lattice (the translations modulo 1 do not belong to a finite subgroup of the circle).

B) Let us consider the following matrices A and B:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We obtain a sequence (τ_n) of transformations which preserve the Lebesgue measure on the torus $\mathbb{T}^3$ by taking for τ_n either $\tau_n x = Ax \bmod 1$, or $\tau_n x = Bx \bmod 1$.

It can be shown that the sequence is exact if each matrix appears infinitely often in the sequence.

C) Hyperbolic automorphisms of the 2-torus

We give now an example with invertible transformations.

Let A be a hyperbolic element in $SL(2,\mathbb{Z})$, (B_n) a sequence in $SL(2,\mathbb{Z})$ such that the sequence $(\text{trace}(B_n))$ is bounded. Let $p \geq 1$ be a fixed integer. We consider the sequence of transformations on the 2-torus defined by:

$$\theta_n x = B_n A^p B_{n-1} A^p ... B_1 A^p x \bmod 1. \tag{1.12}$$

L. Polterovich and Z. Rudnick have called a sequence of the form (1.12) a “kicked system” and defined “stable mixing” for the element A as the property that, for all sequences of kicks (B_k) with bounded trace, there exists p_0 such that the sequence defined by (1.12) is mixing, for every $p \geq p_0$.

They proved ([**PR03**]) that A is stably mixing if and only if A is not conjugate to its inverse.

The following questions can be asked:
- Do we have the CLT for the Hlderian functions ?
- Is there a notion of K-system for a sequence of invertible transformations, which would be satisfied by examples like (1.12) ?
- These questions (stable mixing, CLT,...) can be asked in higher dimension or for other classes of diffeomorphisms.

Before considering in section 3 the example of sequences of expanding maps on the interval $[0,1]$, we prove in the next section some general results on products of operators of “quasi-compact” type.

2. Decorrelation for products of operators

We state in this section some general results which extend to a product of transfer operators classical spectral results for the iterates of a single operator.

Notations - Hypothesis 2.1. *Let $(\mathcal{B}, \| \, \|)$ be a normed space, $\mathcal{V}$ be a subspace of $\mathcal{B}$ equipped with a norm $| \, |_v$ such that $\| \, \| \leq | \, |_v$, $\mathcal{P}$ be a set of contractions of $(\mathcal{B}, \| \, \|)$ leaving the subspace $\mathcal{V}$ invariant.*

We assume the following hypotheses:
(H_1) *The unit ball of $(\mathcal{V}, | \, |_v)$ is relatively compact in $(\mathcal{B}, \| \, \|)$.*
(H_2) *There is a countable family in $\mathcal{V}$ which is dense in $(\mathcal{B}, \| \, \|)$.*
(H_3) *There are an integer $r \geq 1$ and constants $\rho_r \in]0,1[$, $M_0, C_r > 0$ such that:*

$$\forall R \in \mathcal{P}, \ |Rf|_v \leq M_0 |f|_v, \forall f \in \mathcal{V};$$

for all r-tuples $R_1, ..., R_r$ of operators in $\mathcal{P}$:

$$\forall f \in \mathcal{V}, \ |R_r ... R_1 f|_v \leq \rho_r |f|_v + C_r \|f\|. \tag{2.1}$$

The operators we will consider further are adjoint of the operators of composition by expanding transformations of the interval. $\mathcal{B}$ will be $L^1([0,1], m)$, where m is the Lebesgue measure on $[0,1]$, and $\mathcal{V}$ the space of BV (bounded variation) functions. This is the example that we have in mind in this section 2.

Remark 2.2. In applications, we will show (2.1) for some families of operators. Remark that the norm $| \, |'_v$ defined by:

$$| \, f|'_v = |f|_v + \sum_{k=1}^{r-1} \rho_r^{-k/r} \sup_{R_{\ell_k}, ..., R_{\ell_1} \in \mathcal{P}} |R_{\ell_k} ... R_{\ell_1} f|_v$$

is equivalent to the norm $| \, |_v$ and satisfies for a constant C' the inequality:

$$\forall f \in \mathcal{V}, \forall R \in \mathcal{P}, \ |Rf|'_v \leq \rho_r^{1/r} |f|'_v + C' \|f\|.$$

Taking $| \, |'_v$ instead of $| \, |_v$, we can assume during this paragraph that there exists $\rho \in]0,1[$ and $C_0 > 0$ such that the operators R in $\mathcal{P}$ satisfy the inequality:

$$\forall f \in \mathcal{V}, \forall R \in \mathcal{P}, \ |Rf|_v \leq \rho |f|_v + C_0 \|f\|. \tag{2.2}$$

Remark 2.3. If the unit ball $(\mathcal{V}, | \, |_v)$ is not closed in $(\mathcal{B}, \| \, \|)$, we can consider the subspace $\mathcal{V}_1$ of $\mathcal{B}$ defined as follows. For $f \in \mathcal{B}$, we set

$$\|f\|_v = \lim_{\delta \to 0^+} \inf(|\phi|_v, \phi \in \mathcal{V} : \|f - \phi\| < \delta).$$

Let $\mathcal{V}_1 = \{f \in \mathcal{B} : |f|_v < \infty\}$. Now the unit ball of $(\mathcal{V}_1, | \, |_v)$ is compact in $(\mathcal{B}, \| \, \|)$. Replacing $(\mathcal{V}, | \, |_v)$ by $(\mathcal{V}_1, | \, |_v)$, we can therefore assume the *compactness of the unit ball* of $(\mathcal{V}, | \, |_v)$ in $(\mathcal{B}, \| \, \|)$.

From now on we will make this assumption. This implies in particular that, for a sequence (f_n) in $\mathcal{V}$ such that $|f_n|_v \leq C$ and $\|f_n - f\| \to 0$, we have $f \in \mathcal{V}$ and $|f|_v \leq C$.

We define a distance on $\mathcal{P}$ by taking

$$d(R,R') = \sup_{\{f\in\mathcal{V}:|f|_v\le 1\}} \|Rf - R'f\|.$$

For $P \in \mathcal{P}$, we note $B(P,\delta) := \{R \in \mathcal{P} : d(R,P) < \delta\}$.

For convenience we set $R_0 = Id$, the identity operator of $\mathcal{B}$. From (2.2) we get easily the following inequalities:

LEMMA 2.4. *For each $n \ge 1$, for all choices of operators $R_1,...,R_n,P$ in $\mathcal{P}$, all $f \in \mathcal{V}$, we have, setting $M = 1 + C_0(1-\rho)^{-1}$:*

$$|R_n \cdots R_1 f|_v \le \rho^n |f|_v + C_0 \sum_{k=0}^{n-1} \rho^{n-1-k} \|R_k \cdots R_1 R_0 f\|, \tag{2.3}$$

$$|R_n \cdots R_1 f|_v \le M|f|_v,\ n \ge 1, \tag{2.4}$$

$$d(R_n \cdots R_1, P^n) \le M \sum_{j=1}^{n} d(R_j,P),\ n \ge 1. \tag{2.5}$$

LEMMA 2.5. *Let (P_k) be a sequence of operators in $\mathcal{P}$ and $\Pi_n = P_n P_{n-1} ... P_1$. For every strictly increasing sequence of integers there are a subsequence (n_k) of the integers and an operator Λ from $\mathcal{B}$ into $\mathcal{V}$ such that $\Lambda\mathcal{B}$ is contained in $\mathcal{V}$, $dim(\Lambda\mathcal{B}) < +\infty$ and for each $f \in \mathcal{B}$, we have:*

$$\|\Pi_{n_k} f - \Lambda f\| \to 0, \tag{2.6}$$

$$|\Lambda f|_v \le M\|\Lambda f\|. \tag{2.7}$$

Proof: For every $g \in \mathcal{V}$, the set $\{\Pi_n g : n \in I\!N^*\}$ is relatively compact in $(\mathcal{B}, \|\,\|)$ by (H_1) and (2.4). Let $\mathcal{D} = (g_p)_{p\in I\!N}$ be a sequence of elements of $\mathcal{V}$ which is dense in $(\mathcal{B}, \|\,\|)$. Using the diagonal process we obtain a strictly increasing sequence of natural numbers $(n_k) = (\varphi(k))_{k\ge 1}$ such that, for every $p \in I\!N$, the sequence $\left(\Pi_{\varphi(k)} f_p\right)_{k\ge 1}$ converges in $(\mathcal{B}, \|\,\|)$ to a function of $\mathcal{V}$, denoted by Λf_p, satisfying $|\Lambda f_p|_v \le C_0(1-\rho)^{-1}\|f_p\| \le M\|f_p\|$.

Let $f \in \mathcal{B}$ and $(h_p)_{p\in I\!N}$ be a sequence of $\mathcal{D}$ which converges to f in $(\mathcal{B}, \|\,\|)$. The real sequence $(\|h_p\|)_{p\in I\!N}$ converges to $\|f\|$ and $\sup_{p\in I\!N} \|h_p\| < +\infty$. It follows that, for every $p \in I\!N$, $|\Lambda h_p|_v \le M \sup_{n\in I\!N} \|h_n\|$. Taking, if necessary, a subsequence, we can suppose that the sequence $(\Lambda h_p)_{p\in I\!N}$ converges in $(\mathcal{B}, \|\,\|)$ to a function Λf, satisfying $|\Lambda f|_v \le M\ \limsup_p \|h_p\| \le M\|f\|$.

The inequalities

$$\|\Pi_{\varphi(k)} f - \Lambda f\| \le \|\Pi_{\varphi(k)} f - \Pi_{\varphi(k)} h_p\| + \|\Pi_{\varphi(k)} h_p - \Lambda f\| \le \|\,f - h_p\| + \|\Pi_{\varphi(k)} h_p - \Lambda f\|$$

show that, for every $f \in \mathcal{B}$, the sequence $\left(\Pi_{\varphi(k)} f\right)_{k\ge 1}$ converges in $(\mathcal{B}, \|\,\|)$ to Λf.

Now, for each $k, p \in I\!N^*$ such that $k > p$, we can write

$$\Pi_{\varphi(k)} = P_{\varphi(k)} \cdots P_{\varphi(p)+1} \Pi_{\varphi(p)}.$$

As before, taking a subsequence $\big(\varphi(\xi(k))\big)_{k\in I\!N}$ of $\big(\varphi(k)\big)_{k\in I\!N}$ we can suppose that, for every $f \in \mathcal{B}$, the sequence $P_{\varphi(\xi(k))} \cdots P_{\varphi(p)+1} f$ converges in $(\mathcal{B}, \|\,\|)$ to a limit denoted by $\Lambda_p f$ satisfying $|\Lambda_p f|_v \le M\,\|f\|$.

For each $p \in \mathbb{N}^*$, we obtain

$$|\Lambda f|_v = |\Lambda_p \Pi_{\varphi(p)} f|_v \leq M \, \|\Pi_{\varphi(p)} f\| \ ,$$

and consequently

$$|\Lambda f|_v \leq M \, \|\Lambda f\|.$$

The operator Λ defined on $\mathcal{B}$ satisfies (2.6), (2.7), $\Lambda(\mathcal{B}) \subset \mathcal{V}$. By (H_1), (2.7) and Riesz Theorem we have $dim(\Lambda\mathcal{B}) < +\infty$.

□

DEFINITIONS 2.6. In the following, we denote by $\mathcal{V}_0$ a subspace of $\mathcal{V}$ which is *invariant by the operators* $P \in \mathcal{P}$.

We say that a sequence of operators (P_k) in $\mathcal{P}$ is *exact* in $\mathcal{V}_0$ if

$$\lim_n \|P_n P_{n-1} ... P_1 f\| = 0, \forall f \in \mathcal{V}_0. \tag{2.8}$$

A single operator $P \in \mathcal{P}$ is *exact* in $\mathcal{V}_0$ if $\lim_n \|P^n f\| = 0, \forall f \in \mathcal{V}_0$.

• **Decorrelation**:

Property (Dec): We say that a subset of operators $\mathcal{P}_0 \subset \mathcal{P}$ satisfies the decorrelation property (*Dec*) in $\mathcal{V}_0$ if there exist $\lambda \in]0,1[$ and $K > 0$ such that, for all integers $\ell \geq 1$, all ℓ-tuples of operators $R_1, ..., R_\ell$ in $\mathcal{P}_0$:

$$\forall f \in \mathcal{V}_0, \ |R_\ell \cdots R_1 f|_v \leq K \lambda^\ell |f|_v. \tag{2.9}$$

Let $\varepsilon_0 = \frac{(1-\rho)}{2C_0}$, where ρ and C_0 are the constants of (2.2). The following result uses an argument of convolution as in [**CR03**].

PROPOSITION 2.7. *Let $\mathcal{P}_0$ be a subset of $\mathcal{P}$ such that there exists an integer q for which every product of q operators $R_1, ..., R_q$ in $\mathcal{P}_0$ satisfies:*

$$\forall f \in \mathcal{V}_0 \ , \|R_q \cdots R_1 f\| \leq \varepsilon_0 |f|_v. \tag{2.10}$$

Then $\mathcal{P}_0$ verifies the property (Dec) in $\mathcal{V}_0$.

Proof: We can complete the sequence $(R_n \cdots R_1)_{1 \leq n \leq \ell}$ by taking $R_n = R_\ell$, for $n \geq \ell$. Let f be in $\mathcal{V}_0$. We define the sequences $\alpha_f, \zeta, \beta, \beta_q$ (with support in $\mathbb{Z}^+$) by:

$$\begin{aligned}
\alpha_f(n) &= |R_n \cdots R_1 f|_v, \ n \geq 0, \\
\zeta(n) &= C_0 \varepsilon_0 \rho^{n-q-1}, \text{if } n \geq q+1, \ = 0, \text{if } n \leq q, \\
\beta(n) &= B|f|_v \ \rho^n, \ n \geq 0, \ \text{with } B = 1 + C_0 \frac{\rho^{-q} - 1}{1 - \rho},
\end{aligned}$$

$$\beta_q(n) = \begin{cases} \rho^n \, |f|_v + C_0 \sum_{k=0}^{q-1} \rho^{n-1-k} \, |f|_v, & \text{if } n \geq q \\ \rho^n \, |f|_v + C_0 \sum_{k=0}^{n-1} \rho^{n-1-k} \, |f|_v, & \text{if } 1 \leq n \leq q-1 \\ |f|_v, & \text{if } n = 0. \end{cases}$$

We have $\beta_q(n) \leq \beta(n)$, $n \geq 0$, and from (2.3) and (2.10),
for $n \geq q+1$,

$$\begin{aligned} |R_n \cdots R_1 f|_v &\leq \rho^n |f|_v + C_0 \sum_{k=0}^{q-1} \rho^{n-1-k} \|R_k \cdots R_1 f\| \\ &+ C_0 \varepsilon_0 \sum_{k=0}^{n-q-1} \rho^{n-1-q-k} |R_k \cdots R_1 f|_v, \end{aligned}$$

and for $1 \leq n \leq q$:

$$|R_n \cdots R_1 f|_v \leq \rho^n |f|_v + C_0 \sum_{k=0}^{n-1} \rho^{n-1-k} \|R_k \cdots R_1 f\|.$$

Therefore we have: $\alpha_f(n) \leq \beta_q(n) + (\zeta * \alpha_f)(n),\ \forall n \in \mathbb{Z}$, and

$$\alpha_f(n) \leq \sum_{p=0}^{\ell-1} (\zeta^{*p} * \beta_q)(n) + (\zeta^{*\ell} * \alpha_f)(n),\ \forall n \in \mathbb{Z}, \forall \ell \geq 1.$$

Since $(\zeta^{*\ell} * \alpha_f)(n) = 0$, for ℓ such that $\ell(q+1) > n$, this implies

$$|R_n \cdots R_1 f|_v \leq \sum_{p \geq 0} (\zeta^{*p} * \beta_q)(n) \leq \sum_{p \geq 0} (\zeta^{*p} * \beta)(n),\ \forall n \geq 1.$$

For t such that $0 < \rho t < 1$ and $C_0 \varepsilon_0 t^{q+1} < 1 - \rho t$, we have:

$$\begin{aligned} \sum_{p \geq 0} \sum_{n \geq 0} (\zeta^{*p} * \beta)(n)\ t^n &= \sum_{p \geq 0} \big(\sum_{n \geq 0} \zeta(n)\ t^n \big)^p\ \big(\sum_{n \geq 0} \beta(n)\ t^n \big) \\ &= \sum_{p \geq 0} B|f|_v\ (C_0 \varepsilon_0)^p\ t^{p(q+1)}\ \big(\frac{1}{1-\rho t} \big)^{p+1} \\ &= \frac{B|f|_v}{1 - \rho t - C_0 \varepsilon_0 t^{q+1}}. \end{aligned}$$

Let $r(t) = 1 - \rho t - C_0 \varepsilon_0 t^{q+1}$. ¿From the choice of ε_0, we get $C_0 \varepsilon_0 (1-\rho)^{-1} = 1/2$ and therefore $r(1) > 0$.

For $q \geq 1$, the polynomial r has only one real positive root t_0, which is strictly between 1 and $1 + \frac{1}{q+1}$. The other roots have a modulus $> t_0$. Therefore there exists a constant $K > 0$ such that

$$\sum_{p \geq 0} (\zeta^{*p} * \beta)(n) \leq K |f|_v t_0^{-n},\ \forall n \in \mathbb{N}.$$

We deduce that $|R_n \cdots R_1 f|_v$ is bounded by $K \lambda^n |f|_v$, with $\lambda = t_0^{-1}$. The constants K and λ depend only on ρ, C_0 and q.

□

By Lemma 2.5 and Proposition 2.7, we get for an operator P in $\mathcal{P}_0$ the classical spectral properties:

Proposition 2.8. *For all operators $P \in \mathcal{P}$ restricted to $\mathcal{V}$ we have*

$$Pf = L_P f + Q_P f, \tag{2.11}$$

where the spectral radius of Q_P is < 1: there are constants $\gamma_0 \in]0,1[$ and $C_1 > 0$ such that

$$\forall f \in \mathcal{V},\ |Q_P^n f|_v \leq C_1 \gamma_0^n |f|_v, \tag{2.12}$$

L_P is an operator of finite rank of the form

$$L_P(f) = \sum_{j=0}^{p} c_j(f) e_j, \tag{2.13}$$

where the elements $e_j, 1 \leq j \leq p$, $p \in I\!N$, are proper vectors for P with proper values χ_j of modulus 1 and the c_j are linear forms such that $c_j(e_i) = \delta_{i,j}, \forall 1 \leq i, j \leq p$.

If P is exact in $\mathcal{V}_0$, $P|_{\mathcal{V}_0} = Q|_{\mathcal{V}_0}$ and

$$\forall f \in \mathcal{V}_0,\ |P^n f|_v \leq C_1 \gamma_0^n |f|_v. \tag{2.14}$$

If $\mathcal{B}$ is the space $L^1(m)$ for a probability measure m on a space X and if $P \in \mathcal{P}$ is a positive operator, there exists a function $h_P \geq 0$ with maximal support which is P-invariant and the proper values χ_j of P are roots of unity.

Proof: We apply several times Lemma 2.5.

For λ of modulus 1 and f in B, the sequence $\big(S_{n,\lambda} f = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} P^k f\big)_{n \geq 1}$ converges in $(B, \|\ \|)$ either to zero or to a λ-eigenvector $\Pi_\lambda(f)$ of P. Indeed any non-null cluster value of this sequence, which is relatively compact in $(B, \|\ \|)$, is a λ-eigenvector for P. For every integer $p \geq 1$ and every λ-eigenvector h of P, writing $n = \ell p + r$ (Euclidian division), we get:

$$S_{n,\lambda} f - h = \frac{1}{n} \ \big(p \sum_{j=0}^{\ell-1} \lambda^{-j} \ P^j(S_{p,\lambda} f - h) + r \lambda^{-\ell} \ P^\ell(S_{r,\lambda}(f-h))\big)$$

and therefore $\limsup_{n \to +\infty} \|S_{n,\lambda} f - h\| \leq M \|S_{p,\lambda} f - h\|$. This inequality shows that the sequence $\big(S_{n,\lambda} f\big)_{n \geq 1}$ can have only one cluster value ; hence the convergence.

We have that $dim(\Lambda\mathcal{B}) < +\infty$ for any limiting value Λ of (P^n), so that the set $\{\lambda_j : 1 \leq j \leq p\}$ of eigenvalues of modulus 1 of P is finite. The operator $Q_P = P - \sum_{j=1}^p \Pi_{\lambda_j}$ has no eigenvalues of modulus 1.

If $\Lambda = \lim_n Q_P^{\varphi(n)}$ for a subsequence $(\varphi(n))$, we have: $Q_P \Lambda = \Lambda Q_P$ and $|Q^n_{P|\Lambda\mathcal{B}}|_v \to 0$. This implies:

$$\Lambda f = \lim_n Q_P^{\varphi(n+1)-\varphi(n)} Q_P^{\varphi(n)} f = \lim_n Q_P^{\varphi(n+1)-\varphi(n)} \Lambda f = 0.$$

Therefore Q_P is exact in $\mathcal{V}$. By Proposition 2.7 we get the spectral gap for Q_P.

For the last assertion, we use the same arguments as in [**Sc67**] (Appendix).

□

To apply Proposition 2.7 to a subset $\mathcal{P}_0$ of $\mathcal{P}$, we have to check (2.10). We consider two cases: locally, i.e. in a neighbourhood of a given operator, or globally.

• A local result

LEMMA 2.9. *If P is exact in $\mathcal{V}_0$, for every $\varepsilon > 0$, there are an integer $q(\varepsilon) \geq 1$ and a real $\delta(\varepsilon) > 0$ such that, for all products of q operators $R_1, ..., R_q$ in $B(P, \delta(\varepsilon)) \cap \mathcal{P}$, we have:*

$$\forall f \in \mathcal{V}_0, \ \|R_q \cdots R_1 f\| \leq \varepsilon |f|_v.$$

Proof: let $q = q(\varepsilon) \geq 1$ be such that $C_1 \gamma_0^q \leq \frac{\varepsilon}{2}$. For $f \in \mathcal{V}_0$, we have from 2.8:

$$\|P^q f\| \leq \frac{\varepsilon}{2} |f|_v.$$

For all products $R_q \cdots R_1$ such that $R_i \in B(P, \delta(\varepsilon))$, $i = 1, ..., q$, we get by Lemma 2.4:

$$\begin{aligned} \|R_q \cdots R_1 f\| &\leq \|R_q \cdots R_1 f - P^q f\| + \|P^q f\| \\ &\leq qM\delta(\varepsilon)|f|_v + \frac{\varepsilon}{2}|f|_v \leq \varepsilon |f|_v, \end{aligned}$$

for $\delta(\varepsilon)$ such that $qM\delta(\varepsilon) \leq \varepsilon/2$.

□

Applying this result (with $\varepsilon = \varepsilon_0$) and Proposition 2.7 we get:

PROPOSITION 2.10. *If P is exact in $\mathcal{V}_0$, there exists $\delta_0 > 0$ such that the set $\mathcal{P}_0 = B(P, \delta_0)$ satisfies the condition of decorrelation (Dec) in $\mathcal{V}_0$.*

• A non-local result

Let $\mathcal{P}_0$ be a subset of $\mathcal{P}$ such that the following *compactness* condition holds:

Condition (C): For any sequence (R_n) in $\mathcal{P}_0$, there are a subsequence (R_{n_j}) and an operator $R \in \mathcal{P}_0$ such that

$$\forall f \in \mathcal{B}, \ \lim_j \|R_{n_j} f - Rf\| = 0. \tag{2.15}$$

In the next section, we will give examples of families of expanding transformations of the interval for which the set of corresponding transfer operators satisfies this compactness condition and the criterion of the following proposition.

PROPOSITION 2.11. *If $\mathcal{P}_0$ is a set of operators in $\mathcal{P}$ verifying the compactness condition (C) and such that all sequences (P_n) in $\mathcal{P}_0$ are exact in $\mathcal{V}_0$, then it satisfies the decorrelation condition (Dec) in $\mathcal{V}_0$.*

Proof: If (2.10) is not satisfied there are, for each $p \geq 1$, operators $R_{1,p}, \cdots, R_{p,p}$ in $\mathcal{P}_0$ such that $\|R_{p,p} \cdots R_{1,p} f_p\| \geq \varepsilon_0$, for some f_p in $\mathcal{V}_0$ with $|f_p|_v = 1$. As they are contractions of $\mathcal{B}$, this implies:

$$\|R_{\ell,p} \cdots R_{1,p} f_p\| \geq \varepsilon_0, \forall \ell \leq p.$$

By compactness of the unit ball of $\mathcal{V}$ in $\mathcal{B}$ we construct a strictly increasing sequence (p_j) such that (f_{p_j}) converges for the norm $\| \ \|$ to an element g in $\mathcal{V}$ such that $|g|_v \leq 1$. By compactness of $\mathcal{P}_0$ and the diagonal process, there is a subsequence of (R_{r,p_j}) which converges (in the sense of (2.15)) for each r to an operator $\tilde{R}_r \in \mathcal{P}_0$.

Thus we have
$$\|\tilde{R}_\ell \cdots \tilde{R}_1 g\| \geq \varepsilon_0, \forall \ell \geq 1,$$
contrary to the hypothesis. The condition (Dec) follows then from Proposition 2.7.
$\square$

• **Two lemmas**

LEMMA 2.12. *Let $P \in \mathcal{P}$ and $(P_n)_{n\geq 1}$ be operators such that*
$$\lim_n \|P_n f - Pf\| = 0, \forall f \in \mathcal{B}. \tag{2.16}$$
We have, for all integers $r \geq 1$, $\lim_n d(P_n \cdots P_{n-r+1}, P^r) = 0$. There exists a sequence (g_n) in the space $L_P(\mathcal{B})$, image of L_P (cf.(2.11)), such that, uniformly on the unit ball of $\mathcal{V}$:
$$\|P_n \cdots P_1 f - g_n\| \to 0. \tag{2.17}$$

When $\mathcal{B} = L^1(m)$, if $P \in \mathcal{P}$ is exact in $\mathcal{V}_0 = \{f \in \mathcal{V} : m(f) = 0\}$, then (P_n) is exact in $\mathcal{V}_0$ and $\lim_n \|P_n \cdots P_1 f - m(f) h_P\| = 0$, where h_P is P invariant.

Proof: As the P_n's are contractions, the convergence in (2.16) is uniform on the compact sets of $\mathcal{B}$. Therefore we have $\lim_n d(P_n, P) = 0$, which implies the first statement using (2.5).

Let $\varepsilon > 0$. We have, from (2.11) and (2.14) for r big enough,
$$\|P^r P_{n-r} \cdots P_1 f - L_P^r(P_{n-r} \cdots P_1 f)\| = \quad \|Q_P^r P_{n-r} \cdots P_1 f\| \leq C_1 M \gamma_0^r |f|_v \leq \varepsilon.$$
From the first statement, we have, uniformly on the unit ball of $\mathcal{V}$:
$$\lim_n \|P_n P_{n-1} \cdots P_{n-r+1}[P_{n-r} \cdots P_1 f] - P^r[P_{n-r} \cdots P_1 f]\| = 0.$$
This implies (2.17).
$\square$

LEMMA 2.13. *For a constant $C_2 \geq 1$, we have for all integers $p \leq n$:*
$$\begin{aligned}\|P_n \cdots P_1 \varphi - P^n \varphi\| &\leq C_2 \,|\varphi|_v \sum_{k=1}^n \min(d(P_{n-k+1}, P_{n-k}), \gamma_0^{k-1}\} \\ &\leq C_2 \,|\varphi|_v \big(\sum_{k=1}^p d(P_{n-k+1}, P) + (1-\gamma_0)^{-1} \gamma_0^p\big).\end{aligned}$$

Proof: We have from (2.5) and (2.14) :
$$\begin{aligned}\|P_n \cdots P_1 \varphi - P^n \varphi\| &\leq \sum_{k=1}^n \|P^{k-1} P_{n-k+1} \cdots P_1 \varphi - P^k P_{n-k} \cdots P_1 \varphi\| \\ &\leq |\varphi|_v \sum_{k=1}^n \min\{C\, M\, d(P_{n-k+1}, P_{n-k}), C_1\, M\, \gamma_0^{k-1}\}.\end{aligned}$$

$\square$

3. Application to some classes of expanding maps of $[0,1]$

• Classes of expanding maps of the interval

There is a large number of works on expanding maps of the interval, following Lasota-Yorke (1973) ([**LY73**]), Keller ([**Ke80**], [**Ke85**]), Rychlik ([**Ry83**]). (For the central limit theorem for these systems, see in particular the following references: A. Broise [**Br96**], M. Viana [**Vi97**]).

In what follows, we consider the probability space $(X, \mathcal{A}, m)$, where X is $[0,1]$, $\mathcal{A}$ the Borel σ-algebra and m the Lebesgue measure. We apply the results of section 2 to the space $\mathcal{B} = L^1(m)$ with the subspace $\mathcal{V}$ of BV (bounded variation) functions on $[0,1]$.

We write $V(f)$ for the variation of a function $f \in \mathcal{V}$. The space $\mathcal{V}$ is equipped with the norm

$$|f|_v := V(f) + \|f\|_1,$$

where $\| \, \|_1$ is relative to the Lebesgue measure. For $f \in \mathcal{V}$, we have: $\|f\|_\infty \le |f|_v$. The hypotheses (H_1) and (H_2) of section 2 are satisfied.

We consider a class $\mathcal{C}$ of piecewise expanding transformations τ of $I = [0,1]$ and the corresponding set of transfer operators

$$\mathcal{P} = \{P_\tau, \tau \in \mathcal{C}\}.$$

If (τ_n) is a sequence of transformations in $\mathcal{C}$, by composition we get the sequence:

$$\theta_n = \tau_n \circ \tau_{n-1} \cdots \circ \tau_1, n \ge 1.$$

Let us recall the following notations

$$\begin{aligned} \mathcal{A}_k &= \tau_1^{-1} \cdots \tau_k^{-1} \mathcal{A}, \\ \Pi_k &= P_k \cdots P_1, \end{aligned}$$

where $P_k = P_{\tau_k}$.

We assume that the transformations τ in $\mathcal{C}$ satisfy the following hypothesis:

Hypothesis 3.1. There exists a finite or countable partition (I_j) of I such that the restriction of τ to each interval I_j is strictly monotone on I_j and can be extended into a derivable function with a BV derivative on $\overline{I}_j$. The transformations τ satisfy:

$$\gamma(\tau) := \inf_j \inf_{x \in I_j} |\tau'(x)| > 1, \tag{3.1}$$

$$K := \sup_j \sup_{x \ne y \in I_j} \left|\frac{\tau'(x) - \tau'(y)}{x - y}\right| < \infty. \tag{3.2}$$

Let us remark that the subspace $\mathcal{V}_0 = \{f \in \mathcal{V} : m(f) = 0\}$ is invariant by the operators $P_\tau, \tau \in \mathcal{C}$.

To get (2.1) which allows to apply the results of section 2, we have to consider classes $\mathcal{C}_0$ of transformations in $\mathcal{C}$ for which the following condition holds:

Condition (D_r): We will say that a class $\mathcal{C}_0$ in $\mathcal{C}$ satisfies (D_r) if the corresponding set of transfer operators $\mathcal{P}_0 = \{P_\tau, \tau \in \mathcal{C}_0\}$ verifies the conditions (H_3) of section 2.

In particular, if $\mathcal{C}_0$ satisfies (D_r) for some integer $r \geq 1$, each operator P_τ corresponding to $\tau \in \mathcal{C}_0$ satisfies for constants $\rho_r \in]0,1[$, C_r the so-called Lasota-Yorke (or Doeblin-Fortet)[1] inequality:

$$\forall f \in \mathcal{V},\ |P_\tau^r f|_v \leq \rho_r |f|_v + C_r \|f\|_1. \tag{3.3}$$

By proposition 2.8, P_τ has in the space $\mathcal{V}$ an invariant function $h_\tau \geq 0$ of greatest support such that $m(h_\tau) = 1$ and this function is the density of an ACIM for τ.

As a consequence of Lemma 2.5, we have also:

PROPOSITION 3.2. *: For any sequence (τ_n) belonging to a set of transformations $\mathcal{C}_0$ which satisfies (D_r) for some integer $r \geq 1$, the asymptotic σ-algebra $\mathcal{A}_\infty = \bigcap_{k\geq 1} \tau_1^{-1} \cdots \tau_k^{-1} \mathcal{A}$ is finite.*

• A counterexample to stability

If we perturb a transformation τ_0 verifying (3.3), the inequality (3.3) with bounded constants C_r independent from the perturbed transformations τ can be lost. A counterexample has been given by G. Keller in [**Ke82**] and M. Blank and G. Keller in [**BK97**]. Let us recall it.

EXAMPLE 3.3.

Let r and b be two parameters such that $b \geq 1/2$ and $0 < r < 1/4$. We consider the transformation τ_b of $[0,1]$ into itself defined by: $\tau_b(x) = 1 - x/r$, for $0 \leq x \leq r$, $\tau_b(x) = 2b(1-2r)^{-1}(x - r)$, for $r \leq x \leq 1/2$, $\tau_b(x) = \tau_b(1-x)$, for $1/2 \leq x \leq 1$.

Each transformation τ_b has an unique ACIM (cf. [**Ko75**]). Let h_b be its density. For $1/2 < b \leq 1-2r$, we have: $\tau_b([1-b, b]) \subset [1-b, b]$, which implies that h_b has its support in the interval $[1-b, b]$. If (b_n) is a sequence such that $1/2 < b_n \leq 1-2r$ and $\lim_n b_n = 1/2$, the sequence of invariant measures $(h_{b_n} m)$ converges weakly to the measure $\delta_{1/2}$, the Dirac mass at the point $1/2$. Therefore the transformation $\tau_{1/2}$ is not stochastically stable in $\{\tau_b, 1/2 < b \leq 1-2r\}$.

This counterexample to stochastic stability gives also a counterexample to the property of sequential stability 1.5. It is characteristic of the obstruction to stochastic or sequential stability. If we write to simplify $\tau_n = \tau_{b_n}$, we get, using the notations of 1.1 that we can have $\lim \|\Pi_n 1\|_\infty = +\infty$, if the convergence of (b_n) to $1/2$ is slow enough. In particular in that case, the family $(\Pi_n 1)$ is not bounded in variation.

[1]In their paper of 1937 [**DF37**], Wolfgang Doeblin and Robert Fortet introduced the technique of what later has been called "quasi-compact operators". For the study of the "chanes liaisons compltes", a concept due to O. Onicescu and Gh. Mihoc, they used an inequality of the type of 3.3. In [**LY73**] A. Lasota and J. A. Yorke proved and used this type of inequality (in the BV-norm) in the context of dynamical systems, for expanding maps of the interval.

• **Classes where (D_r) holds**

We give now a series of examples where the condition (D_r) (i.e. inequality (2.1)) can be obtained. To get it, the method is the same as for a single transformation or a random product of transformations (cf. [**LY73**], [**Ke80**],[**Li95**],[**Br96**], [**Vi97**]), [**Bu99**]).

We will consider the following assumption on a transformation τ. We suppose that for a subdivision $0 = a_0 < a_1 < a_2 < \ldots < a_p = 1$ of $[0,1]$, the restriction of τ to $I_j =]a_j, a_{j+1}[$ is C^1 and strictly monotone. Let σ_j be the inverse application of the restriction of τ to I_j. Write $b_j^+ = \lim_{x \to a_j^+} \tau(x)$ (resp. $b_j^- = \lim_{x \to a_j^-} \tau(x)$). The condition is:

$$\forall n \geq 0, \forall j, \tau^n(b_j^{\pm}) \notin \{a_0, a_1, \ldots, a_p\}. \tag{3.4}$$

THEOREM 3.4. *Condition (D_r) is satisfied by each following family $\mathcal{C}_0$ of transformations:*

a) $\mathcal{C}_0$ is a class of transformations $\tau \in \mathcal{C}$ such that the coefficient $\gamma(\tau)$ of dilatation defined by (3.1) verifies $\gamma(\tau) \geq 2 + a$, $a > 0$ independent of τ;

b) $\mathcal{C}_0$ is a (convenient) neighbourhood in $\mathcal{C}$ of a transformation τ verifying (3.4);

c) $\mathcal{C}_0$ is the family of transformations $\tau : x \to \beta x \bmod 1$ (β-transformations) such that $\beta \geq 1 + a$, $a > 0$ independent of τ;

Proof:

1) Let τ be a transformation $\mathcal{C}$. The corresponding transfer operator P_τ is given by:

$$P_\tau f(x) = \sum_j f(\sigma_j x) \frac{1}{|\tau'(\sigma_j x)|} 1_{\tau(I_j)}(x).$$

If φ is a function on $[0,1]$ and $J =]u, v[\subset [0,1]$, the variation of $\varphi 1_J$ can be bounded via the following inequality where $[u,v] \subset [c,d] \subset [0,1]$:

$$\begin{aligned} V(\varphi\, 1_J) &\leq V_{[u,v]}(\varphi) + |\varphi(u)| + |\varphi(v)| \leq V_{[u,v]}(\varphi) + V_{[c,d]}(\varphi) + 2 \inf_{t \in [c,d]} |\varphi(t)| \\ &\leq V_{[u,v]}(\varphi) + V_{[c,d]}(\varphi) + \frac{2}{m([c,d])} \int_{[c,d]} |\varphi(t)|\, dt. \end{aligned} \tag{3.5}$$

If $[c,d] = [u,v]$, the inequality reduces to:

$$\begin{aligned} V(\varphi\, 1_J) &\leq V_{[u,v]}(\varphi) + |\varphi(u)| + |\varphi(v)| \leq V_{[u,v]}(\varphi) + V_{[u,v]}(\varphi) + 2 \inf_{t \in [u,v]} |\varphi(t)| \\ &\leq 2V_{[u,v]}(\varphi) + \frac{2}{m([u,v])} \int_{[u,v]} |\varphi(t)|\, dt. \end{aligned} \tag{3.6}$$

2 a) We apply first (3.6) for the intervals $[c,d]$ and $\tau(I_j) =]\alpha_j, \beta_j[$. This gives for the variation of $P_\tau f$:

$$\begin{aligned}
V(P_\tau f) &\leq \sum_j V[(\frac{f}{\tau'}) \circ \sigma_j \ 1_{\tau(I_j)}] \\
&\leq \sum_j V_{[\alpha_j,\beta_j]} \ [(\frac{f}{\tau'}) \circ \sigma_j] + \sum_j [|(\frac{f}{\tau'})(\sigma_j \alpha_j)| + |(\frac{f}{\tau'})(\sigma_j \beta_j)|] \\
&\leq 2 \sum_j V_{[\alpha_j,\beta_j]} \ [(\frac{f}{\tau'}) \circ \sigma_j] + \sum_j \frac{1}{m(\tau(I_j))} \int_{[\alpha_j,\beta_j]} |(\frac{f}{\tau'})(\sigma_j x)| \ dx \\
&\leq 2 \sum_j V_{[\alpha_j,\beta_j]} \ [(\frac{f}{\tau'}) \circ \sigma_j] + \sum_j \frac{1}{m(\tau(I_j))} \int_{[a_j,a_{j+1}]} |f| \ dx.
\end{aligned}$$

The first term on the right (cf. [**Br96**]) is less than

$$\frac{2}{\gamma} V(f) + 2\frac{K}{\gamma^2} \|f\|_1.$$

The second term is less than $\frac{2}{\delta}\|f\|_1$, where $\delta = \inf_j m(\tau I_j)$. Therefore we get:

$$V(P_\tau f) \leq \frac{2}{\gamma} V(f) + 2\frac{K}{\gamma^2} \|f\|_1 + \frac{2}{\delta}\|f\|_1. \tag{3.7}$$

The inequality (3.7) implies that (D_r) is satisfied for $r = 1$ in the case a) of the proposition.

2 b) For this case, we refer to Viana [**Vi97**].

2 c) *β-transformations*

Let $a > 0$ and β_k, $k \geq 1$, be real numbers such that $\beta_k \geq 1 + a$. Denote by $\tau_k : x \to \beta_k x \bmod 1$, the corresponding β-transformations and write $\theta_r(x) = \tau_r ... \tau_1 x$. We show the existence of $r > 0$ depending only on a such that (D_r) is verified.

Let us consider the partition $\mathcal{P}_n$ of $[0,1]$ into monotonicity intervals of θ_n for a given n. We call **full** intervals of rank n the open intervals $J \in \mathcal{P}_n$ such that the transformation θ_n applies J surjectively on $I =]0,1[$.

Let $J_1^p, ..., J_t^p$ denote the full intervals in increasing order and $J_{k,1}, ... J_{k,\ell(k)}$, for $k = 1, ..., t$, the non-full consecutive intervals between J_k^p and J_{k+1}^p.

Let $w(r) = \max_{k=1,...,t} \ell(k)$ be the maximal number of non-full intervals separated by two full intervals (we include the case of contiguous full intervals at the left of the end point 1).

If $[u,v] = J_{k,j}$ is a non-full interval, we bound the variation on this interval by an application of (3.5) to $[u,v]$ and to the interval $[c,d] = \tilde{J}_k^p = J_k^p \cup_{j=1}^{\ell(k)} J_{k,j}$ which is the union of the full interval J_k^p (at the left of $[u,v]$) and of the non-full intervals which are consecutive to J_k^p.

Let $\pi_r = \beta_1 ... \beta_r$. As $m(J) = \pi_r^{-1}$ if J is a full interval and the intervals $\tilde{J}_k^p$ are pairwise disjoint, we get:

$$\begin{aligned} V(P_r...P_1 f) &\leq \frac{1}{\pi_r} \sum_{J \in \mathcal{P}_r} V_J(f) + 2\frac{w(r)}{\pi_r} \sum_k [V(\tilde{J}_k^p, f) + \frac{1}{m(\tilde{J}_k^p)} \int_{\tilde{J}_k^p} |f| \; dt \;] \\ &\leq \frac{1 + 2w(r)}{\pi_r} V(f) + 2w(r) \|f\|_1. \end{aligned}$$

For β-transformations, we have $w(r) \leq r$. Indeed let J_k^p be a full interval and $J_{k,1}, ..., J_{k,\ell(k)}$ non-full intervals following J_k^p at the right. At the next step J_k^p can give rise possibly at the right to a monotonicity non-full interval for θ_{r+1} ; on the other hand, if one of the intervals $J_{k,j}$ gives rise to more then one monotonicity interval for θ_{r+1}, among these intervals at least one is full and at most one is not full. This shows that $w(r+1) \leq w(r) + 1$. Therefore we have $w(r) \leq r, \forall r \geq 1$. If we choose r such that $1 + 2r < \pi_r$ we get (D_r).

□

- **Exactness of some sequences of expanding maps**

A) Transformations $\tau : x \to \beta x + \alpha \bmod 1$, with $\beta > 2$.

We take for $\mathcal{C}_0$ the family of transformations of the interval $I = [0,1]$ of the form $\tau : x \to \beta x + \alpha \bmod 1$, with $\beta \geq 2 + a$, where a is a fixed > 0 real.

In the following lemma, we extend a result of Wilkinson [**Wi74**] for a single transformation $\tau : x \to \beta x + \alpha \bmod 1$ to the case of a sequence (τ_k).

LEMMA 3.5. *For every $\varepsilon > 0$, there exists an integer $r \geq 0$ such that, for every $n \geq 1$, we can cover I, up to a set of measure less then ε, by full intervals of rank between n and $n + r$.*

Proof: Let π_n be the product $\pi_n = \beta_n \beta_{n-1} ... \beta_1$, $T(n)$ the number of atoms of the partition $\mathcal{P}_n$, $F(n)$ the number of full intervals of rank n

If J is a full interval of rank n, we have: $\pi_n m(J) = 1$, and therefore (summing on full intervals of rank n) we get $F(n) = \pi_n \sum_J m(J) \leq \pi_n$.

Since an interval of rank $n-1$ gives at rank n at most 2 non-full intervals we have: $T(n) \leq \pi_n + 2T(n-1)$; hence, for a constant C:

$$T(n) \leq \pi_n + 2\pi_{n-1} + ... + 2^{n-1} \leq \pi_n (1 + \frac{2}{\beta_n} + \frac{2^2}{\beta_n \beta_{n-1}} + ... + \frac{2^{n-1}}{\beta_n \beta_{n-1} .. \beta_1}) \leq C \pi_n.$$

Let $\varepsilon > 0$ be given and n an integer ≥ 1. Let us take first the partial covering of $[0,1]$ by full intervals of rank n. There remains non-full intervals of rank n (at most $T(n)$) which we partially cover by full intervals of rank $n+1$. This step gives rise to at most $2T(n)$ non-full intervals. Using this procedure up until $n + r$, there remains a non-covered set which is formed of non-full intervals of measure at most $\frac{1}{\pi_{n+r}}$ and whose total measure is less then

$$\frac{2^r T(n)}{\pi_{n+r}} \leq \frac{C 2^r \pi_n}{\pi_{n+r}} = \frac{C 2^r}{\beta_{n+r} \beta_{n+r-1} \cdots \beta_{n+1}} \leq C (\frac{2}{2+a})^r \leq \varepsilon,$$

for r big enough (independent from n).

□

B) Transformations $\tau : x \to \beta x \bmod 1$.

Let us take now for $\mathcal{C}_0$ the family of transformations of $[0,1]$ of the form $\tau : x \to \beta x \bmod 1$, with $\beta \geq 1 + a$, where a is a fixed > 0 real.

By the same argument, changing only the recurrence formula into : $T(n) \leq \pi_n + T(n-1)$, we have $T(n) \leq \sum_1^n \pi_k$ and therefore:

$$\frac{T(n)}{\pi_{n+r}} \leq \frac{\sum_1^n \pi_k}{\pi_{n+r}} = \frac{1}{\beta_{n+r}\beta_{n+r-1}...\beta_1}(\pi_n + \pi_{n-1} + ... + 1)$$
$$\leq \quad \frac{1}{\beta_{n+r}\beta_{n+r-1}...\beta_{n+1}}(1 + \frac{1}{\beta_n} + \frac{1}{\beta_n\beta_{n-1}} + ... + \frac{1}{\beta_n\beta_{n-1}..\beta_1}) \leq C(1+a)^{-r}.$$

Theorem 3.6. *For both families of transformations A) and B), any sequence* (τ_n) *is exact.*

Proof: The proof is that of [**Ro61**] for the iterates of a single β-transformation.

Let A be in the asymptotic σ-algebra with $m(A) > 0$ Let us show that $m(A) = 1$. We have $A = \theta_n^{-1}\theta_n A$. Let $\varepsilon > 0$.

By Lemma 3.5 the family of full intervals (of arbitrary rank) generates the Borel σ-algebra. Therefore there exists a full interval J such that

$$m(J \cap A) \geq (1 - \varepsilon)m(J).$$

(If not, we would have $m(B \cap A) \leq (1 - \varepsilon)m(B)$, for every B, in particular for $B = A$.)

Let n be the rank of J. The restriction of θ_n to J is an affine bijection from J onto $]0,1[$. We have:

$$m(\theta_n(J - J \cap A)) = m(J - J \cap A)/m(J) = 1 - m(J \cap A)/m(J) \leq \varepsilon;$$

therefore:

$$\begin{aligned} m(\theta_n A) \geq m(\theta_n(J \cap A)) \quad &\geq \quad m(\theta_n J) - m(\theta_n(J - J \cap A)) \\ &\geq \quad m(\theta_n J) - \varepsilon = 1 - \epsilon, \\ m(\theta_n^{-1}(\theta_n A)^c) \quad &= \quad \int 1_{(\theta_n A)^c}\, \Pi_n 1\, dm \leq C\varepsilon. \end{aligned}$$

□

- **Decorrelation, law of large numbers**

Let $\mathcal{C}_0$ be a set of transformations of $[0,1]$ such that:
1) $\mathcal{C}_0$ one of the families satisfying the statement of Theorem 3.4;
2) any sequence (τ_n) in $\mathcal{C}_0$ is exact;
3) $\mathcal{C}_0$ verifies a compactness condition: if (σ_n) is a sequence in $\mathcal{C}_0$, there exists a subsequence (σ_{n_j}) and a transformation $\tau \in \mathcal{C}_0$ such that

$$\lim_j \|f \circ \sigma_{n_j} - f \circ \tau\|_1 = 0, \forall f \in L^1(m). \tag{3.8}$$

This condition implies the compactness condition (C) for the corresponding set of transfer operators $\mathcal{P}_0 = \{P_\sigma, \sigma \in \mathcal{C}_0\}$. In the previous paragraph we provided some examples of such sets of transformations.

The hypothesis of Proposition 2.11 is satisfied and therefore Proposition 2.7 can be applied. *Therefore the condition of decorrelation (Dec) is satisfied by* $\mathcal{C}_0$.

Consider now transformations τ_n, $n = 1, 2, ...$, in the class $\mathcal{C}_0$. Let f be a function in $\mathcal{V}$. Using the notations (1.1), we set

$$\tilde{f}_k := f - m(f\Pi_k 1). \tag{3.9}$$

It follows from (1.10) that:

$$\begin{aligned} \mathbb{E}(T_1 \cdots T_k \tilde{f}_k \; T_1 \cdots T_\ell \tilde{f}_\ell) &= \mathbb{E}(T_1 \cdots T_k \tilde{f}_k \; T_1 \cdots T_k (\frac{P_k \cdots P_{\ell+1} \tilde{f}_\ell \; \Pi_\ell 1)}{\Pi_k 1})) \\ &= \mathbb{E}(\tilde{f}_k \; P_k \cdots P_{\ell+1} (\tilde{f}_\ell \; \Pi_\ell 1)). \end{aligned}$$

As $\tilde{f}_\ell \; \Pi_\ell 1 \in \mathcal{V}_0$ and

$$|\tilde{f}_\ell \; \Pi_\ell 1|_v \le 3M |f|_v, \tag{3.10}$$

the condition (*Dec*) implies:

$$|\int T_1 \cdots T_k \; \tilde{f}_k \; T_1 \cdots T_\ell \; \tilde{f}_\ell \; dm| \le 3MD' \lambda^{-|k-\ell|} |f|_v \|f\|_1.$$

It is well known that, if (Z_n) is a sequence of centered square integrable random variables such that $|\mathbb{E}(Z_n Z_{n+\ell})| \le \varepsilon_\ell$, where (ε_n) is a summable sequence, the law of large numbers holds for the sequence (Z_n). This implies the following law of large numbers:

THEOREM 3.7. *Under the previous conditions, we have, for $f \in \mathcal{V}$ and m-a.e. x,*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [f(\tau_k \cdots \tau_1 x) - \int T_1 \cdots T_k f \; dm] = 0.$$

• Application in a neighbourhood of a transformation, equidistribution

Due to Proposition 2.10 and Theorem 3.4, the result of Theorem 3.7 is valid in a neighbourhood of an exact transformation τ in $\mathcal{C}$ which satisfies (3.4). Let τ_n, $n = 1, 2, ...$, be transformations in $\mathcal{C}$ such that $\lim_n |\tau_n x - \tau x| = 0$, for each $x \in I$.

The exact transformation τ has an unique ACIM with density h and we have by Lemma 2.12 $\lim_n \int f \; \Pi_k 1 dm = \int f \; h \; dm$; therefore:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \int T_1 \cdots T_k f \; dm = \int f \; h \; dm.$$

We deduce from it the following equidistribution theorem (*sequential stability*):

THEOREM 3.8. *1) If $\tau = \lim_n \tau_n$, if τ is exact and verifies (3.4), then for m-a.a. x, the asymptotic distribution of the sequence $(\theta_n(x) = \tau_n \cdots \tau_1 x)_{n \ge 0}$ is given by the measure hm: for every BV function f, we have*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f(\tau_k \cdots \tau_1 x) = \int_0^1 f \, h \; dm.$$

2) If (β_n) is a sequence such that $\lim_n \beta_n = \beta > 1$, and $(\theta_n(x))_{n\geq 0}$ is defined by $\theta_0(x) = x$, $\theta_{n+1}(x) = (\beta_n \theta_n(x))$ mod $1, n \geq 0$, then we have the same conclusion: for almost all x, the sequence $(\theta_n(x))_{n\geq 0}$ is distributed according to the measure $h\, m$, where h is the density of the ACIM for $\tau : x \to \beta x$ mod 1.

In particular if β is an integer ≥ 2, the sequence $(\theta_n(x))_{n\geq 0}$ is uniformly distributed on $[0, 1]$ for a.a. x.

• Rate of convergence of $(\Pi_n\varphi)$ to $m(\varphi)\, h$, case of β-transformations

The convergence given by Lemma 2.12 is qualitative. For the β-transformations, the rate of convergence is related to the rate of convergence of (β_n) towards β. To prove it, we need a measure of the regularity of functions.

For a real $t > 0$ and f a bounded Borel function, we set

$$\begin{aligned} w(f, x, t) &= \sup_{|y-x|\leq t} |f(y) - f(x)|, \\ \widetilde{w}(f, t) = \int_0^1 w(f, x, t)\, dm(x) &= \int_0^1 \sup_{|y-x|\leq t} |f(y) - f(x)|\, dm(x). \end{aligned}$$

By Fubini's Theorem, we get:

$$\widetilde{w}(f, t) \leq 2tV(f). \tag{3.11}$$

Lemma 3.9. *There exists a constant C such that for any two reals $\beta_1, \beta_2 > 1$ with P_1, P_2 the transfer operators corresponding resp. to the transformations $x \to \beta_1 x$ mod 1, $x \to \beta_2 x$ mod 1, we have:*

$$d(P_1, P_2) \leq C|\beta_1 - \beta_2|. \tag{3.12}$$

Proof: To simplify we assume that $\beta_2 > \beta_1$ and $[\beta_2] = [\beta_1]$.

We have:

$$\begin{aligned} |P_2 f(x) - P_1 f(x)| &= |\sum_{k=0}^{[\beta_1]-1} \frac{1}{\beta_2} f(\frac{x+k}{\beta_2}) + \frac{1}{\beta_2} f(\frac{x+[\beta_1]}{\beta_2}) 1_{[0,\{\beta_2\}]}(x) \\ &\quad - \sum_{k=0}^{[\beta_1]-1} \frac{1}{\beta_1} f(\frac{x+k}{\beta_1}) - \frac{1}{\beta_1} f(\frac{x+[\beta_1]}{\beta_1}) 1_{[0,\{\beta_1\}]}(x)| \\ &\leq (1 - \frac{\beta_2}{\beta_1}) P_2|f|(x) + \sum_{k=0}^{[\beta_1]-1} \frac{1}{\beta_1} |f(\frac{x+k}{\beta_2}) - f(\frac{x+k}{\beta_1})| \\ &\quad + \frac{1}{\beta_1} |f(\frac{x+[\beta_1]}{\beta_1})| 1_{[0,\{\beta_1\}]}(x)| + \frac{1}{\beta_1} |f(\frac{x+[\beta_1]}{\beta_2})| 1_{[0,\{\beta_2\}]}(x)| \\ &\leq \frac{|\beta_2 - \beta_1|}{\beta_1} P_2|f|(x) + P_1 w(f, ., \frac{1}{\beta_2} - \frac{1}{\beta_1})(x) \\ &\quad + \frac{1}{\beta_1} \|f\|_\infty\, |1_{[0,\{\beta_1\}]}(x) - 1_{[0,\{\beta_2\}]}(x)|. \end{aligned}$$

Since m is P-invariant, we have:

$$\int Pw(f, ., \frac{1}{\beta_2} - \frac{1}{\beta_1})(x)\, dm(x) = \tilde{w}(f, \frac{1}{\beta_2} - \frac{1}{\beta_1}).$$

Integrating in the previous equality and using 3.11, we get for some constant C:

$$\|P_2 f - P_1 f\|_1 \leq C|f|_v|\beta_2 - \beta_1|.$$

□

Using this Lemma and Lemma 2.13, we can now give a rate of convergence of the sequence $(\Pi_n\varphi)$ to $m(\varphi)h$, for a BV function φ. Recall that the constant γ_0 as defined in Proposition 2.8, inequality (2.14).

LEMMA 3.10. *There exists a constant $C_1 \geq 1$ such that for all integers $p \leq n$*

$$\begin{aligned} \|P_n \cdots P_1\, \varphi - P^n\, \varphi\|_1 &\leq C_1\, |\varphi|_v \sum_{k=1}^{n} \min\{|\beta_{n-k+1} - \beta|, \gamma_0^{k-1}\} \\ &\leq C_1\, |\varphi|_v \,\big(\sum_{k=1}^{p} |\beta_{n-k+1} - \beta| + (1-\gamma_0)^{-1}\, \gamma_0^p\big). \end{aligned}$$

As a corollary of the previous result we get:

COROLLARY 3.11. *If $|\beta_n - \beta| < \frac{1}{n^\theta}$, with $\theta > 0$, we have, for a constant A independent from n and from $\varphi \in \mathcal{V}$:*

$$\|\Pi_n\varphi - m(\varphi)\, h\|_1 \leq \frac{A \log n}{n^\theta} |\varphi|_v.$$

4. A Borel-Cantelli lemma

A law of large numbers of the type of the Borel-Cantelli Lemma, stronger than Theorem 3.7, can be obtained under a condition of minoration for the sequence $(\Pi_n 1)$, which can be checked for the β-transformations, when the β_n's are in a convenient neighbourhood of a fixed β.

Condition (Min): There exists $\delta > 0$ such that $\Pi_n 1(x) \geq \delta$, $\forall x \in [0,1], \forall n \geq 0$.

This condition and boundedness of the sequence of functions $(\Pi_n 1)$ imply that the integrals $\int T_1 \cdots T_k f\; dm$ and $\int f\; dm$ are of the same order, i.e. for $f \geq 0$ and all $k \geq 1$,

$$\delta m(f) \leq m(T_1 \cdots T_k f) \leq \sup_n \|\Pi_n 1\|_\infty m(f).$$

THEOREM 4.1. *Assuming the conditions (Dec) and (Min), if $(f_n)_{n\geq 0}$ is a sequence of positive BV functions such that*

$$\sum_{n\geq 0} m(f_n) = +\infty \quad \textit{and} \quad \sup_{n\geq 0} |f_n|_v < +\infty,$$

then the sequence $(F_n)_{n\geq 0}$ defined by

$$F_n(x) = \frac{\sum_{k=0}^n f_k(\tau_k \cdots \tau_1 x)}{\sum_{k=0}^n m(T_1 \cdots T_k f_k)}, n \geq 0, \tag{4.1}$$

converges m-p.p. to 1.

Proof: 1) We show that there is a constant C such that

$$\int_E (F_n - 1)^2 \, dm \leq \frac{C}{\sum_{k=0}^n m(T_1 \cdots T_k f_k)}.$$

Write $\tilde{f}_k = f_k - m(f_k \Pi_k 1), \forall k \geq 0$. We have for some constants C_1, C_2, for each $n \geq 1$:

$$\begin{aligned}
\Big(\sum_{k=0}^n m(T_1 \cdots T_k f_k)\Big)^2 & \int_E (F_n - 1)^2 \, dm \\
& \leq 2 \sum_{0 \leq k \leq \ell \leq n} \int_E T_1 \cdots T_k \tilde{f}_k \; T_1 \cdots T_\ell \tilde{f}_\ell \, dm \\
& \leq C_1 \sum_{0 \leq \ell \leq n} \big(\sum_{0 \leq k \leq \ell} \lambda^{-(\ell-k)} |f_k|_v \big) \; m(f_\ell) \\
& \leq C_2 \sup_j |f_j|_v \sum_{0 \leq \ell \leq n} m(f_\ell) \\
& \leq C_2 \frac{1}{\delta} \sup_j |f_j|_v \sum_{0 \leq \ell \leq n} m(T_1 \cdots T_\ell f_\ell),
\end{aligned}$$

hence the result.

2) As the functions (f_n) are uniformly bounded, we can assume $\|f_n\|_\infty \leq 1$. Let ψ be the sequence defined by $\psi(n) = \inf\{\ell : \sum_{k=0}^\ell m(T_1 \cdots T_k f_k) \geq n^2\}, n \geq 1$. We have

$$n^2 \leq \sum_{k=0}^{\psi(n)} m(T_1 \cdots T_k f_k) \leq (n+1)^2, n \geq 1.$$

The subsequence $(F_{\psi(n)})$ converges m-a.s. to 1, since

$$\sum_{n \geq 1} \int_E |F_{\psi(n)} - 1|^2 \, dm \leq \sum_{n \geq 1} \frac{C}{n^2} < +\infty.$$

For $n \in I\!N$, let $r = r(n)$ such that $\psi(r) \leq n \leq \psi(r+1)$. The inequalities

$$(\frac{r}{r+2})^2 \, F_{\psi(r)} \leq F_n \leq \frac{\sum_{k=0}^{\psi(r+1)} m(T_1 \cdots T_k f_k)}{\sum_{k=0}^{\psi(r)} m(T_1 \cdots T_k f_k)} F_{\psi(r+1)} \leq (\frac{r+2}{r})^2 \, F_{\psi(r+1)}$$

imply the convergence of $(F_n)_{n \geq 0}$ m-a.s. towards 1.

□

Remark 4.2. The previous "Borel-Cantelli Lemma" applied to sequences of regular sets gives information on the visits of regular sets of small measure by the sequence $(\theta_n(x))$.

Let (B_n) be a sequence of sets such that $\sup_n |1_{B_n}|_v < \infty$. Then if (*Dec*) and (*Min*) are satisfied, we have either: $\sum_k m(B_k) < +\infty$ or

$$\sum_{k \geq 0} 1_{B_k}(\theta_k x) = +\infty, \text{ for a.a. } x.$$

For example in the case of $[0, 1]$, we can take for (B_n) a sequence of intervals of length $1/n$. We get that, infinitely often, $\theta_n x \in B_n$.

For $r > 0$ and $a \in]0, 1[$, let $k(x, r) := \inf\{\ell : \theta_\ell(x) \in [a - r, a + r]\}$.

Let γ be any real > 1, $M = \sup_n \|\Pi_n 1\|_\infty$. Let us take $\varepsilon > 0$ such that $\delta(\gamma - 1) > 2M\gamma\varepsilon$.

By (4.1) there exists $N = N(\gamma, \varepsilon, x)$ such that, for $n \geq N$,

$$\begin{aligned}\sum_n^{n^\gamma} 1_{B_k}(\theta_k x) &\geq (1-\varepsilon)\Big(\sum_1^{n^\gamma} m(\theta_k^{-1} B_k)\Big) - (1+\varepsilon)\Big(\sum_1^{n} m(\theta_k^{-1} B_k)\Big) \\ &\geq \delta \sum_n^{n^\gamma} m(B_k) - 2M\varepsilon\Big(\sum_1^{n^\gamma} m(B_k)\Big) \geq \delta(\gamma-1)\mathrm{Log} n - 2M\varepsilon\gamma \mathrm{Log} n.\end{aligned}$$

This implies that $k(x, 1/n) \leq n^\gamma$, for $n \geq N(\gamma, \varepsilon, x)$ and therefore for a.a. x:

$$\limsup_{r\to 0} \frac{\mathrm{Log} k(x,r)}{\mathrm{Log} 1/r} \leq 1.$$

• **Minoration of $\Pi_n 1$ (case of β-transformations)**

In this paragraph, we show that the condition (*Min*) is satisfied in the class of β-transformations for a neighbourhood of each β-transformation.

We consider reals $\beta_n > 1$ and the corresponding β-transformations.

Let $\chi_0(x) = 1$ and, for $n \geq 1$

$$\chi_n(x) = 1 + \sum_{j=1}^{n} \beta_n^{-1} \beta_{n-1}^{-1} \ldots \beta_{n-j+1}^{-1}\ 1_{\{x < \tau_n \tau_{n-1} \ldots \tau_{n-j+1} 1\}}. \tag{4.2}$$

We have:

$$\forall n \geq 0,\ P_{n+1}\chi_n(x) = \sum_{k\geq 0} \beta_{n+1}^{-1} \chi_n\Big(\frac{x+k}{\beta_{n+1}}\Big) 1_I\Big(\frac{x+k}{\beta_{n+1}}\Big) = \chi_{n+1}(x) + c_n,$$

where c_n is a constant. By integration with respect to the Lebesgue measure, we get that $c_n = m(\chi_n) - m(\chi_{n+1})$.

On the other hand, if β_n is such that $|\beta_n - \beta| \leq a$, for some $a > 0$ such that $\beta - a > 1$, then the functions χ_n satisfy:

$$1 \leq \chi_n(x) \leq \frac{\beta - a}{\beta - a - 1} := M, \forall x \in [0, 1].$$

When $\beta_k = \beta > 1$, for each k, we get:

$$0 \leq \chi_{n+1} - \chi_n \leq \beta^{-(n+1)} \text{ and } \lim_n c_n = 0.$$

In that case, the sequence (χ_n) converges to the sum

$$\chi = 1 + \sum_{j=1}^{\infty} \beta^{-j}\ 1_{\{x < \tau^j 1\}}$$

which is P-invariant and gives, up to a factor, the density of the ACIM for the β-transformation. This shows also that the density is bounded from below by the constant $\frac{\beta-1}{\beta} > 0$ (cf. Renyi [**Re57**], Parry [**Pa60**]).

In the general case, if the β_n's are close to a fixed β, as in the estimation of the rate of convergence of $(\Pi_n\varphi)$, one can show that $\|\chi_{n+1} - \chi_n\|_1$ is small and therefore also the constant c_n, for a small enough.

Iterating, we get:

$$\begin{aligned}&P_n P_{n-1} ... P_{n-k} \chi_{n-k-1}\\ &= \chi_n + c_{n-1} + c_{n-2} P_n 1 + c_{n-3} P_n P_{n-1} 1 + ... + c_{n-k-1} P_n P_{n-1} ... P_{n-k+1} 1,\end{aligned}$$

hence

$$|P_n P_{n-1} ... P_{n-k} \chi_{n-k-1} - \chi_n| \leq C \sum_{j=1}^{k+1} |c_{n-j}|. \tag{4.3}$$

Proposition 4.3. *For every* $\beta > 1$, *there exists* $a > 0, \delta > 0$ *such that, if* $\beta_n \in [\beta - a, \beta + a]$, *then* $\Pi_n 1(x) \geq \delta$.

Proof: Recall that we have $|P_n ... P_{n-k} 1\|_\infty \leq C$, independently of n and k, $|m(\chi_j) - m(\chi_{j+1})| < \varepsilon$ for j big enough. On an other hand we have (cf. (2.14)):

$$|\Pi_n 1 - P_n ... P_{n-r} 1|_v \leq C\gamma_0^r.$$

We fix r such that $MC\gamma_0^r \leq 1/4$. For n big enough it follows from (4.3):

$$P_n P_{n-1} ... P_{n-r} \chi_{n-r-1} \geq \chi_n - r\varepsilon \geq 1 - r\varepsilon,$$

hence:

$$P_n P_{n-1} ... P_{n-r} 1 \geq M^{-1} P_n P_{n-1} ... P_{n-r} \chi_{n-r-1} \geq M^{-1}(1 - r\varepsilon)$$

and finally, taking $\delta = \frac{1}{2M}$ and $\varepsilon \leq \frac{1}{4r}$, we obtain the inequality:

$$\Pi_n 1 \geq M^{-1}(1 - r\varepsilon) - C\gamma_0^r \geq \delta.$$

□

5. Central limit theorems

In the following, we assume that the transformations τ_n belong to a family of transformations $\mathcal{C}_0$ such that the conditions (*Dec*) and (*Min*) are satisfied.

As we have seen, this is the case for any $\beta_0 > 1$, when $\mathcal{C}_0$ is the set of β-transformations for β in a suitable neighbourhood of $\beta_0 > 1$.

We will show the convergence towards the normal law for the sums

$$\sum_{k=0}^{n-1} f(\tau_k \cdots \tau_1 x)$$

after centering and normalisation. (We have to center at each step of the iteration since there is no joint invariant measure for the transformations τ_n.) We consider also the more general case of a sequence of functions f_n.

We define the operators Q_n, for $n \geq 1$, by

$$g \to Q_n g = \frac{P_n(g \Pi_{n-1} 1)}{\Pi_n 1}.$$

Let (f_n) be a sequence in $\mathcal{V}$ such that $\sup_{n \geq 0} |f_n|_v < +\infty$. Write

$$\tilde{f}_n = f_n - m(T_1 \cdots T_n f_n).$$

Let h_n be defined by the relations $h_{n+1} = Q_{n+1}\tilde{f}_n + Q_{n+1}h_n$, with $h_0 = 0$. We get:

$$\begin{aligned} h_n &= Q_n\tilde{f}_{n-1} + Q_nQ_{n-1}\tilde{f}_{n-2} + ... + Q_nQ_{n-1}...Q_1\tilde{f}_0 \\ &= \frac{1}{\Pi_n 1}[P_n(\tilde{f}_{n-1}\Pi_{n-1}1) + P_nP_{n-1}(\tilde{f}_{n-2}\Pi_{n-2}1) + ... + P_nP_{n-1}...P_1(\tilde{f}_0\Pi_0 1)]. \end{aligned}$$

The functions $\tilde{f}_{n-k}\Pi_{n-k}1$ belong to $\mathcal{V}_0$. Therefore, under condition (Dec), we have an exponential decay for the norm $|\ |_v$ of the general term of the previous sums and the sequence (h_n) is bounded for the norm $|\ |_v$. In particular it is bounded for the uniform norm. We write

$$\varphi_n = \tilde{f}_n + h_n - T_{n+1}h_{n+1}, \tag{5.1}$$

$$U_n = T_1...T_n\varphi_n. \tag{5.2}$$

¿From (1.10), the sequence (U_n) is a sequence of reversed martingale for the filtration $(\mathcal{A}_n)$ and we have

$$\sum_0^{n-1} T_1...T_k\tilde{f}_k = \sum_0^{n-1} U_k + T_1\cdots T_nh_n.$$

Therefore we can replace $\sum_0^{n-1} T_1...T_k\tilde{f}_k$ by a reversed martingale, the error term being bounded. We can now apply a theorem of B.M. Brown ([**Br71**]) (see 5.8 below) on martingales to get the CLT.

We write $S_n = \sum_{k=0}^{n-1} T_1\cdots T_k\tilde{f}_k$.

THEOREM 5.1. *Let (f_n) be a sequence in $\mathcal{V}$ such that* $\sup_{n\geq 0}|f_n|_v < +\infty$. *Assuming (Dec) and (Min), we have:*
- either the norms $\|S_n\|_2$ are bounded and in that case, for a.e. x, the sequence

$$\sum_{k=0}^{n-1} \tilde{f}_k(\tau_k\cdots\tau_1 x), n{\geq}1$$

is bounded,
- or the sequence

$$\Big(\frac{f_0 - m(f_0) + T_1f_1 - m(T_1f_1) + \ldots + T_1\cdots T_{n-1}\,f_{n-1} - m(T_1\cdots T_{n-1}\,f_{n-1})}{\|S_n\|_2}\Big)_{n\geq 1}$$

converges in law to $N(0,1)$.

Proof: Let:

$$\sigma_n^2 = \sum_{k=0}^{n-1} I\!E[U_k^2], \quad V_n = \sum_{k=0}^{n-1} I\!E[U_k^2|\mathcal{A}_{k+1}].$$

We have to check conditions i) and ii) of Theorem 5.8 :
$i)$ the sequence of v.a.r. $(\sigma_n^{-2}\ V_n)_{n\geq 1}$ converges in probability to 1 ;
$ii)$ for every $\varepsilon > 0$, $\lim_{n\to+\infty} \sigma_n^{-2} \sum_{k=0}^{n-1} I\!E[U_k^2 1_{\{|U_k|>\varepsilon\sigma_n\}}] = 0$.

The difference $|\|S_n\|_2 - \|\sum_{k=0}^{n-1} U_k\|_2| = |\|S_n\|_2 - (\sum_{k=0}^{n-1}\ I\!E[U_k^2])^{\frac{1}{2}}| \leq \|S_n - \sum_{k=0}^{n-1} U_k\|_2$ is bounded. We have

$$I\!E[U_k^2|\mathcal{A}_{k+1}] = T_1\cdots T_kT_{k+1}\big(\frac{P_{k+1}(\varphi_k^2\ \Pi_k 1)}{\Pi_{k+1}1}\big).$$

If the sequence (σ_n) tends to $+\infty$, $ii)$ is satisfied since the functions U_k are uniformly bounded. For the first condition, we apply the law of large numbers (Theorem 4.1) to the sequence

$$(\frac{P_{k+1}(\varphi_k^2\ \Pi_k 1)}{\Pi_{k+1}1}) = (\frac{P_{k+1}[(\tilde{f}_k + h_k - T_{k+1}h_{k+1})^2\Pi_k 1]}{\Pi_{k+1}1}).$$

Using the fact that the functions $\Pi_n 1$ are bounded from below from 0, we get

$$\sup_k |\frac{P_{k+1}(\varphi_k^2\ \Pi_k 1)}{\Pi_{k+1}1}|_v < \infty$$

and we can apply 4.1, under the assumption

$$\sum_0^\infty \int (T_k \cdots T_1 \tilde{f}_k + T_k \cdots T_1 h_k - T_{k+1} \cdots T_1 h_{k+1})^2\ dm = +\infty.$$

In that case, we have $\lim_n \sigma_n = +\infty$ and $\lim_n \|S_n\|_2 = +\infty$.

On the contrary if this series converges, we have by a martingale theorem that the series

$$\sum_{k=0}^{n-1} [T_k \cdots T_1 \tilde{f}_k + T_k \cdots T_1 h_k - T_{k+1} \cdots T_1 h_{k+1}]$$

converges for a.a. x and the sum $\sum_{k=0}^{n-1} f_k(\tau_k \cdots \tau_1 x) - (m(f_0) + m(T_1 f_1) + \ldots + m(T_1 \cdots T_{n-1}\, f_{n-1}))$ can be written as the sum of a (for a.a. x) bounded sequence and of $T_k \cdots T_1 h_k - T_{k+1} \cdots T_1 h_{k+1}$ which is uniformly bounded in k.

□

Remark 5.2. The previous statement can be made more precise (locally) in the class of β-transformations and for a fixed function f. Denote by ${}^hP$ the relativised transfer operator corresponding to $x \to \beta x$ mod 1 for a given $\beta > 1$ and by ${}^hG$ its potential (cf. below).

In a neighbourhood of a β-transformation, if f is not a coboundary for the transformation $x \to \beta x$ mod 1, the functions $f - h_k - T_{k+1}h_{k+1}$, which are close (in L^2 norm) to $f - {}^hP^hGf + T^hP^hGf$, have norms bounded from below. This shows that the variance in the previous theorem is of order $\sqrt{n}$.

As a particular case of the previous theorem we have, in the case of the iterates of a single transformation of the interval which belongs to the class $\mathcal{C}$ and whose ACIM has a strictly positive density, the following result: if (f_n) is a sequence of BV functions such that $\sup_{n\geq 0} |f_n|_v < +\infty$, then the sums $\sum_0^{n-1} T^k f_k$ after centering and normalisation satisfy a CLT.

- **CLT when** $\lim_n \tau_n = \tau$

We can give a more precise formulation of the CLT for a fixed BV function f, when $\lim_n \tau_n = \tau$.

We suppose from now on that the P_τ-invariant normalised function h is m-a.e. **strictly positive**. This is the case for the β-transformations as we have shown, but not always true for transformations of the form $x \to \beta x + \alpha$ mod 1, as in the following example:

$\beta = \frac{1+\sqrt{5}}{2}, \alpha = \frac{3-\beta}{2}$, where the density h is zero on interval $]\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{4}[$.

We consider the "relativised" operator ${}^hPf = \frac{P(h\,f)}{h}$ and its "potential" defined, for f in $\mathcal{V}$ such that $m(hf) = 0$ by:

$$ {}^hGf = \sum_{k\geq 0} ({}^hP)^k f = \sum_{k\geq 0} \frac{P^k(h\,f)}{h}. $$

The probability $h\,m$ is ${}^hP$-invariant.

We set $\xi = {}^hGf$, so that f can be written $f = \xi - {}^hP\xi$. The function ξ has still a bounded variation.

THEOREM 5.3. *Let $f \in \mathcal{V}$ be such that $m(h\,f) = 0$. Let*

$$ \sigma^2 = m\big(h\,({}^hGf)^2\big) - m\Big[h\,\Big({}^hP^hGf\Big)^2\Big] \tag{5.3} $$

$$ = \int_0^1 {}^hP\Big[\Big({}^hGf(\cdot) - {}^hP^hGf(x)\Big)^2\Big](x)\,h(x)\,dx. \tag{5.4} $$

If f is not a coboundary for $x \to \beta x \bmod 1$, we have: $\sigma^2 > 0$ and for every density $\varphi \in \mathcal{V}$ such that $m(\varphi) = 1$, under the probability $\varphi\,m$, the sequence

$$ \Big(\frac{1}{\sqrt{n}}(f - m(\varphi f) + T_1 f - m(\varphi T_1 f) + \ldots + T_1 \cdots T_{n-1}\,f - m(\varphi\,T_1 \cdots T_{n-1}\,f))\Big)_{n\geq 1} $$

converges in distribution to $\mathcal{N}(0, \sigma^2)$.

For example for the β-transformations, taking into account Corollary 3.11, we have:

COROLLARY 5.4. *Suppose that $|\beta_n - \beta| < \frac{1}{n^\theta}$, with $\theta > \frac{1}{2}$. Let $f \in \mathcal{V}$ such that $m(h\,f) = 0$. Suppose $\sigma^2 \neq 0$. Then for every $\varphi \in \mathcal{V}$ such that $m(\varphi) = 1$, under the probability $\varphi\,m$, the sequence of real random variables*

$$ \Big(\frac{1}{\sqrt{n}}(f + T_1 f + \ldots + T_1 \cdots T_{n-1}\,f)\Big)_{n\geq 1} $$

converges in distribution to $\mathcal{N}(0, \sigma^2)$.

The proof of Theorem 5.3 will be given in several lemmas. We begin with some notations:

For $\eta \in \mathcal{V}$ and $k \in \mathbb{N}$, let $\eta_k = \eta - \mathbb{E}_{\varphi m}[T_1 \cdots T_k\,\eta] = \eta - m(\eta\,\Pi_k\varphi)$. We set:

$$ \begin{aligned} U_k &= T_1 \cdots T_k\,(\xi_k) - \mathbb{E}_{\varphi m}[T_1 \cdots T_k\,(\xi_k)\,|\,\mathcal{A}_{k+1}], \\ W_k &= \mathbb{E}_{\varphi m}[T_1 \cdots T_k\,(\xi_k)\,|\,\mathcal{A}_{k+1}] - T_1 \cdots T_{k+1}\,\big({}^hP\xi\big)_{k+1}. \end{aligned} $$

With these notations, we have:

$$ \sum_{k=0}^{n-1} \big(T_1 \cdots T_k\,f_k - (U_k + W_k)\big) = T_1 \cdots T_n\,\big({}^hP\xi\big)_n - \big({}^hP\xi\big)_0. $$

LEMMA 5.5. *Let ψ be an integrable function, (g_n) and (φ_n) two sequences of functions such that:* $\sup_n \|g_n\|_\infty = M < \infty, \|g_n\varphi_n\|_1 \to 0, \|\varphi_n - \psi\|_1 \to 0$. *We have then:* $\|g_n 1_{\{\psi>0\}}\|_1 \to 0$.

Proof: We apply the inequality

$$\begin{aligned}\|g_n 1_{\{\psi>0\}}\|_1 &\leq M\ m\big(\{0<\psi<\epsilon\}\big) + \frac{1}{\varepsilon}\int |g_n||\psi|\ dm \\ &\leq M\ m(\{\psi<\epsilon\}\cap\{\psi>0\}) + \frac{1}{\varepsilon}\int |g_n||\psi-\varphi_n|\ dm + \frac{1}{\varepsilon}\int |g_n||\varphi_n|\ dm.\end{aligned}$$

□

LEMMA 5.6. *The sequence* $(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_k)_{n\geq 1}$ *converges in* $\mathbb{L}^2(\varphi\, m)$*-norm to 0.*

Proof: For $f\in\mathcal{V}$, we have $\varphi\, m$-a.e.

$$\begin{aligned}\mathbb{E}_{\varphi m}[T_1\cdots T_k\, f\,|\,\mathcal{A}_{k+1}] &= T_1\cdots T_{k+1}\Big(\frac{P_{k+1}\big(f\,\Pi_k\varphi\big)}{\Pi_{k+1}\varphi}\,1_{\{\Pi_{k+1}\varphi>0\}}\Big); \\ \varphi\, m(\{T_1\cdots T_{k+1}\Pi_{k+1}\varphi=0\}) &= \mathbb{E}_m[1_{\{\Pi_{k+1}\varphi=0\}}\,\Pi_{k+1}\varphi]=0.\end{aligned}$$

Hence $W_k = T_1\cdots T_{k+1}\, g_{k+1}$ with

$$g_k = \frac{P_k\big((\xi)_{k-1}\,\Pi_k\varphi\big)}{\Pi_k\varphi}1_{\{\Pi_k\varphi>0\}} - \big({}^hP\xi\big)_k.$$

We have: $m(g_k\,\Pi_k\varphi)=0, \forall k\geq 1$ and $\sup_{k\geq 1}\big|g_k\,\Pi_k\varphi\big|_v = L<+\infty$. Moreover the inequalities:

$$\begin{aligned}\mathbb{E}_m\Big[|g_n|\,\Pi_n\varphi\Big] &= \mathbb{E}_m\Big[\Big|P_n\big(\xi_{n-1}\,\Pi_{n-1}\varphi\big)-\big({}^hP\xi\big)_n\,\Pi_n\varphi\Big|\Big] \\ &\leq \mathbb{E}_m\Big[\Big|P_n\big(\xi\,\Pi_{n-1}\varphi\big)-{}^hP\xi\,\Pi_n\varphi\Big|\Big] \\ &\quad +\big|m\big(\xi\,\Pi_{n-1}\varphi\big)-m\big({}^hP\xi\,\Pi_n\varphi\big)\big| \\ &\leq \mathbb{E}_m\Big[\Big|P_n\big(\xi\,\Pi_{n-1}\varphi\big)-P\big(\xi\,\Pi_{n-1}\varphi\big)\Big|\Big] \\ &\quad +\mathbb{E}_m\Big[\Big|P\big(\xi\,\Pi_{n-1}\varphi\big)-P\big(h\,\xi\big)\Big|\Big] \\ &\quad +\mathbb{E}\Big[\Big|{}^hP\xi\,\big(h-\Pi_n\varphi\big)\Big|\Big] + \big|m\big(\xi\,\Pi_{n-1}\varphi\big)-m\big({}^hP\xi\,\Pi_n\varphi\big)\big|\end{aligned}$$

show that $\lim_{n\to+\infty}\|g_n\,\Pi_n\varphi\|_{\mathbb{L}^1(m)}=0$. [Remark that ${}^hP\xi$ is bounded by $\|\xi\|_\infty$].

As $\sup_{k\geq 0}\|g_k\|_\infty<+\infty$, using Lemma 5.5, this implies:

$$\lim_{n\to+\infty}\|h\,g_n\|_{\mathbb{L}^1(m)}=0 \text{ and } \lim_{n\to+\infty}\|g_n\|_{\mathbb{L}^1(m)}=0.$$

For $1\leq\ell<k$, we have then:

$$\begin{aligned}\mathbb{E}_{\varphi m}[W_kW_\ell] &= \mathbb{E}_m[g_{k+1}P_{k+1}\cdots P_{\ell+2}\left(g_{\ell+1}\,\Pi_{\ell+1}\varphi\right)] \\ &\leq \sup_{j\geq\ell+1}\|g_{j+1}\|_{\mathbb{L}^1(m)}\ \|P_{k+1}\cdots P_{\ell+2}\left(g_{\ell+1}\,\Pi_{\ell+1}\varphi\right)\|_v \\ &\leq D\,\lambda^{k-\ell}\left|g_{\ell+1}\,\Pi_{\ell+1}\varphi\right|_{\mathcal{V}}\ \sup_{j\geq\ell+1}\|g_{j+1}\|_{\mathbb{L}^1(m)} \\ &\leq D\,L\lambda^{k-\ell}\sup_{j\geq\ell+1}\|g_{j+1}\|_{\mathbb{L}^1(m)},\end{aligned}$$

and $\mathbb{E}_{\varphi m}[W_k^2] \leq L\,\|g_{k+1}\|_{\mathbb{L}^1(m)}$. This implies the result.

□

LEMMA 5.7. *The sequence* $(\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_{\varphi m}[U_k^2|\mathcal{G}_{k+1}])_{n\geq 1}$ *converges in* $\mathbb{L}^1(\varphi\, m)$ *norm to* σ^2 *given by (5.4).*

Proof: We have:

$$\mathbb{E}_{\varphi m}[U_k^2 \,|\, \mathcal{G}_{k+1}] = \mathbb{E}_{\varphi m}[T_1 \cdots T_k\, \xi_k^2 \,|\, \mathcal{G}_{k+1}] - \big(\mathbb{E}_{\varphi m}[T_1 \cdots T_k\, \xi_k \,|\, \mathcal{A}_{k+1}]\big)^2.$$

By the martingale property, we have convergence to zero in $\mathbb{L}^2(\varphi\, m)$-norm and $\varphi\, m$-a.e. of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1}\big(T_1 \cdots T_k\, \xi_k^2 - \mathbb{E}_{\varphi m}[T_1 \cdots T_k\, \xi_k^2 \,|\, \mathcal{A}_{k+1}]\big),$$

so that the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}_{\varphi m}[T_1 \cdots T_k\, \xi_k^2 \,|\, \mathcal{A}_{k+1}]$$

has, for the $\mathbb{L}^1(\varphi, m)$-norm and $\varphi\, m$-a.e., the same limit as

$$\frac{1}{n}\sum_{k=1}^{n} T_1 \cdots T_k\, \xi_k^2.$$

Therefore it converges (Theorem 3.8) to $m(h\, \xi^2) - \big(m(h\, \xi)\big)^2$.

On the other hand the convergence of $\|g_n\, \Pi_n \varphi\|_{\mathbb{L}^1(m)}$ to zero (cf. proof of Lemma 5.6) implies that the sequence of r.r.v.

$$\mathbb{E}_{\varphi m}[T_1 \cdots T_n\, \xi_n \,|\, \mathcal{A}_{n+1}] - T_1 \cdots T_{n+1}\big({}^hP\xi\big)_{n+1}$$

converges in $\mathbb{L}^1(\varphi\, m)$-norm to zero. As these r.r.v. are uniformly bounded, this implies that

$$\big(\mathbb{E}_{\varphi m}[T_1 \cdots T_n\, \xi_n |\, \mathcal{A}_{n+1}]\big)^2 - T_1 \cdots T_{n+1}\big(({}^hP\xi)_{n+1}\big)^2$$

converges in $\mathbb{L}^1(\varphi\, m)$ to 0.

We deduce from it that the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}_{\varphi m}[T_1 \cdots T_k\, \xi_k |\, \mathcal{A}_{k+1}]\big)^2$$

converges for the $\mathbb{L}^1(\varphi\, m)$-norm to $m\Big(h\,\big({}^hP\xi\big)^2\Big) - \big(m(h\, \xi)\big)^2$. This implies the result.

□

To conclude the proof of Theorem 5.3 we apply the following result, whose proof is analogous to that of the theorem of B.M. Brown [**Br71**] for direct martingales.

THEOREM 5.8. *Let $\big(U_n, \mathcal{G}_n\big)_{n\geq 0}$ be a sequence of differences of square integrable reversed martingales, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \geq 0$, let*

$$S_n = U_0 + \ldots + U_{n-1}, \quad \sigma_n^2 = \sum_{k=0}^{n-1} \mathbb{E}[U_k^2] \quad \text{and} \quad V_n = \sum_{k=0}^{n-1} \mathbb{E}[U_k^2 | \mathcal{A}_{k+1}].$$

Let us assume the following two conditions:

i) the sequence of r.r.v. $(\sigma_n^{-2}\ V_n)_{n\geq 1}$ converges in probability to 1.

ii)For each $\varepsilon > 0$, $\lim_{n\to+\infty} \sigma_n^{-2} \sum_{k=0}^{n-1} \mathbb{E}[U_k^2 1_{\{|U_k|>\varepsilon\sigma_n\}}] = 0$.

Then we have:

$$\lim_{n\to+\infty} \sup_{\alpha\in\mathbb{R}} \big|\mathbb{P}[\frac{S_n}{\sigma_n} \leq \alpha] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}\ dx\big| \ = \ 0.$$

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