ERGODIC COCYCLES, POINTS WITH NON UNIFORM DISTRIBUTION IN INFINITE BILLIARDS

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Abstract. The dynamic of the directional billiard with periodically distributed rectangular obstacles in the plane reduces to that of a skew product defined by a cocycle with values in $\mathbb{Z}^2$ over some interval exchange transformations.

This is a motivation to investigate the construction of recurrent ergodic cocycles over interval exchange transformations. In the first part of the talk, we will present a class of such cocycles over IET of periodic type (joint work with Krzysztof Frączek).

For the models provided by the billiard, there are special directions where the directional flows in the periodic rectangular billiard can be reduced to a cocycle over an irrational rotation. In a second part, we will present ergodicity results for this particular model (joint work with Eugene Gutkin).

Finally, we will apply to this model the construction of conformal measures for a rotation, and show the existence of points with non uniformly distributed orbits in the plane.

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Introduction

We consider the billiard in the plane with $\mathbb{Z}^2$-periodically distributed obstacles. The obstacles are rectangles with their center at the center of each unit cell.

The billiard flow and the billiard transformation commute with the action of $\mathbb{Z}^2$ on the configuration space. By taking the quotient modulo $\mathbb{Z}^2$, we obtain a flow and a map on a space of finite volume and the flow is an extension of the quotient flow.

The dynamic of the directional billiard with periodically distributed rectangular obstacles in the plane reduces to that of a skew product defined by a cocycle with values in $\mathbb{Z}^2$ over some interval exchange transformations.

Some classical notations and notions

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel probability space. Let m be the Lebesgue measure on $\mathbb{R}^\ell$, $\ell \geq 1$.

To a measurable function $\varphi : X \to \mathbb{R}^\ell$ corresponds a *cocycle* over T, denoted by (φ_n) or $(\varphi(n, .))$

$$\varphi_n(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x), & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -(\varphi(T^n x) + \varphi(T^{n+1}x) + \ldots + \varphi(T^{-1}x)), & \text{if } n < 0. \end{cases}$$

The associated *skew product* on $X \times \mathbb{R}^\ell$ is

$$T_\varphi(x, g) \;=\; (Tx, y + \varphi(x)). \tag{1}$$

The cocycle (φ_n) can be viewed as a *"stationary" walk* in $\mathbb{R}^\ell$ over the dynamical system (X, μ, T). It is *recurrent* if for a.e. x $(\varphi_n(x))$ returns infinitely often in any neighborhood of 0. The map T_φ is then conservative for the invariant σ-finite measure $\mu \times m$. If $(X \times \mathbb{R}^\ell, \mu \times m, T_\varphi)$ is *ergodic*, we say that the cocycle (φ_n) (or simply the "cocycle φ") is ergodic.

A problem is the construction of recurrent ergodic cocycles defined over a given dynamical system by regular functions φ with values in $\mathbb{R}^\ell$. There is an important literature on skew products over an irrational rotation on the circle, and several classes of ergodic cocycles with values in $\mathbb{R}$ or $\mathbb{R}^\ell$ are known in that case.

Skew products appear in a natural way in the study of the billiard flow in the plane with $\mathbb{Z}^2$ periodically distributed obstacles. For instance when the obstacles are rectangles, they provide skew products over interval exchange transformations (abbreviated as IETs) (with 20 intervals permuted). Recurrence (for dim > 1) and ergodicity of the models associated to the billiard are mainly open questions. Nevertheless a first step is the construction of recurrent ergodic cocycles over some classes of IETs. See also a recent results by Patrick Hooper, Pascal Hubert and Barak Weiss for cocycles associated to translation surfaces.

1. Cocycles over IET of periodic type

(results taken in a joint work with Krzysztof Frączek)

For the rotations on the circle, a special class consists in the rotations with bounded partial quotients, in particular with a periodic continued fraction expansion. For IETs, it is natural to consider the interval exchange transformations of periodic type.

1.1. Interval exchange transformations, Rauzy induction.

$\mathcal{A}$ an alphabet with d elements; $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$ a vector in $\mathbb{R}_+^{\mathcal{A}}$;
$\pi = (\pi_0, \pi_1)$ an irreducible pair of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, \dots, d\}$, $\varepsilon = 0, 1$,
with $\pi_0^{-1}\{1, \dots, k\} \neq \pi_1^{-1}\{1, \dots, k\}$, $\forall 1 \leq k < d$

$$|\lambda| = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha, \quad I = [0, |\lambda|[\tag{2}$$

$$I_\alpha = [\ell_\alpha, r_\alpha[, \text{ where } \ell_\alpha = \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \quad r_\alpha = \sum_{\pi_0(\beta)\leq\pi_0(\alpha)} \lambda_\beta. \tag{3}$$

Then $|I_\alpha| = \lambda_\alpha$. Denote by Ω_π the matrix $[\Omega_{\alpha\,\beta}]_{\alpha,\beta\in\mathcal{A}}$

$$\Omega_{\alpha\,\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta), \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta), \\ 0 & \text{in all other cases.} \end{cases}$$

$T_{(\pi,\lambda)} : [0, |\lambda|) \to [0, |\lambda|)$ denotes the *interval exchange transformation* (IET) on d intervals I_α, $\alpha \in \mathcal{A}$, rearranged according to the permutation $\pi_1^{-1} \circ \pi_0$, i.e.

$$T_{(\pi,\lambda)}x = x + w_\alpha, \text{ for } x \in I_\alpha, \text{ where } w = \Omega_\pi \lambda.$$

Let $T = T_{(\pi,\lambda)}$ be an IET satisfying Keane's condition (also called IDOC), i.e.

$$T^m_{(\pi,\lambda)}\ell_\alpha \neq \ell_\beta, \ \forall m \geq 1, \forall \alpha, \beta \in \mathcal{A} \text{ with } \pi_0(\beta) \neq 1,$$

Let

$$\tilde{I} = \left[0, \max\left(l_{\pi_0^{-1}(d)}, l_{\pi_1^{-1}(d)}\right)\right)$$

and denote by $\mathcal{R}(T) = \tilde{T} : \tilde{I} \to \tilde{I}$ the first return map of T to the interval $\tilde{I}$. Set

$$\varepsilon = \varepsilon(\pi, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}, \\ 1 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}. \end{cases}$$

We define the pair $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1)$ by

$$\begin{aligned} \tilde{\pi}_\varepsilon(\alpha) &= \pi_\varepsilon(\alpha) \text{ for all } \alpha \in \mathcal{A}, \\ \tilde{\pi}_{1-\varepsilon}(\alpha) &= \begin{cases} \pi_{1-\varepsilon}(\alpha) & \text{if} \quad \pi_{1-\varepsilon}(\alpha) \leq \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d), \\ \pi_{1-\varepsilon}(\alpha) + 1 & \text{if} \quad \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}\pi_\varepsilon^{-1}(d) + 1 & \text{if} \quad \pi_{1-\varepsilon}(\alpha) = d. \end{cases} \end{aligned}$$

It was shown by Rauzy that $\tilde{T}$ is the IET on d-intervals $\tilde{T} = T_{(\tilde{\pi},\tilde{\lambda})}$ with $\tilde{\lambda} = \Theta^{-1}(\pi,\lambda)\lambda$, where

$$\Theta(T) = \Theta(\pi,\lambda) = I + E_{\pi_\varepsilon^{-1}(d)\,\pi_{1-\varepsilon}^{-1}(d)} \in Sl(d,\mathbb{Z}).$$

The IET $\tilde{T}$ fulfills the Keane condition as well. Therefore we can iterate the renormalization procedure and generate the sequence of IETs $T^{(n)} = \mathcal{R}^n(T)$ for $n \geq 0$. It corresponds to the pair $\pi^{(n)} = (\pi_0^{(n)}, \pi_1^{(n)})$ and the vector $\lambda^{(n)} = (\lambda_\alpha^{(n)})_{\alpha\in\mathcal{A}}$. Then $T^{(n)}$ is the first return map of T to the interval $I^{(n)} = [0, |\lambda^{(n)}|)$ and

$$\lambda = \Theta^{(n)}(T)\lambda^{(n)} \text{ with } \Theta^{(n)}(T) = \Theta(T)\cdot\Theta(T^{(1)})\cdot\ldots\cdot\Theta(T^{(n-1)}).$$

1.2. **IETs of periodic type, growth of BV cocycles.**

Definition 1.1. *An IET T is of* periodic type *if there exists $p > 0$ such that $\Theta(T^{(n+p)}) = \Theta(T^{(n)})$ for every $n \geq 0$ and $A := \Theta^{(p)}(T)$ has strictly positive entries.*

Every IET of periodic type is uniquely ergodic.

A procedure giving an explicit construction of IETs of periodic type has been introduced by Ya.G. Sinai and C. Ulcigrai. The construction is based on choosing closed paths on the graph giving the Rauzy classes.

Let $Sp(A)$ be the set of eigenvalues of A. Consider the Lyapunov exponents of A:

$$\{\log|\rho| : \rho \in Sp(A)\} = \theta_1 > \theta_2 \geq \theta_3 \geq \ldots \geq \theta_g \geq 0 = \ldots = 0 \geq -\theta_g \geq \ldots \geq -\theta_3 \geq -\theta_2 > -\theta_1,$$

Growth of BV cocycles

Sufficient conditions for the recurrence of (φ_n) with values in $\mathbb{R}^\ell$ are related to its growth.

For an irrational rotation $T : x \to x + \alpha \bmod 1$, as it is well known, the growth of φ_n is controlled by the Denjoy-Koksma inequality: if φ is a zero mean function on $X = \mathbb{R}/\mathbb{Z}$ with bounded variation $\mathrm{Var}(\varphi)$, and (q_n) the denominators (of the convergents) given by the continued fraction expansion of α, then the following inequality holds:

$$|\sum_{j=0}^{q_n-1} \varphi(x + j\alpha)| \leq \mathrm{Var}(\varphi), \forall x \in X. \tag{4}$$

This inequality implies obviously recurrence of the cocycle φ_n and if α has bounded partial quotients $\sum_{j=0}^{n-1} \varphi(x + j\alpha) = O(\log n)$ uniformly in $x \in X$.

It is much more difficult to get a precise upper bound for the growth of a cocycle over an IET. The following theorem, which can be viewed as a special case of Zorich's results on the growth of ergodic sums over an IET, gives for an IET of periodic type a control on the growth of a BV cocycle in term of the Lyapunov exponents of the matrix A.

Theorem 1.2. *Suppose that $T_{(\pi,\lambda)} : I \to I$ is an IET of periodic type, $0 \leq \theta_2 < \theta_1$ are the two largest Lyapunov exponents, and M is the maximal size of Jordan blocks in the Jordan decomposition of its periodic matrix A. Then there exists $C > 0$ such that*

$$\|\varphi_n\|_{\sup} \leq C \cdot \log^{M+1} n \cdot n^{\theta_2/\theta_1} \cdot \mathrm{Var}(\varphi), \forall n \geq 0$$

for every function $\varphi : I \to \mathbb{R}$ of bounded variation with zero mean.

1.3. **Examples of ergodic cocycles.** We need examples with $\theta_2(T)/\theta_1(T)$ small. There are examples with arbitrary small values of this ratio for IET's of 4 intervals.

Notations Denote by $\mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the set of piecewise linear (with constant slope $s = s(\varphi)$) functions $\varphi : I \to \mathbb{R}$ such that $\varphi(x) = sx + c_\alpha$ for $x \in I_\alpha$.

Theorem 1.3. *Let $T : I \to I$ be an IET of periodic type. If $\varphi \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi) \neq 0$, then the skew product T_φ is ergodic.*

Now we consider cocycles taking values in $\mathbb{R}^\ell$, $\ell \geq 1$. Suppose that $\varphi : I \to \mathbb{R}^\ell$ is a piecewise linear cocycle with zero mean such that the slope $s(\varphi) \in \mathbb{R}^\ell$ is non-zero. Then, by an appropriate choice of coordinates, we obtain $s(\varphi_1) \neq 0$ and $s(\varphi_2) = 0$, where $\varphi = (\varphi_1, \varphi_2)$ and $\varphi_1 : I \to \mathbb{R}$, $\varphi_2 : I \to \mathbb{R}^{\ell-1}$. Thus φ_2 is piecewise constant.

Theorem 1.4. *Suppose that $T : I \to I$ is an IET of periodic type such that $\theta_2(T)/\theta_1(T) < 1/\ell$. Let $\varphi_1 \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R})$, $\varphi_2 \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$ be piecewise linear cocycles with zero mean such that $s(\varphi_1) \neq 0$ and $s(\varphi_2) = 0$. If the cocycle $\varphi_2 : I \to \mathbb{R}^{\ell-1}$ is ergodic, then the cocycle $\varphi = (\varphi_1, \varphi_2) : I \to \mathbb{R}^\ell$ is ergodic as well.*

Theorem 1.5. *Let $T_j : I^{(j)} \to I^{(j)}$ be an interval exchange transformation of periodic type such that $\theta_2(T_j)/\theta_1(T_j) < 1/\ell$ for $j = 1, \ldots, \ell$. Suppose that the Cartesian product $T_1 \times \ldots \times T_\ell$ is ergodic. If $\varphi_j \in \mathrm{PL}(\sqcup_{\alpha\in\mathcal{A}_|} I^{(j)}_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi_j) \neq 0$ for $j = 1, \ldots, \ell$, then the Cartesian product $(T_1)_{\varphi_1} \times \ldots \times (T_\ell)_{\varphi_\ell}$ is ergodic.*

2. **Rectangular billiard in the plane**

Now we consider the models given by the directional billiard with $\mathbb{Z}^2$-periodically distributed rectangular obstacles. (joint work with Eugene Gutkin).

The IET provided by the periodic billiard are difficult to study. But there are special directions where the directional flows in the periodic rectangular billiard can be reduced to a cocycle over an irrational rotation. In this second part, we will present ergodicity results for this particular model.

We will study the rectangular Lorenz gas *wind-tree model* (cf. Hardy-Weber (1980)).

Notation Let $0 < a,\ b < 1$. For $(m,n) \in \mathbb{Z}^2$ let $R_{(m,n)}(a,b) \subset \mathbb{R}^2$ be the $a \times b$ rectangle centered at (m,n) whose sides are parallel to the coordinate axes. The Lorenz gas with rectangular obstacles of size $a \times b$ corresponds to the polygonal surface

$$\tilde{P}(a,b) = \mathbb{R}^2 \setminus \bigcup_{(m,n)\in\mathbb{Z}^2} R_{(m,n)}(a,b). \tag{5}$$

The quotient surface $P(a,b) = \tilde{P}(a,b)/\mathbb{Z}^2$ is the unit torus with a rectangular hole.

The surface $P(a,b)$ is rational. It is invariant by the group R_2 generated by 2 orthogonal reflections with axes with the angle $\pi/2$. We identify U/R_2 with $[0,\pi/2]$.

For $\theta \in [0,\pi/2]$ we denote by $(\tilde{X}_\theta, \tilde{\tau}_\theta, \tilde{\nu}_\theta)$ and $(X_\theta, \tau_\theta, \nu_\theta)$ the corresponding directional billiard map) for $\tilde{P}(a,b)$ and $P(a,b)$ respectively.

2.1. **Rational directions and small obstacles.**

A direction $\theta \in [0, \pi/2]$ is rational if $\tan\theta \in \mathbb{Q}$. Rational directions $\theta(p,q) = arctan(q/p)$ correspond to pairs $(p,q) \in \mathbb{N}$ with relatively prime p,q. We will use the notation (p,q) instead of $\theta(p,q)$.

Let $R(a,b) = ABCD$ be the rectangle in the unit torus. Let $\theta \in (0, \pi/2)$. The space X_θ consists of unit vectors pointing outward, whose base points belong to $ABCD$ and whose directions belong to the set $\{\pm\theta, \pi \pm \theta\}$.

We say that the Lorenz gas $\tilde{P}(a,b)$ has *small obstacles with respect to* (p,q) if the geodesics in $P(a,b)$ emanating from A or C in the direction $\theta(p,q)$ return to either point without encountering $R(a,b)$ on the way. The picture corresponds to the angle $\pi/4$.

The small obstacles condition is satisfied iff

$$qa + pb \leq 1. \tag{6}$$

Remark that the inequality in equation (6) is strict iff the directional geodesic flow has a set of positive measure of orbits that do not encounter obstacles.

In what follows we fix (p,q) and assume that the inequality equation (6) is satisfied. We identify $X_{(p,q)}$ with 2 copies of the rectangle $ABCD$; the copy denoted by $X_+ = (ABCD)_+$ (resp. $X_- = (ABCD)_-$) carries the outward pointing vectors in the directions $\theta, \pi + \theta$ (resp. $\pi - \theta, 2\pi - \theta$). We will now investigate the Poincaré map $\tau_{(p,q)} : X_{(p,q)} \to X_{(p,q)}$.

it *1.* There are natural identifications of X_+ and X_- with the circle $\mathbb{R}/\mathbb{Z}$ endowed with distinguished points A, B, C, D; their relative positions are given by

$$|AB| = |CD| = \frac{qa}{2(qa+pb)}, \ |BC| = |DA| = \frac{pb}{2(qa+pb)}. \tag{7}$$

2. Set $\tau_\pm = \tau|_{X_\pm}$. Then $\tau_+ : X_+ \to X_-$, $\tau_- : X_- \to X_+$. Set $S = \mathbb{R}/\mathbb{Z}$. With the identifications $X_\pm = S$, the maps $\tau_+ : S \to S$ and $\tau_- : S \to S$ are the orthogonal reflections about the axes AC and BD respectively. The maps $\tau_-\tau_+ : X_+ \to X_+$ and $\tau_+\tau_- : X_- \to X_-$ are the rotations of S by $\dfrac{qa}{(qa+pb)}$ and $\dfrac{pb}{(qa+pb)}$ respectively.

Set $Z_\pm = S_\pm \times \mathbb{Z}^2$ and $Z = Z_+ \cup Z_-$. We set $\tau = \tau_{(p,q)}$. Then $\tau : Z \to Z$ is the Poincaré map; it interchanges the sets Z_+, Z_-. We use the notation $\tau_\pm(x,y) = (\tau_\pm(x), y + \varphi_\pm(x))$. Thus, $\varphi_\pm : S \to \mathbb{Z}^2$ are the displacement functions.

Now we describe the transformation $\tau^2 : Z \to Z$. We set $\tau^2_\pm = \tau^2|_{Z_\pm}$. Recall that we have identified S and $\mathbb{R}/\mathbb{Z}$.

$$\begin{aligned}
(\tau^2)_+(x,g) &= (x + \frac{qa}{qa+pb}, g + \psi_+(x)),\\
(\tau^2)_-(x,g) &= (x + \frac{pb}{qa+pb}, g + \psi_-(x)).
\end{aligned}$$

The displacement functions $\psi_\pm$ take values $(\pm 2p, 0), (0, \pm 2q)$. Each $\psi_\pm$ determines a partition of S into four intervals such that $\psi_\pm = const$ on each interval.

Let $\alpha = \frac{qa}{qa+pb}$. The function $\psi_+ : [0,1] \to \mathbb{Z}^2$ is given by

$$\psi_+(x) = \begin{cases} (0,2q) & \text{on }]0, \frac{1}{2} - \frac{\alpha}{2}[, \\ (2p,0) & \text{on }]\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}[, \\ (0,-2q) & \text{on }]\frac{1}{2}, 1 - \frac{\alpha}{2}[, \\ (-2p,0) & \text{on }]1 - \frac{\alpha}{2}, 1[. \end{cases} \tag{8}$$

Each function $\psi_\pm : [0,1] \to \mathbb{Z}^2$ takes four values which generate the subgroup $H_{(p,q)} = 2p\mathbb{Z} \oplus 2q\mathbb{Z} \subset \mathbb{Z}^2$. Using the isomorphism $(a,b) \mapsto (2pa, 2qb)$ of $\mathbb{Z}^2$ and $H_{(p,q)}$, we replace the displacement functions ψ_+ and ψ_- by piecewise constant functions on $[0,1]$ that do not depend on p, q. Let Ψ be the function corresponding to ψ_+. Then

$$\Psi(x) = \begin{cases} (0,1) & \text{on }]0, \frac{1}{2} - \frac{\alpha}{2}[, \\ (1,0) & \text{on }]\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}[, \\ (0,-1) & \text{on }]\frac{1}{2}, 1 - \frac{\alpha}{2}[, \\ (-1,0) & \text{on }]1 - \frac{\alpha}{2}, 1[. \end{cases} \tag{9}$$

We have analogous formulas for ψ_-.

2.2. **Ergodicity of the directional billiard for the Lebesgue measure.** Let $0 < \alpha < 1$. Let ρ_α be the rotation by α. We define piecewise constant functions $\gamma, \zeta : X \to \mathbb{Z}$ by

$$\gamma = 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}, \tag{10}$$

$$\zeta = 1_{[0,\frac{1}{2}-\frac{\alpha}{2}[} - 1_{[\frac{1}{2},1-\frac{\alpha}{2}[}. \tag{11}$$

Theorem 2.1. *The cocycle defined by $\Psi : X \to \mathbb{Z}^2$ is ergodic over any irrational rotation. This implies an equirepartition: on a set of infinite measure, the ratio of the number of hits by a geodesic of two obstacles (m, n) and (m', n'), tends to 1 when $t \to \infty$, if $\frac{m-m'}{p}, \frac{n-n'}{q}$ are integers of the same parity. On the complement of this set, the number of hits is 0 (transience).*

Proof. Set $\beta = \frac{1}{2} - \frac{\alpha}{2}$. The coordinates $\psi^1, \psi^2 : X \to \mathbb{Z}$ of Ψ satisfy

$$\psi^1(x) + \psi^2(x) = \gamma(x),\ \psi^1(x) - \psi^2(x) = \gamma(x + \beta).$$

We use that *out of any four consecutive denominators at least one is odd and satisfies $q_n\|q_n\alpha\| < 1/2$. If q is odd and $q\|q\alpha\| < 1/2$, then for all x we have $\sum_{j=0}^{q-1}\gamma(x+j\alpha) = \pm 1$.* Therefore, for an infinite sequence (n_k), we have $\gamma(q_{n_k}, x), \gamma(q_{n_k}, x + \beta) \in \{\pm 1\}$. This implies the existence of measurable sets $A_k \subset X$ satisfying $\mathrm{Leb}(A_k) > \frac{1}{4}$, and such that for $x \in A_k$ the vector function $(\gamma(q_{n_k}, x), \gamma(q_{n_k}, x + \beta))$ is constant, with values: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, or $(-1, -1)$.

Thus, for $x \in A_k$, either we have $\psi^1(q_{n_k}, x) = 1$ and $\psi^2(q_{n_k}, x) = 0$, or we have $\psi^1(q_{n_k}, x) = 0$ and $\psi^2(q_{n_k}, x) = 1$, or we have the opposite value.

Hence, one of the group elements $(1,0),(0,1) \in \mathbb{Z}^2$, or the opposite, is a quasi-period for the cocycle (Ψ_n) (this notion of quasi-period will be discussed more precisely later). Suppose first that $(1,0)$ is a quasi-period, and hence a period, by Lemma 3.2. Let f be a $\rho_{\alpha,\Psi}$-invariant function. It defines a $\rho_{\alpha,\zeta}$-invariant function on $X \times \mathbb{Z}$, where $\zeta : X \to \mathbb{Z}$ is given by equation (10): by Theorem 2.2 below, $f = const$, so that (Ψ_n) is ergodic. □

Theorem 2.2. *Let $\zeta : X \to \mathbb{Z}$ be given by equation* (10)*. Then the cocycle (ζ_n) over ρ_α is ergodic.*

Sketch of the proof. Let p_n/q_n be the convergents of α. Let $p'_n, q'_n \in \mathbb{N}$ be such that $q_n = 2q'_n$ or $q_n = 2q'_n + 1$ and $p_n = 2p'_n$ or $p_n = 2p'_n + 1$, depending on the parity. Set $\alpha = p_n/q_n + \theta_n$.

The set of discontinuities of ζ is $\{0, \beta = \frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}, \beta' = 1 - \frac{\alpha}{2}\}$; the respective jumps are $1, -1, -1, 1$. If t is a discontinuity of ζ, the set of corresponding discontinuities of ζ_q is $D_t := \{t - j\alpha : j = 0, ..., q-1\}$.

Depending on the parities of p_n, q_n, we define partitions $\{0, \beta, \frac{1}{2}, \beta'\} = P_1 \cup P_2$, as follows:

1) For q_n odd and p_n even, we set $P_1 = \{0, \beta'\}$, $P_2 = \{\frac{1}{2}, \beta\}$;

2) For q_n even and p_n odd, we set $P_1 = \{0, \frac{1}{2}\}$, $P_2 = \{\beta, \beta'\}$;

3) For q_n odd and p_n odd, we set $P_1 = \{0, \beta\}$, $P_2 = \{\frac{1}{2}, \beta'\}$.

Discontinuities of ζ_{q_n} coming from points in the same element of the partition are very close to each other; well separated when coming from points in distinct elements of the partition.

See figures We will consider case 2).

a) For each integer r there is $j_1(r) \in \{0, ..., q-1\}$ such that $-j_1(r)p = r \bmod q$. $D_0 = \{\frac{r}{q} - j_1(r)\theta : \ r = 0, ..., q-1\}$.

b) Let $t = \frac{1}{2}$. For each integer r there is $j_2(r) \in \{0, ..., q-1\}$ such that $q' - j_2(r)p = r \bmod q$. $D_{\frac{1}{2}} = \{\frac{r}{q} - j_2(r)\theta, \ r = 0, ..., q-1\}$.

c) Let $t = 1 - \frac{\alpha}{2}$. For each integer r there is $j_3(r) \in \{0, ..., q-1\}$ such that $-(p' + 1 + j_3(r)p) = r \bmod q$. $D_{1-\frac{\alpha}{2}} = \{\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta : \ r = 0, ..., q-1\}$.

d) Let $t = \frac{1}{2} - \frac{\alpha}{2}$. For each integer r there is $j_4(r) \in \{0, ..., q-1\}$ such that $q' - (p' + 1 + j_4(r)p) = r \bmod q$. $D_{\frac{1}{2}-\frac{\alpha}{2}} = \{\frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta : \ r = 0, ..., q-1\}$.

The set of discontinuities of ζ_q is $D_0 \cup D_{\frac{1}{2}} \cup D_{1-\frac{\alpha}{2}} \cup D_{\frac{1}{2}-\frac{\alpha}{2}}$. Observe that in all cases $|j_i(r)\theta| \leq |q\theta|$; since $(j_2 - j_1)p = \frac{1}{2}q \bmod q$ and $(j_4 - j_3)p = \frac{1}{2}q \bmod q$, we have

$$j_2(r) = j_1(r) \pm \frac{1}{2}q, \ j_4(r) = j_3(r) \pm \frac{1}{2}q.$$

We are going to determine the values taken by the cocycle $\zeta_q(x)$ for x in a neighborhood of the typical interval $[\frac{r}{q}, \frac{r+1}{q}]$, where r is an integer in $\{1, ..., q-1\}$. Assume, for concreteness, that $\theta < 0$, $j_1 = j_2 + \frac{1}{2}q$, $j_4 = j_3 + \frac{1}{2}q$. Let x start at $\frac{r}{q}$ and let it move to the right; set $\zeta_q(x) = a$. The value of the cocycle $\zeta_q(x)$ is constant until x crosses the discontinuity (corresponding to $t = 0$) at $\frac{r}{q} - j_1(r)\theta$, where $\zeta_q(x)$ increases by 1. After that the cocycle does not change until x crosses the discontinuity at $\frac{r}{q} - j_2(r)\theta$ (corresponding to $t = \frac{1}{2}$) where the cocycle decreases by 1, returning to the value a.

These two first discontinuities occur before x crosses the two other discontinuities under the condition that $|j_i(r)\theta|$ is less than $\frac{1}{2q}$. This takes place if $q^2|\theta| < \frac{1}{2}$, a condition which holds below because we consider the case when $q^2|\theta|$ is small. As x continues to move to the right, the cocycle remains at the value a until, near $\frac{r}{q} + \frac{1}{2q}$, it increases by 1 at the point $\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta$ (a discontinuity corresponding to $t = 1 - \frac{\alpha}{2}$) and then decreases by 1 at $\frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta$ (a discontinuity corresponding to $t = \frac{1}{2} - \frac{\alpha}{2}$). Therefore, we have

$$
\begin{aligned}
\zeta_q &= a \pm 1 \text{ on }]\frac{r}{q} - j_1(r)\theta, \frac{r}{q} - j_2(r)\theta[, \\
\zeta_q &= a \pm 1 \text{ on }]\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta, \frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta[. \\
\zeta_q &= a, \text{ elsewhere.}
\end{aligned}
$$

This analysis is valid for every interval $[\frac{r}{q}, \frac{r+1}{q}]$. In particular, this implies that $\zeta_q = a$ on a subset of large measure in $[0, 1]$. Since the mean value of ζ_q is zero, we have $a = 0$.

Suppose that one of the cases 1), 2), 3) holds for an infinite sequence (n_k) such that $a_{n_k+1} \to \infty$. If this condition is not satisfied, then it is easier to show that 1 is a quasi-period of the cocycle. If case 1) occurs infinitely often, then it is easy to show that 1 is a quasi-period for (ζ_n). See figure 3.

Suppose now that case 2) occurs infinitely often. For $1 \leq r \leq q_{n_k} - 1$ set

$$\begin{aligned} I_{k,r} &=]\frac{r}{q_{n_k}} - j_1(r)\theta_{n_k}, \frac{r}{q_{n_k}} - j_2(r)\theta_{n_k}[, \\ J_{k,r} &=]\frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - (j_3(r) + \frac{1}{2})\theta_{n_k}, \frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - (j_4(r) + \frac{1}{2})\theta_{n_k}[. \end{aligned}$$

These intervals have length $\frac{1}{2}q_{n_k}|\theta_{n_k}|$. At the scale $\frac{1}{q_{n_k}}$, they are close to $\frac{r}{q_{n_k}}$ and to $\frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}}$ respectively. Outside of these intervals, $\zeta(q_{n_k}, \cdot) = 0$. Set $t_k = \lfloor \delta a_{n_k+1} \rfloor$, for $\delta \in]0, \frac{1}{4}[$. Set

$$A_k = \cup_{j=0}^{q_{n_k}-1} \cup_{s=0}^{t_k-1} (I_{k,j} - sq_{n_k}\alpha), \;\; B_k = \cup_{j=0}^{q_{n_k}-1} \cup_{s=0}^{t_k-1} (J_{n_k,j} - sq_{n_k}\alpha).$$

The distance between the intervals $I_{n_k,r}$ and $J_{n_k,r}$ is at least $\frac{1}{2q_{n_k}} - q_{n_k}|\theta_{n_k}|$. Since, by the choice of t_k, the translated intervals in the definition of A_k and B_k do not overlap.

Let us consider the cocycle at time $t_k q_{n_k}$. We have $\zeta(t_k q_{n_k}, x) = \sum_{s=0}^{t_k-1} \zeta(q_{n_k}, x + sq_{n_k}\alpha)$. By the previous analysis of the values of ζ_q, we have $\zeta(t_k q_{n_k}, \cdot) = \pm 1$ on A_k and B_k. Also

$$\mathrm{Leb}(A_k) = \frac{1}{2} t_k q_{n_k} q_{n_k} |\theta_{n_k}| \geq \frac{1}{2}\delta a_{n_k+1} \frac{q_{n_k}}{q_{n_k+1}} \geq \frac{1}{2}\delta > 0.$$

Since $t_k q_{n_k}\alpha \bmod 1 \to 0$, and since on A_k the cocycle takes at most the two values ± 1, we have shown that 1 or -1 is a quasi-period for the cocycle (ζ_n). The claim now follows, by Lemma 3.2.

3. **Cocycles and Maharam measures**

Finally, we will apply to this model the construction of conformal measures for a rotation on $X = \mathbb{T}^1$, and show the existence of non uniformly distributed orbits in the plane.

3.1. **Conformal measures.**

We recall how to construct such measures and the corresponding locally finite τ_φ-invariant measures.

Let ζ be a continuous real valued positive function on X. For an ergodic rotation τ, if $\int_X \log\zeta \ dx = 0$, then there exists a quasi-invariant probability measure ν such that $\tau\nu = \zeta\nu$. If $\log\zeta$ has bounded variation, ν is not atomic and unique. Therefore:

If φ is a function on the circle with values in $\mathbb{R}^d$ with bounded variation and null integral, then for every ergodic rotation τ on the circle and for every exponential χ on $\mathbb{R}^d$, there exists a unique non-atomic probability measure μ_χ (called $\chi \circ \varphi$-conformal) such that

$$\tau\mu_\chi = \chi \circ \varphi \circ \tau^{-1} \ \mu_\chi. \tag{12}$$

The following measure (Maharam measure) is a τ_φ-invariant locally finite measure.

$$\lambda_\chi(dx, dy) := \mu_\chi(dx) \times \chi(y)dy. \tag{13}$$

The τ_φ-invariant ergodic locally finite measures have been described for some cocycles over a 1-dim rotation by J. Aaronson, H. Nakada, O. Sarig, R. Solomyak. They proved that for $\varphi = 1_{[0,\beta]} - \beta$, if τ is a rotation by α, assuming an arithmetical condition on (α, β), then the τ_φ-invariant ergodic measures on $X \times \mathbb{R}$ are given by (13). (See also (C.) for extensions.)

The methods do not apply immediately to the case of a centered $\mathbb{Z}$-valued cocycle. We will use the notion of quasi-period.

Definition 3.1. Let $(X, \mathcal{A}, \mu, \tau)$ be a dynamical system, where μ is quasi-invariant. We say that an element $c \in \mathbb{R}^d$ is a *quasi-period* of a cocycle $(\varphi(n, .))$ with values in $\mathbb{R}^d$ if there exist $\delta > 0$, $C > 0$, and sequences $(\ell_n)_{n\geq 1}$, $(\varepsilon_n > 0)_{n\geq 1}$, with $\lim_n \varepsilon_n = 0$, such that

$$C^{-1} \leq \tfrac{d(\tau^{\ell_n}\mu)}{d\mu} \leq C. \tag{14}$$

$$\lim_n \tau^{\ell_n} x = x, \forall x \in X, \quad \mu\{x : d(\varphi_{\ell_n}(x), c) < \varepsilon_n\} > \delta. \tag{15}$$

Let χ be an exponential on $\mathbb{R}^d$ and μ_χ the probability measure ($\chi \circ \varphi$-conformal) such that $\tau\mu_\chi = \chi \circ \varphi \circ \tau^{-1}\, \mu_\chi$. The measure $\lambda(dx, dy) := \mu(dx)\chi(y)dy$ on $X \times \mathbb{R}^d$ is invariant.

If φ is BV, and if the sequence (ℓ_n) is extracted from the sequence (q_n) associated to α, then (14) is satisfied, with $C = e^{Var(\varphi)}$, as a consequence of the inequality of Denjoy-Koksma.

Lemma 3.2. *Any quasi-period of the cocycle $(\varphi(n, .))$ over (X, μ_χ, τ) is a period of the τ_φ-invariant functions (relatively to the measure λ_χ).*

3.2. **Application to the billiard.** In the "good" case 1) of the previous discussion, the Maharam measures λ_χ can be shown to be ergodic for the directional billiard map.

Therefore, for points in the support of the conformal measure, the equirepartion mentioned above does not hold. For these points, the ergodic theorem implies that there is a half-plane (defined by the exponential χ) which is more frequently visited by the ball moving according the directional billiard map, with rational directions and the small obstacles property.