Functional CLT for stationary random fields: some examples and methods

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Introduction

Our framework will be $\mathbb{Z}^d$-dynamical systems, in the sense of ergodic theory, that is: probability spaces $(E, \mathcal{F}, \mu)$ (μ can be sometimes $\mathbb{P}$) with d commuting measure preserving maps $T_1, ..., T_d$. The $\mathbb{Z}^d$-action generated by $T_1, ..., T_d$ will be denoted Θ. For $x \in E$, we write: $T^{\underline{k}}x = T_1^{k_1} ... T_d^{k_d} x$, with the notation $\underline{k} = (k_1, ..., k_d)$.

Some classes of $\mathbb{Z}^d$-dynamical systems:

1) A class of models can be described with a "local" structure by the following data:
S a set of "local states", $\Omega = S^{\mathbb{Z}^d}$ the configuration space, $X_{\underline{k}} : \omega \to \omega_{\underline{k}}$ the coordinate maps, $\mathcal{F}$ the σ-field on Ω generated by the cylindric sets. The shift on coordinates is a natural $\mathbb{Z}^d$-action Θ on Ω. If a probability measure μ on $(\Omega, \mathcal{F})$ is invariant under the shift, $(\Omega, \mathcal{F}, \mu, \Theta)$ is $\mathbb{Z}^d$-dynamical system.

The simplest examples are the $\mathbb{Z}^d$-Bernoulli schemes defined as the shift action on $\Omega = S^{\mathbb{Z}^d}$ endowed with a product probability measure $p^{\otimes \mathbb{Z}^d}$, where p is some probability vector on S. The method of specification given by a potential in statistical mechanics gives a way to construct invariant measures of Markovian type.

For a real $f \in L^2(\Omega, \mathcal{F}, \mu)$ with $\mu(f) = 0$, the family $(T^{\underline{k}} f)_{\underline{k} \in \mathbb{Z}^d}$, where $(T^{\underline{k}} f)(x) = f(T^{\underline{k}} x)$, is a strictly stationary real random field indexed by $\mathbb{Z}^2$ with finite second moments and zero mean.

2) Another class of $\mathbb{Z}^d$-dynamical system is given by algebraic models. Examples:

- commuting endomorphisms on compact abelian groups;
- translations and flows on homogeneous spaces.

In the first case, the space is a compact abelian group G endowed with its Haar measure μ and the maps $T_1, ..., T_d$ are commuting surjective algebraic endomorphisms on G. If they are not invertible, we can extend them to a bigger group to get an invertible $\mathbb{Z}^d$-action.

In the algebraic examples, there is no "local" structure. We get such a local structure by taking finite partitions of the space, like a Markov partition in the case of a single hyperbolic automorphism on a torus, but for a multi-dimensional action this seems less easy. As above, for a real "observable" $f \in L^2(\mu)$ with $\mu(f) = 0$, $(T^{\underline{k}} f)_{k \in \mathbb{Z}^k}$ is a strictly stationary real random field with finite second moments and zero mean.

In general, we need a regularity assumption on the observables: in terms of dependence on far coordinates for the model with a local structure, in terms of Hölder regularity for the smooth models.

Notation: The set of finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{V}$. For D in $\mathcal{V}$, $|D|$ is the cardinal of D and $S_D f$ is

$$S_D f(x) = \sum_{\underline{\ell} \in D} f(T^{\underline{\ell}} x). \tag{1}$$

In what follows, (D_n) is an increasing sequence in $\mathcal{V}$, with boundary ∂D_n such that $\lim_{n\to\infty} |\partial D_n|/|D_n| = 0$. (a condition equivalent to the Følner property).

A question is to investigate if a Central Limit Theorem (CLT) holds for the normalized sums $S_{D_n} f(x)/\|S_{D_n} f\|_2$. In some cases a stronger result, the functional CLT, can be expected.

The results are often given for an increasing sequence of rectangles. But triangular domains or balls are also natural. Another method is the summation along the trajectories of a random walk on $\mathbb{Z}^d$.

Our goal in the talk is to mention some models and examples and to describe methods which can be applied to the CLT or the FCLT.

For $d = 1$, an important tool is the martingale property which can be used in many models. For $d > 1$, in some cases, a martingale method can still be used, but even in very simple models like the action of commuting endomorphisms or automorphisms on the torus, this may be impossible. Another difficulty in the study of some algebraic multidimensional models is the possible lack of mixing, even for an action with a strong stochasticity.

After recalling some properties for multidimensional dynamical systems, I will describe some examples belonging to two classes mentioned above: 1) from statistical mechanics, 2) from algebraic origin. The goal is not to give general conditions implying a limit theorem, but to look at explicit examples.

In the last part, I will describe some methods of proof for the CLT and the FCLT.

Remark: Methods of martingale and mixing methods are adequate sometimes to dynamical systems which have property K and they have been used in some models in statistical mechanics. But other methods are to be used for systems like the algebraic ones.

1. Some properties

1.1 Spectral properties and variance

If for every $f \in L^2_0(E, \mu)$ the spectral measure of f for the Θ-action has a density (denoted by φ_f), then we have the Lebesgue spectrum. Equivalently, there is a family $(\psi_j)_{j \in J}$ such that the collection of functions $\{T^{\underline{k}} \psi_j, \underline{k} \in \mathbb{Z}^d, j \in J\}$ is an orthonormal basis in $L^2_0(\mu)$. The set of indices J is countable in the standard case.

For "observables" satisfying an assumption of absolute summability of the decorrelations:

$$\sum_{\underline{k} \in \mathbb{Z}^d} \left| \int_E T^{\underline{k}} f \, \overline{f} \, d\mu \right| < \infty, \tag{2}$$

the spectral density of f is continuous, i.e., there is $\varphi_f \in C(\mathbb{T}^d)$ (even in the space $AC(\mathbb{T}^d)$) such that

$$\int_E T^{\underline{k}} f \, \overline{f} \, d\mu = \int_{\mathbb{T}^d} e^{2\pi i \langle \underline{k}, \underline{t} \rangle} \, \varphi_f \, d\underline{t}, \ \forall \underline{k} \in \mathbb{Z}^d. \tag{3}$$

Summation on sets $(D_n) \subset \mathbb{Z}^d$*, Variance.* The computation of the variance is based on the normalized non-negative kernel

$$K(D_n)(t) = |D_n|^{-1} \left| \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle} \right|^2. \tag{4}$$

The Følner property, i.e.,

$$\lim_n \frac{|D_n \cap (D_n - \underline{p})|}{|D_n|} = 1, \forall \underline{p} \in \mathbb{Z}^d$$

is equivalent to

$$\int_{\mathbb{T}^d} K(D_n)(\underline{t})\, \varphi_f(\underline{t})\, d\underline{t} = |D_n|^{-1} \int |S_{D_n} f(x)|^2 \, d\mu \underset{n\to\infty}{\to} \varphi_f(0). \qquad (5)$$

Therefore, if $\varphi_f(0) \neq 0$, the sums $S_{D_n} f$ must be normalised by dividing by a quantity of order $|D_n|^{\frac{1}{2}}$.

Non-degeneracy? In some cases, it can be shown that it occurs iff the observable is a''mixed coboundary''.

Sets of summation: a way to construct a natural family of Følner sequences of sets.

The Lebesgue measure in $\mathbb{R}^d$ is denoted by λ. Let Δ be a set in $[0,1]^d$. The set $\mathbb{Z}^d \bigcap n\Delta$ of integral points in $n\Delta$ is denoted by Δ_n.

If Δ is Jordan-measurable, then $\lim_n \frac{|\Delta_n|}{n^d} = \lambda(\Delta)$. Moreover, the sequence $(\Delta_n)_{n\geq 1}$ satisfies the Følner property.

We need to define what means "functional CLT" (or "invariance principle"). First we have to embed the sums in a continuous time model:

Let $\mathcal{J}^+$ be the class of Jordan-measurable set such that $(t\Delta)_{t\in[0,1]}$ is an increasing family. We consider the sums depending on the parameter $t \in [0,1]$;

$$S_{n,t}^{\Delta} f = \sum_{\underline{\ell} \in tn\Delta} T^{\underline{\ell}} f,$$

or the "smoothed" sums (where for $\underline{\ell} \in \mathbb{Z}^d$, $R(\underline{\ell})$ is the unit cube $\{\underline{t} : \ell_1 \leq t_1 \leq \ell_1 + 1, ..., k_d \leq t_d \leq \ell_d + 1\}$):

$$\tilde{S}_{n,t}^{\Delta} f := \sum_{\underline{\ell}} \lambda(nt\Delta \cap R(\underline{\ell})) T^{\underline{\ell}} f. \tag{6}$$

They are r.v.s with values in the space of continuous functions on $[0,1]$. For the smoothed sums, one says that a functional limit theorem holds if, after normalisation, convergence in distribution holds in the space of continuous functions on $[0,1]$ endowed with the uniform norm to $(W_t)_{t\in[0,1]}$ the standard Brownian.

Note that we can also embed the sums in a process with a multi-dimensional continuous time model.

1.2 K-systems

For simplicity take $d = 2$ and consider an action Θ of $\mathbb{Z}^2$ defined by two commuting automorphisms T_1 and T_2 of a probability space $(E, \mathcal{F}, \mu)$. The entropy of Θ can be defined. The Pinsker factor is the largest factor with zero entropy. The action Θ has a completely positive entropy if its Pinsker factor is trivial.

The notion of K-systems can be defined for a $\mathbb{Z}^2$-action in the following way: For a sub σ-algebra $\mathcal{B}$ of $\mathcal{F}$, we denote by $\mathcal{B}^{T_i}$, $i = 1, 2$, the σ-algebra generated by $(\cup_{n=-N}^{N} T_i^n \mathcal{B}, N \geq 1)$. Let us choose the lexicographic order on $\mathbb{Z}^2$ given by:

$$(p, n) \leq (p', n') \Leftrightarrow [(p = p' \text{ and } n \leq n') \text{ or } (p < p')].$$

A sub-σ-algebra $\mathcal{F}_0$ is a K-σ-algebra if

(i) $\mathcal{F}_0$ is increasing in the sense of the lexicographic order for the $\mathbb{Z}^2$-action : $(p, n) \leq (0, 0) \Rightarrow T_1^{-p} T_2^{-n} \mathcal{F}_0 \subset \mathcal{F}_0$;

(ii) $\mathcal{F}_0$ is generating: the σ-algebra generated by $\cup_{p,n} T_1^{-p} T_2^{-n} \mathcal{F}_0$ is $\mathcal{F}$;

(iii) the remote past of $\mathcal{F}_0$ is trivial, i.e.,

$$\bigcap_p T_2^p \mathcal{F}_0 = T_1 \mathcal{F}_0^{T_2} \text{ and } \bigcap_p T_1^p \mathcal{F}_0^{T_2} \text{ is trivial.}$$

A system is K if there is a K-σ-algebra in $\mathcal{F}$.

Due to the choice of the lexicographic order, there is a dissymmetry in the definition. But it can be shown that the property itself doesn't depend on the choice of the order between the coordinates. The existence of such a sub σ-algebra implies completely positive entropy and conversely Kamiński (1981) proved the existence of a K-sub σ-algebra, when the action has completely positive entropy.

If α_0 is a finite partition, denoting by $V_{\underline{0}}$ the set of all $\underline{k} \in \mathbb{Z}^2$ smaller than $\underline{0}$ in the lexicographic order. Then $\mathcal{F}_0 = \vee_{\underline{k} \in V_{\underline{0}}} T^{\underline{k}} \alpha_0$ is the sub-σ-algebra of the past generated by α_0 under the action of T_1, T_2.

In the model $\Omega = S^{\mathbb{Z}^d}$, α_0 can be taken as the partition associated to the first coordinate $X_{\underline{0}}$.

Taking an orthonormal basis $(\psi_j)_{j \in J}$ of $L^2_0(\mathcal{F}_0, \mu) \setminus L^2_0(T_2^{-1}\mathcal{F}_0, \mu)$, the collection of functions $\{T^{\underline{k}}\psi_j, \underline{k} \in \mathbb{Z}^2, j \in J\}$ is an orthonormal basis of $L^2_0(\mu)$ (hence Lebesgue spectrum, mixing of all orders).

If μ is trivial on the tail field $\mathcal{F}_\infty := \bigcap_{V \in \mathcal{V}} \mathcal{F}_{V^c}$, then the K-property holds. This is a stronger property than the non-symmetric analogous K-property. In dimension 1, it corresponds to the 2-sided K-property.

The simplest examples of K-systems for a $\mathbb{Z}^2$-action are the $\mathbb{Z}^2$-Bernoulli schemes. Observe that, on a manifold, commuting actions by d measure preserving smooth maps generate d-dimensional systems of d-dimensional entropy zero if $d \geq 2$.

Now we give examples from statistical mechanics.

2. Examples 2.1 from statistical mechanics

We recall briefly the notation for a Gibbs measure on $\Omega = S^{\mathbb{Z}^d}$ associated to a specification. For $V \subset \mathbb{Z}^d$, $\mathcal{F}_V$ is the sub σ-algebra of events depending only on the sites in V, and X_V the projection on the set of configurations in V.

A specification is a family Λ of conditional probabilities indexed by $V \in \mathcal{V}$ and interpreted as the probability to have ω inside V given η outside V:

$$\Lambda = (\lambda_V(\omega|\eta), \omega \in \Omega_V, \eta \in \Omega)_{V \in \mathcal{V}},$$

with $\lambda_V(\omega|.)$ $\mathcal{F}_{V^c}$-measurable and $\lambda_V(.|\eta)$ a probability on Ω_V.

A probability measure μ on Ω is specified by Λ if, for all $V \in \mathcal{V}$,

$$\mu\Big(X_V = \omega \,|\mathcal{F}_{V^c}\Big) = \lambda_V(\omega|\,.), \mu - \text{a.s.}, \forall \omega \in \Omega_V.$$

If Λ is stationary $(\lambda_V(\omega|\eta) = \lambda_{V+\underline{k}}(\omega|T^{-\underline{k}}\eta), \forall \underline{k})$, we denote by $\mathcal{G}_\Theta(\Lambda)$ the set of stationary Gibbs measures specified by Λ.

Each positive continuous specification Λ s.t. $\mathcal{G}(\Lambda) \neq \emptyset$ coincides with the system of Gibbs distribution associated to an interaction potential U.

Two probability measures μ and ν in $\mathcal{G}(\Lambda)$ coincide iff they coincide on the remote σ-field $\mathcal{F}_\infty$. Thus, when it holds, an uniqueness theorem for a given model in statistical mechanics provides an example of multidimensional K-system.

Example: Ising model: The specification $\Lambda_{T,h}$ associated to $U = U_{T,h}$ depending on two parameters defined by

$$\begin{aligned} U(A,\omega) \quad &= -\beta\omega_{\underline{s}}\,\omega_{\underline{t}}, \text{ if } A = \{\underline{s},\underline{t}\}, \text{ diam} A = 1, \\ &= -h\omega_{\underline{s}}, \text{ if } A = \{\underline{s}\}, = 0 \text{ otherwise.} \end{aligned}$$

$\beta > 0$ corresponds to the ferromagnetic and $\beta < 0$ to the anti-ferromagnetic model, h is the external magnetic field and $|\beta|^{-1}$ is proportional to the temperature T.

1) if Λ is defined by the d-dimensional Ising ferromagnet for large β and $h = 0$, $d \geq 2$, then: the extremal points of $\mathcal{G}_\Theta(\Lambda)$ consist in two distinct measures μ^+, μ^- describing the pure phases of the ferromagnet. This gives two K-systems $(\Omega, \mathcal{F}, \mu^+, \Theta)$, $(\Omega, \mathcal{F}, \mu^-, \Theta)$ isomorphic via the spin flip $\omega \to -\omega$.

2) If Λ is defined by the d-dimensional Ising ferromagnet with $h \neq 0$, then $\mathcal{G}(\Lambda) = \{\mu\}$ is reduced to a single measure and $(\Omega, \mathcal{F}, \mu, \Theta)$ is a K-system.

2.2 Examples from algebraic origin

As mention in the introduction, we will discuss examples given by commuting endomorphisms on compact abelian groups. Other examples are given by flows on homogeneous spaces, with recent results on the CLT obtained by M. Björklund and A. Gorodnik.

Let G be a compact abelian group G endowed with its Haar measure μ and commuting surjective algebraic endomorphisms $T_1, ..., T_d$. Call Θ the action generated by $T_1, ..., T_d$ viewed as a $\mathbb{Z}^d$- or $\mathbb{N}^d$-action.

In case for example of commuting matrices on tori, the entropy is 0, therefore even if there is a strong stochasticity like mixing of all orders, for $d > 1$, they do not generate K-systems.

K. Schmidt and T. Ward (1993) proved that
If the group G is zero-dimensional, Θ has completely positive entropy iff Θ is r-mixing for every $r \geq 2$.

If the group G is connected, then Θ is r-mixing for every $r \geq 2$.

As shown by F. Ledrappier (1978), there are non-connected groups G with 2-mixing algebraic actions Θ which are not 3-mixing.

Ledrappier's example is a special case of a general construction of $\mathbb{Z}^d$-actions by automorphisms on shift invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$.

These $\mathbb{Z}^d$-actions are examples of mixing actions which are not mixing of all orders. Nevertheless for a special class of such actions, it is possible to show that the non-mixing r-configurations (or patterns) in $(\mathbb{Z}^d)^r$ are rare in some sense, and deduce a CLT for these $\mathbb{Z}^d$-actions (JP C., G Cohen (2016)).

3. Martingale-like methods for the CLT

3.1 K-systems and CLT under $\mathbb{Z}^2$-actions for rotated sums

Of course the K-property does not suffice to provide for free a CLT, even if it is a favorable framework. Nevertheless, if we make a rotation on the action in the ergodic sums over cubes, the CLT is satisfied, without assumption on the observable, for almost all rotations. This surprising result was obtained by M. Peligrad and W. B. Wu (2010) in dimension 1. With Guy Cohen, we gave a simplified proof and its extension for multidimensional processes.

Let us briefly explain how to obtain a CLT for rotated sums of L_2^0-functions on $\mathbb{Z}^d$-systems, $d > 1$, which have the K-property.

Take $d = 2$. The ergodic sums of the rotated process over the square $R_N = [0, N-1] \times [0, N-1]$ (like a periodogram) are

$$S_N^{\theta_1,\theta 2} f = \sum_{(k,\ell)\in R_N} e^{2\pi i \langle k\theta_1 + \ell\theta_2\rangle} T_1^k T_2^\ell f.$$

Let $\mathcal{F}_0$ be a K-sub-σ-algebra. As already mention, if $\mathcal{K}_0$ is the subspace in $L^2(\mathcal{F}_0)$ orthogonal of $L_0^2(T_2\mathcal{F}_0)$ and $(\psi_j)_{j\in J}$ an orthonormal basis of $\mathcal{K}_0$, then $(T^{\underline{k}}\psi_j)_{j\in J, \underline{k}\in\mathbb{Z}^2}$ is an orthonormal basis of $L_0^2(\Omega, \mu)$.

For $f \in L_2^0$, setting $a_{j,\underline{n}} := \langle f, T^{\underline{n}}\psi_j\rangle$, for $j \in J$, let γ_j be an everywhere finite square integrable function on $\mathbb{T}^2$ with Fourier coefficients $a_{j,\underline{n}}$. The spectral measure of f for the $\mathbb{Z}^2$ action has a density given by $\varphi_f = \sum_{j\in J} |\gamma_j|^2$. Since

$$\int_{\mathbb{T}^2} \sum_{j\in J} |\gamma_j(\theta_1,\theta_2)|^2 \; d\theta_1 d\theta_2 = \sum_{j\in J} \sum_{\underline{n}\in\mathbb{Z}^2} |a_{j,\underline{n}}|^2$$
$$= \int \varphi_f(\theta_1,\theta_2) \; d\theta_1 d\theta_2 = \|f\|^2 < \infty,$$

the set $\Lambda_0 := \{\theta \in \mathbb{T}^2 : \sum_{j\in J} |\gamma_j(\theta)|^2 < \infty\}$ has full measure.

$$\text{For } \theta \in \Lambda_0, \text{ let } M_\theta f := \sum_j \gamma_j(\theta)\, \psi_j \in \mathcal{K}_0. \tag{7}$$

Using that, if φ is integrable, then $\lim_{N\to\infty} F_{N,N} * \varphi = \varphi$ a.e., where $F_{N,N}$ is the bi-dimensional Fejèr's kernel. one can show that the set

$$\Lambda(f) = \{\theta : \lim_N \frac{1}{|R_N|} \left\| \sum_{\underline{k}\in R_N} \mathrm{e}^{2\pi i(k_1\theta_1+k_2\theta_2)}\, T_1^{k_1} T_2^{k_2}(f - M_\theta f) \right\|_2^2 = 0\}. \tag{8}$$

has full measure in $\mathbb{T}^2$.

Martingale approximation: Let f be in $L^2_0(\mu)$ and $M_{(\theta_1,\theta_2)} \in \mathcal{K}_0$ be associated to f for a.e. (θ_1, θ_2) by (7). The martingale property is satisfied by $M_{(\theta_1,\theta_2)} f$:

$$\mathbb{E}(M_{(\theta_1,\theta_2)} f \circ T_1^{k_1} T_2^{k_2} | T_1^{-k'_1} T_2^{-k'_2} \mathcal{A}_0) = 0, \ \forall\, \underline{k} > \underline{k}', \tag{9}$$

Using a CLT for martingales, we obtain:

$$\frac{1}{n} \sum_{\underline{k} \in R_n} M_\theta(T_1^{k_1} T_2^{k_2} \omega) \overset{distr}{\longrightarrow} \mathcal{N}(0, \Gamma(\theta))$$

with respect to μ, where $\Gamma(\theta)$ is the covariance matrix

$$\Gamma(\theta) = \begin{pmatrix} \frac{1}{2}\varphi_f(\theta_1, \theta_2) & 0 \\ 0 & \frac{1}{2}\varphi_f(\theta_1, \theta_2) \end{pmatrix}. \tag{10}$$

Now by the martingale approximation given by (8) it follows:

Theorem 1. *Let $(\Omega, \mathcal{F}, \mu, (T_1, T_2))$ be a 2-dimensional K-system. Let f be in $L^2_0(\mu)$ with spectral density φ_f. Then for Lebesgue-a.e. θ the asymptotic distribution (with respect to μ) of*

$$\left(\frac{1}{n} \sum_{\underline{k} \in R_n} \cos 2\pi(k_1\theta_1 + k_2\theta_2)\, T_1^{k_1} T_2^{k_2} f, \ \frac{1}{n} \sum_{\underline{k} \in R_n} \sin 2\pi(k_1\theta_1 + k_2\theta_2)\, T_1^{k_1} T_2^{k_2} f)\right)$$

is the centered normal law in $\mathbb{R}^2$ with covariance matrix (10).

3.2 Method of mixing, Dedecker's results

Let us mention briefly E. Bolthausen (1982) who has a simple proof for a CLT under mixing assumptions, which should apply to models in statistical mechanics, although this is not stated clearly in his paper.

Mention also the results of B.S. Nahapetian (1980) and B.S. Nahapetian and A.N. Petrosian (1992): "Martingale difference, Gibbs random fields and Central Limit Theorem" with ad-hoc hypotheses on their model.

All these results could probably be extended to a FCLT.

Now I describe some of the results of J. Dedecker (1998, 2001) on the CLT and FCLT with an application to Gibbs measures.

Dedecker's results

Let $(X_i)_{i\in\mathbb{Z}^d}$ be a strictly stationary field of real-valued random variables with mean zero and finite variance.

For $k \geq 1$ and $\underline{i} \in \mathbb{Z}^d$, let

$$\begin{aligned} V_{\underline{i}}^1 &:= \{\underline{j} \in \mathbb{Z}^d : \underline{j} < \underline{i}\}, \\ V_{\underline{i}}^k &:= V_{\underline{i}}^1 \cap \{\underline{j} \in \mathbb{Z}^d : \max_{1\leq t\leq d} |i_t - j_t| \geq k\} \\ \mathcal{F}_{|k|} &:= \sigma\{X_{\underline{i}},\ \underline{i} \leq \underline{0},\ \max_{1\leq t\leq d} |i_t| \geq \max_{1\leq t\leq d} |k_t|\}. \end{aligned}$$

Condition used for the FCLT

(conditioning by σ-algebras generated over lower-left quadrants):

$$\sum_{\underline{k}\in V_{\underline{0}}^1} \|X_{\underline{k}}\, \mathbb{E}(X_{\underline{0}}|\mathcal{F}_{|k|})\|_p < \infty. \tag{11}$$

If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, the smoothed partial sums $\{S_n(A), A \in \mathcal{A}\}$ are

$$S_n(A) = \sum_{\underline{i}\in\mathbb{Z}^d} \lambda(nA \cap R(\underline{i}))\, X_{\underline{i}}.$$

We say that the sequence $n^{-d/2} S_n(A)$ satisfies a functional central limit theorem if it converges in distribution to a mixture of set-indexed Brownian motions in the space $C(A)$ (that is: the limiting process is of the form ηW, where W is a standard Brownian motion and η is a nonnegative random variable independent of W).

For any $\underline{t} \in [0,1]^d$, the lower-left quadrant $[0,\underline{t}]$ with upper corner at $\underline{t}$ is $[0,\underline{t}] := [0,t_1] \times ... \times [0,t_d]$. Denote by $\mathcal{Q}_d$ the collection of lower-left quadrants in $[0,1]^d$.

If (11) holds with $p = 1$, the finite-dimensional convergence of $\{n^{-d/2} S_n(A)\}$ is a consequence of a central limit theorem established in Dedecker (1998).

Theorem 2. *(Dedecker (2001)) Let $(X_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$ be a strictly stationary field of centered random variables. Assume that there exists $p > 1$ such that $\|X_{\underline{0}}^2\|_p$ is finite and the L_p criterion (11) is satisfied. Then*

(a) $\sum_{\underline{k}\in\mathbb{Z}^d} \|\mathbb{E}(X_{\underline{0}}X_{\underline{k}}|\mathcal{I})\|_p < \infty$. Let $\eta = \sum_{\underline{k}\in\mathbb{Z}^d} \mathbb{E}(X_{\underline{0}}X_{\underline{k}}|\mathcal{I})$, where $\mathcal{I}$ is the σ-algebra of invariant sets;

(b) the sequence $\{n^{-d/2} S_n(\underline{t}) : \underline{t} \in [0,1]^d\}$ converges in distribution in $C(\mathcal{Q}_d)$ to $\sqrt{\eta}W$, where W is a standard Brownian motion indexed by $\mathcal{Q}_d$ and independent of $\mathcal{I}$.

Gibbs example (Dedecker (2001)) Application to spin systems

Assume that the random variable X_0 is bounded and that the distribution of the random field is a Gibbs measure associated to a specification Λ given by a finite-range potential.

Suppose that the family the Gibbs specifications Λ satisfies the *weak mixing condition* (Dobrushin and Shlosman (1985), Martinelli and Olivieri (1994)). This condition implies that, if in a finite V we consider the Gibbs state with boundary condition τ, then a local modification of the boundary condition τ has an influence on the corresponding Gibbs measure which decays exponentially fast inside V with the distance from the boundary of V.

Under this assumption there is ergodicity and uniqueness. Moreover, there exist two positive constants C_1 and C_2 such that

$$\|\mathbb{E}(X_{\underline{0}} \mid \mathcal{F}_{V_{\underline{i}}^k}) - \mathbb{E}(X_{\underline{0}})\|_\infty \leq C_1 \exp(-C_2\, k). \tag{12}$$

Set $Y = (X_{\underline{i}} - \mathbb{E}(X_{\underline{i}}))_{\underline{i}\in\mathbb{Z}^d}$. From inequality (12) it follows that the L^1 criterion is satisfied. Consequently the theorem applies to the stationary random field Y, with $\eta = \sigma^2 = \sum_{k\in\mathbb{Z}^d} Cov(X_0, X_k)$.

Example: $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$, Ising model with parameters h, T.

As recalled previously, there exists a critical temperature T_c and a uniqueness region $\mathcal{U}$

$$\mathcal{U} = \{(h, T) \in \mathbb{R} \times [0, \infty[: h \neq 0 \text{ or } T > T_c\}.$$

The family $\Lambda_{T,h}$ is weak mixing in the following regions of $\mathcal{U}$:
(a) for any temperature $T > T_c$ (Higuchi (1993)),
b) for low temperature and arbitrarily small (not vanishing) field h provided that h/T is large enough (Martinelli and Olivieri (1994));
(c) for any (h, T) in $\mathcal{U}$ if $d = 2$ (Schonmann and Shlosman (1995).

Therefore, we get the FCLT for this model when the parameters belong to the uniqueness region $\mathcal{U}$.

4. Methods for algebraic examples

4.1 Mixing of order r for $\mathbb{N}^d$-actions

A word about the question of mixing of order r which is crucial in the study of limit theorems for $\mathbb{N}^d$ or $\mathbb{Z}^d$ algebraic actions.

For a general $\mathbb{N}^d$-action, $\underline{\ell} \to T^{\underline{\ell}}$, preserving a probability measure μ, the property of mixing of order $r \geq 2$ is that, for any r-tuple of bounded measurable functions $f_1, ..., f_r$ with 0 integral, every $\varepsilon > 0$, there is M such that

$$\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M, \forall j \neq j' \Rightarrow |\int T^{\underline{\ell}_1} f_1 ... T^{\underline{\ell}_r} f_r \, d\mu| < \varepsilon. \tag{13}$$

For the action by endomorphisms on G, mixing of order r is equivalent to: for every set $\mathcal{K} = \{\chi_1, ..., \chi_r\}$ of r characters $\neq \chi_0$, there is $M > 0$ such that $\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M$ for $j \neq j'$ implies $T^{\underline{\ell}_1}\chi_1 ... T^{\underline{\ell}_r}\chi_r \neq \chi_0$. An r-tuple $(\underline{\ell}_1, ..., \underline{\ell}_r)$ is said to be mixing for $\mathcal{K}$ if

$$T^{\underline{\ell}_1}\chi_1 ... T^{\underline{\ell}_r}\chi_r \neq \chi_0. \tag{14}$$

r-mixing for an $\mathbb{N}^d$-action by endomorphisms is equivalent to: for every set $\mathcal{K}$ of r non trivial characters, there is M s.t. $\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M, \forall j \neq j' \Rightarrow (\underline{\ell}_1, ..., \underline{\ell}_r)$ is mixing for $\mathcal{K}$.

As already mention, by K. Schmidt and T. Ward (1992) we know that every 2-mixing $\mathbb{Z}^d$-action by automorphisms or semi-group of endomorphisms on a compact connected abelian group G is mixing of all orders.

Let us see on an example how this is related to a certain type of Diophantine equations.

An example: action by $\times 2$, $\times 3 \mod 1$ on $\mathbb{T}^1$

A set $\mathcal{K}$ of non zero characters on $\mathbb{T}^1$ is given by an r-tuple $\{k_1, ..., k_r\}$ of non zero integers. Equation (14) reads $k_1 2^{a_1} 3^{b_1} + ... + k_r 2^{a_r} 3^{b_r} = 0$. Hence we consider equations written in the form:

$$k_1 2^{a_1} 3^{b_1} + ... + k_r 2^{a_r} 3^{b_r} = 1, \ ((a_1, b_1), ..., (a_r, b_r)) \in (\mathbb{Z}^2)^r. \quad (15)$$

It is known that, for a given set $k_1, ..., k_r$, there is only a finite number of r-tuples $((a_1, b_1), ..., (a_r, b_r))$ solutions of (15), if no proper subsum vanishes. This implies that the (invertible extension) 2-dimensional action generated by $\times 2, \times 3$ is mixing of all orders.

This fact on solutions of (15) is a special case of a theorem on S-units:

Theorem 3. *(Evertse, Schlickewei, W. Schmidt) Let K be an algebraically closed field of characteristic 0 and let $r \in \mathbb{N}$. Let Γ_r be a subgroup of the multiplicative group $(K^*)^r$ of finite rank ρ. For any $(a_1, ..., a_r) \in (K^*)^r$, the number $A(a_1, ..., a_r, \Gamma_r)$ of solutions $x = (x_1, ..., x_r) \in \Gamma_r$ of the equation*

$$a_1 x_1 + ... + a_r x_r = 1, \tag{16}$$

such that no proper subsum of $a_1 x_1 + ... + a_r x_r$ vanishes, satisfies the estimate $A(a_1, ..., a_r, \Gamma) \leq \exp((6r)^{3r}(\rho + 1))$.

In the above example, we have $x_i = 2^{a_i} 3^{b_i}$ and we are looking at the number of solutions in Equation (16) where the unknown is $x = (x_1, ..., x_r)$ and $a_i = k_i$ are fixed integers.

Corollary 4. *For any finite subset E of Γ_r the number $N(E, r)$ of solutions $x = (x_1, ..., x_r)$ of (16) such that $x_i \in E, \forall i$, is $\leq C|E|^{r'}$, if $r = 2r' + 1$ is odd or $r = 2r'$ is even.*

At the opposite, in the non connected case (for example for endomorphisms of shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$), there are infinitely many non-mixing r-tuples, for $r \geq 3$.

4.2 Matrices and automorphisms of the torus

Finally, for the algebraic models, we will present the special case of matrices acting of the torus $G = \mathbb{T}^\rho$.

Every B in the semi-group $\mathcal{M}^*(\rho, \mathbb{Z})$ of non singular $\rho \times \rho$ matrices with coefficients in $\mathbb{Z}$ defines a surjective endomorphism of $\mathbb{T}^\rho$ and a measure preserving transformation on $(\mathbb{T}^\rho, \mu)$. It defines also a dual endomorphism of the group of characters $\widehat{\mathbb{T}^\rho}$ identified with $\mathbb{Z}^\rho$ (action by the transposed of B). When B is in the group $GL(\rho, \mathbb{Z})$ of matrices with coefficients in $\mathbb{Z}$ and determinant ± 1, it defines an automorphism of $\mathbb{T}^\rho$.

It is well known that $A \in \mathcal{M}^*(\rho, \mathbb{Z})$ acts ergodically on $(\mathbb{T}^\rho, \mu)$ if and only if A has no eigenvalue root of unity.

Let $A_1, ..., A_d$ be d commuting matrices in $\mathcal{M}^*(\rho, \mathbb{Z})$. Let us write $T_j x = A_j x \mod 1$, for $x \in \mathbb{T}^\rho$

With the notation $T^{\underline{\ell}} x = A_1^{\ell_1} ... A_d^{\ell_d} x \mod 1$, we get a $\mathbb{Z}^d$-action $(T^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ on $(\mathbb{T}^\rho, \mu)$, which is totally ergodic if $A^{\underline{\ell}}$ has no eigenvalue root of unity for $\underline{\ell} \neq \underline{0}$.

Regularity assumption for a function f on $\mathbb{T}^\rho$ (with $\alpha > d$):

$$\|f(.+\tau_1, \cdots, .+\tau_\rho) - f\|_2 \leq C(f)\, (\ln \frac{1}{\delta})^{-\alpha}, \text{ for } |\tau_1|, ..., |\tau_\rho| \leq \delta, \forall \delta > 0.$$

Proposition 5. *Let f be in $L_0^2(\mathbb{T}^\rho)$ satisfying the previous regularity condition for some $\alpha > d$. Then there are finite constants B_1, B_2 such that $|\langle T^{\underline{\ell}} f, f\rangle| \leq B_1 \|f\|_2 \|\underline{\ell}\|^{-\alpha}$, $\forall \underline{\ell} \neq \underline{0}$, the spectral density φ_f is continuous, $\sum_{\underline{\ell} \in \mathbb{Z}^d} |\langle T^{\underline{\ell}} f, f\rangle| < \infty$ and $\|\varphi_f\|_\infty \leq B_2 \|f\|_2$.*

For a compact abelian group, either connected or for a special family of non connected groups, with G. Cohen, we have shown a central limit theorem for summation of $(T^{\underline{k}} f)_{\underline{k} \in \mathbb{Z}^d}$ either over sets or along a random walk. I would like to describe briefly an improvement giving a FCLT obtained in collaboration G. Cohen.

Let us mention that a FCLT has been proved by a martingale method for the action generated by commuting exact algebraic endomorphisms (like $\times 2$, $\times 3$). for sums on rectangles by C. Cuny, J. Dedecker and D. Volný. But for the case of general matrices and more complicated sets, an algebraic method seems necessary.

4.3 Moments, FCLT for commuting matrices on the torus

Let $(A_j)_{j \in S}$ be a finite family of d commuting non singular matrices in $Gl(\rho, \mathbb{Z})$.

Let D be a finite set in $\mathbb{Z}^d$. First we establish a bound on the moments of order r for the function $f(x) = 2\cos(\langle \gamma, x\rangle)$, for $\gamma \in \mathbb{Z}^\rho \setminus \{0\}$, i.e.,

$$m_r(D) := \int \Big(\sum_{\underline{\ell} \in D} (\mathrm{e}^{2\pi i \langle A^{\underline{\ell}} \gamma, x\rangle} + \mathrm{e}^{-2\pi i \langle A^{\underline{\ell}} \gamma, x\rangle}) \Big)^r d\mu. \tag{17}$$

By applying Corollary 4 to the multiplicative group generated by the eigenvalues of A_j, we get:

$$|m_{2r'+1}(D)| = o(|D|^{r'}), \ m_{2r'}(D) = O(|D|^{r'}). \tag{18}$$

For every r, $f \to \|f\|_{r,D} := \|S_D f\|_r = \| \sum_{D_n} T^{\underline{k}} f\|$ is a semi-norm. Therefore we can use the sub-additivity.

Let $N_r(n, f) := |D_n|^{-1/2} \, \|S_{D_n} f\|_r$. For every $r \geq 2$, by (18) there is a constant C_r:

$$N_r(n, f) \leq \sum_{\chi \in \widehat{G}} |\widehat{f}(\chi)| \, N_r(n, \chi) \leq C_r \sum |\widehat{f}(\chi)|. \tag{19}$$

Proposition 6. *(convergence of the moments) For $f \in AC_0(\mathbb{T}^\rho)$, for every $p \geq 2$, there is C_p such that, for every $D \subset \mathbb{Z}^d$:*

$$|\int (S_D f)^p \, d\mu| \leq C_p |D|^{p/2}, \forall p \geq 1. \tag{20}$$

Moreover, we have, for any increasing Følner sequence (D_n) of sets in $\mathbb{Z}^d$:

$$\lim_n |D_n|^{-1} \int (S_{D_n} f)^2 \, d\mu = \sigma_f^2 = \varphi_f(0),$$

and, if $\sigma_f^2 = \varphi_f(0) \neq 0$, convergence toward the moments c_p of the normal distribution:

$$\lim_n \frac{\int (S_{D_n} f)^p \, d\mu}{|D_n|^{p/2}} = c_p \, \sigma_f^p, \text{ if } p \text{ is even}, = 0 \text{ if } p \text{ is odd}. \tag{21}$$

A standard method of proof for a FCLT for a process $(Y_n(t))$ is

1) Convergence of the finite dimensional distributions:

$$\forall (t_k \in [0,1])_{1 \leq k \leq r},\ (Y_n(t_1), ..., Y_n(t_r)) \underset{n \to \infty}{\Longrightarrow} (W_{t_1}, ..., W_{t_r}).$$

2) Tightness of $(Y_n(t))$: i.e., for a sequence of random variables $(Y_n(t))$ indexed by $t \in [0,1]$.

$$\forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_n \mathbb{P}(\sup_{|t-s| \leq \delta} |Y_n(t) - Y_n(s)| \geq \varepsilon) = 0. \tag{22}$$

Lemma 7. *The family* $(n^{-d/2} S_{t,n}^{\Delta} f)$ *is tight, for* Δ *in* $\mathcal{J}^+$, *for every* f *such that, for a constant* C

$$m_4(S_D f) = \int |S_D f|^4 \, d\mu \leq C|D|^2, \forall D \subset \mathbb{Z}^d.$$

Theorem 8. *(FCLT for the torus) Let* $(X_{\underline{\ell}}) = (T^{\underline{\ell}} f) = (f(A^{\underline{\ell}}.))$ *be a random field defined by a totally ergodic* d*-dimensional action* $(T^{\underline{\ell}})_{\underline{\ell} \in \mathbb{N}^d}$ *on the torus* $\mathbb{T}^\rho$ *by commuting endomorphisms* $A_1, ..., A_d$ *and a real function* f *in* $AC_0(\mathbb{T}^\rho)$.

For every set Δ *in* $\mathcal{J}^+$, *the process* $\left(n^{-d/2} S_{n,t}^{\Delta} f\right)_{n \geq 1}$ *satisfies a functional CLT.*

Thank you for your attention!