Asymptotic distributions in dynamical systems: Gaussian laws and other laws

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Abstract

The two basic limit theorems in Probability Theory, the law of large numbers (LLN) and the central limit theorem (CLT), have their counterparts in Ergodic Theory, since Birkhoff's Theorem in 1931 for the LLN, much later for the Central Limit Theorem (CLT). After some early results due to Fortet, Kac, Doeblin, it is mainly from the beginning of the years '60, with results of Sinaï ([Sin60]) on the geodesic flow, Leonov ([Leo60]) on the ergodic automorphisms of the torus, that examples of smooth dynamical systems appear in the literature where a CLT is obtained for classes of regular functions. These systems are of hyperbolic type. There are few known examples where a non-standard asymptotic distribution for the normalized ergodic sums can be shown.

In this lecture, we explain first, on a simple example of dilating map, the martingale method, one of the methods used to get the CLT for ergodic sums for hyperbolic maps. In contrast, we also mention some results on the behavior of the ergodic sums related to rotations or extensions of rotations. In the third section we give an application of the CLT: the recurrence of dispersive billiards with periodic obstacles in the plane.

There are some recent results with non-standard distributions. In the last section, we will present dynamical systems with slow decay of correlations (S. Le Borgne [Leb06]) and an example of a weakly hyperbolic system for which stable non-Gaussian laws are obtained as asymptotic distribution (S. Gouëzel [Gou04]).

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1 From Probability Theory to Ergodic Theory: pointwise limit theorems and convergence in distribution for dynamical systems

Let $X = (X_n)_{n \geq 0}$ be a real random process, i.e. a sequence of measurable real functions on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The process of partial sums is:

$$S_n := X_0 + ... + X_{n-1}.$$

In probability theory, there are two basic classical results on the process of partial sums for a sequence of *independent identically distributed random variables* (**iid rv**):

- *the strong law of large numbers (SLLN)*:
if $\mathbb{E}(|X_0|) < \infty$, then:

$$\lim_n \frac{S_n}{n} = \mathbb{E}(X_0), \text{ a.s. };$$

- *the central limit theorem (CLT)*:
if $\mathbb{E}(|X_0|^2) < \infty$, $\mathbb{E}(X_0) = 0$, $\sigma^2 = var(X_0) = \mathbb{E}(X_0^2) \neq 0$, then:

$$\forall [a, b] \subset \mathbb{R}, \lim_n \mathbb{P}(\frac{S_n}{\sqrt{n}} \in [a, b]) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt.$$

We will denote this convergence in distribution by

$$\frac{S_n f}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

In these basic convergence theorems, there are two kinds of hypotheses: stationarity and independence. Can we weaken the assumptions ?

• Stationarity

The first direction is in assuming only stationarity.

Let $(E, \mathcal{A}, \mu, T)$ be a *(measure preserving) dynamical system*:
. $(E, \mathcal{A}, \mu)$ is a probability space,
. $T : E \to E$ a measurable transformation which preserves μ.

Taking "an observable", i.e. a measurable function f defined on E, we obtain a process $(X_n) = (f \circ T^n)_{n \geq 0}$ which is stationary (for every p the distribution of $(X_n, ..., X_{n+p-1})$ does not depend on n).

It is useful to observe that this is the general form of a stationary process: if (X_n) is a stationary process, by considering the product space $\Omega = \mathbb{R}^{\mathbb{N}}$ equipped with the probability measure $\mathbb{P}$ whose marginals are the distributions of $(X_0, ..., X_{p-1}), p \geq 1$, and the shift T

defined on Ω by $T(\omega_0, \omega_1, \omega_2..., ...) = (\omega_1, \omega_2, ...)$, then we can write $X_n = X_0 \circ T^n$, i.e. the process (X_n) has the form $(X_0 \circ T^n)$.

We denote by $S_n f$ the ergodic sums:

$$S_n f(x) = f(x) + f(Tx) + ... + f(T^{n-1}x).$$

For a function defined on E, we write Tf for $f \circ T$, i.e. $Tf(x) = f(Tx)$, $x \in E$.

Birkhoff's ergodic theorem gives a general law of large numbers:

$$\lim_{n\to\infty} \frac{1}{n} S_n f(x) = \tilde{f}, \text{ for } \mu\text{−a.e } x, \ \forall f \in L^1(\mu),$$

where $\tilde{f}$ is T-invariant. Moreover, we have $\tilde{f} = \int_E f \, d\mu$, if T is *ergodic*, i.e.

for every measurable set A such that $T^{-1}A = A$ then $\mu(A) = 0$ or 1.

By this fundamental result, the LLN, that is the pointwise convergence of the ergodic means $(n^{-1}S_n f)$ towards the mean value $\int f \, d\mu$, follows from the ergodicity of the system.

For example, let T be the rotation $x \to Tx = x + \alpha \bmod 1$ on $\mathbb{T}^1$ equipped with the Lebesgue measure μ. If $\alpha \notin \mathbb{Q}$, for $f \in L^1(\mu)$, we have for μ-a.e. x:

$$\frac{1}{n} \sum_{k=0}^{n-1} f(x + k\alpha \bmod 1) \to \int f \, dy.$$

Likewise, if we take for T the map $x \to Tx = 2x \bmod 1$, we have for μ-a.e. x:

$$\frac{1}{n} \sum_{k=0}^{n-1} f(2^k x \bmod 1) \to \int f \, dy.$$

In both cases, it is easy to check ergodicity by Fourier analysis.

Now can we go further and, for example, obtain rate of convergence ?

A rate of convergence ?

Let $(E, \mathcal{A}, \mu, T)$ be a dynamical system and $f \in L^1(\mu)$ such that $\int f \, d\mu = 0$. To be more effective, the convergence in the SLLN or in Birkhoff's ergodic theorem should be accompanied by a speed of convergence. In other words, is there a rate of convergence in the ergodic theorem and a way to get the exact rate via a convergence in distribution as in the CLT for i.i.d. r.v.s ?

This leads to the question: for a given function f does there exist a sequence $(a_n) \uparrow^\infty$ with $\lim_n \frac{1}{n} a_n = 0$ such that:

$(a_n^{-1} \, S_n f)$ converges in distribution towards a non degenerate probability measure π on $\mathbb{R}$? (what we denote by $a_n^{-1} \, S_n f \xrightarrow{d} \pi$)

If the answer is yes, then we get that the fluctuations of the ergodic sums for f are exactly given by the sequence (a_n). The non-degeneracy of the limiting distribution shows that there are effectively fluctuations and that (a_n) is the right order.

If we stay at the level of the space of all integrable functions, then it appears that the answer is negative. It is not possible to get a rate of convergence in the ergodic theorem without some assumptions on the function f: given a non-degenerate dynamical system (E, T) there are integrable functions f on E for which the speed of convergence in the ergodic theorem for the process $(f \circ T^n)$ is arbitrarily slow. (cf. U. Krengel ([Kre65]).

It will be necessary to consider particular dynamical systems and particular classes of functions with some regularity, like Hölderian functions for hyperbolic transformations on a smooth manifold, or functions with bounded variation for dilating maps of the interval.

The behavior, for the convergence in distribution, depends strongly on the properties of the transformation and on the class of functions. This contrasts with pointwise limit theorems, where many convergence results have an universal character. As we have seen the law of large number is general: it holds for any measure preserving ergodic system. Likewise, there are constructions of universal sequences (n_k) such that we have a pointwise ergodic theorem along (n_k). But a result like the CLT is specific to the choice of the system and to the class of functions.

Variance

In the formulation of the CLT for a random process, an important quantity is the variance. For iid square integrable variables, with 0 mean, due to orthogonality we have $\|X_0 + .. + X_{n-1}\|^2 = n\mathbb{E}(|X_0|^2)$. Therefore, in that case, the normalization of the sums $S_n = X_0 + .. + X_{n-1}$ is by $\sqrt{n}$.

In the general case, the variance is related to the correlation, which is defined for a dynamical system $(E, \mathcal{A}, \mu, T)$ and a function $f \in L^2(\mu)$ by

$$(\langle T^n f, f\rangle, n \in \mathbb{N}).$$

By stationarity, we get:

$$\frac{1}{n}\|S_n f\|_2^2 = \|f\|_2^2 + 2\,\Re e(\sum_{k=1}^{n-1}(1 - \frac{k}{n})\langle T^k f, f\rangle).$$

If the speed of decorrelation is such that $\sum_n |\langle T^n f, f\rangle| < +\infty$, we can pass to the limit and we get that, as in the CLT for iid variables, the variance $\sigma^2(f)$ is finite, where

$$\sigma^2(f) := \lim_n \|\frac{S_n f}{\sqrt{n}}\|_2^2 = \|f\|_2^2 + 2\,\Re e(\sum_{k=1}^{\infty}\langle T^k f, f\rangle).$$

This shows the existence of a standard variance, like in the independent case, if the rate of decorrelation is exponential, or like $n^{-\beta}$ with $\beta > 1$, In that case, any limiting distribution for the normalized sums $(n^{-\frac{1}{2}}S_n f)$ has a "moment of order 2" and this excludes stable laws of index < 2. The common intuition is that, once we have a good rate of decorrelation, we should have asymptotically a normal law. This is not always the case (we will see a example to the contrary), but what is true is that, in many situations, the same method gives fast decorrelation and the CLT.

If we have a *slower decay of correlations*: $|\langle T^k f, f\rangle| \leq Cn^{-\beta}$, with $0 < \beta < 1$, the following inequality holds:

$$n^{-2+\beta}\|S_n f\|_2^2 \leq \frac{1}{n^{1-\beta}}\|f\|_2^2 + \frac{2C}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})(\frac{k}{n})^{-\beta} \leq C_1,$$

and the normalization for $\|S_n f\|_2$ is (at most) by $n^{1-\beta/2}$.

For example, if the decay is of order $n^{-1/2}$, $|\langle T^n f, f\rangle| \sim \frac{C}{\sqrt{n}}$, the normalization of S_n is by $n^{3/4}$.

Remark that the fact that the sums take big values, leads to a rather strong normalization ; but this doesn't mean that it is the right normalization for a convergence in distribution. We will see an example with the intermittent model later.

Before giving examples, we recall an important tool: the notion of martingale.

• Martingales

A square integrable process $(X_n)_{n\in\mathbb{Z}}$ is called a *martingale increment* if it satisfies:

$$\mathbb{E}(X_n|X_{n-1}, X_{n-2}....) = 0, \forall n \in \mathbb{Z}.$$

Such a process can be viewed as a model of "fair" game, since the expectation of (algebraic) gain knowing the past is 0. The process of the sums $(S_n = X_0+X_1+...X_{n-1})$ is a martingale.

Usually there is an increasing sequence $(\mathcal{F}_n)$ of σ-algebras such that X_n is measurable with respect to $\mathcal{F}_n$ and $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = 0$, $\forall n$.

Let $V_n := \sum_{k=1}^n \mathbb{E}[X_k^2|\mathcal{F}_{k-1}]$. When (X_n) is a martingale increment, the process $(S_n)_{n\geq 0}$ has orthogonal increments and we have $\sigma_n^2 := \mathbb{E}[S_n^2] = \sum_{k=1}^n \mathbb{E}[X_k^2] = \mathbb{E}[V_n]$.

We are going to apply to dynamical systems the following result (due to Billinsley and Ibragimov in the ergodic stationary case and to B.M. Brown for this general form):

Theorem 1.1 *Let (X_n) be a square integrable martingale increment process such that:*
i) $\sigma_n^{-2}V_n \to 1$, in probability,
ii) $\forall\varepsilon > 0$, $\lim_{n\to\infty} \sigma_n^{-2}\sum_{k=1}^n \mathbb{E}[X_k^2 1_{\{|X_k|>\varepsilon\sigma_n\}}] = 0$ (Lindeberg's condition).
Then the sums $s_n := X_0 + ... + X_{n-1}$ satisfies the CLT: $\sigma_n^{-1}S_n \xrightarrow{d} \mathcal{N}(0,1)$.

The same result is true for a "reverse martingale" (where in the definition of a reverse martingale we have a decreasing sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}(X_n|\mathcal{F}_{n+1}) = 0, \forall n \in \mathbb{N}$).

If we can "reduced" a function f over a dynamical system to a martingale increment, then, as in the iid case, we get $n^{-\frac{1}{2}}$ as normalization and a Gaussian law as asymptotic law.

Reduction to a martingale increment

If a function f, for a dynamical system $(E, \mathcal{A}, \mu, T)$, satisfies for some sequence $(a_n) \uparrow +\infty$ and some non degenerate distribution $\mathcal{D}$

$$a_n^{-1} S_n f \xrightarrow{d} \mathcal{D},$$

this remains true if we modify f by a **coboundary**, i.e. if we replace f by $g = f + h - Th$, with a measurable h. This is due to cancelation in the sums, since we have:

$$S_n g = S_n f + h - T^n h.$$

A function of the form $h - Th$ is called a coboundary.

A method to get the CLT, which is due to Gordin (1969) [Gor69], is to reduce the ergodic sums to a martingale up to a coboundary when it is possible by the following result (stated here in the non-invertible case) and then apply the Billinsley-Ibragimov-Brown Theorem. (See Hall and Heyde for martingale limit theory [HaHe80].)

Theorem 1.2 : *Let $(E, \mathcal{A}, \mu, T)$ be a (non-invertible) ergodic dynamical system, f a function in $L^2(\mu)$ such that $\int_E f \, d\mu = 0$ and*

$$\sum_{n \geq 0} ||\mathbb{E}(f|T^{-n}\mathcal{A})||_2 < \infty,$$

Then there are g and h in $L^2(\mu)$ such that $f = g + h - Th$, g generates a reverse martingale and f satisfies a CLT with variance $\sigma^2 = \|g\|_2^2$.

There is an analogous statement in the invertible case.

Let us now give examples, starting with stochastic systems.

2 Examples of asymptotic distributions in dynamical systems

2.1 Expanding maps

Example $x \to Tx = 2x \text{ mod } 1$

This transformation on $[0, 1]$, equipped with the Lebesgue measure μ and the Borel σ-algebra $\mathcal{A}$, is a "prototype" of hyperbolic transformation.

The transformation T is not invertible. The σ-algebra $T^{-1}\mathcal{A}$ is the σ-algebra of "symmetric" sets (i.e. invariant by $x \to x + \frac{1}{2}$).

There is a natural isomorphism between the transformation $x \to Tx = 2x \bmod 1$ and the shift acting in the product space $\{0,1\}^{\mathbb{N}}$ equipped with the product measure which makes the sequence of coordinate applications a Bernoulli sequence.

Let f be a regular function (Hölderian or with bounded variation) on $[0,1]$ with $\mu(f) = 0$. In general, we are not any more in the independent case. Nevertheless the CLT still holds for the sums $S_n f(x) = \sum_0^{n-1} f(2^k x \bmod 1)$:

Theorem 2.1

$$\frac{S_n f}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where the variance $\sigma^2 = 0$ if and only if f is a coboundary.

We obtain Theorem 2.1 by the martingale method. The idea is as follows. Let P be the adjoint operator of T, i.e.

$$\int f\ Tg\ \ d\mu = \int Pf\ g\ d\mu, \forall g \in L^\infty, f \in L^1.$$

Here we have a very simple expression for P

$$Pf(x) = \frac{1}{2}(f(\frac{x}{2}) + f(\frac{x}{2} + \frac{1}{2})).$$

We can write:

$$\mathbb{E}(f|T^{-1}\mathcal{A})(x) = Pf(Tx) = TPf(x) = \frac{1}{2}(f(x) + f(x + \frac{1}{2})).$$

It is easy to see that, for a continuous function f, $\lim_n P^n f = \int f\ d\mu$. If f is Hölderian of order $\gamma \in]0,1]$ (i.e. such that $|f(x) - f(y)| \leq C|x-y|^\gamma$) with $\mu(f) = 0$, there is a rate of convergence and the series $\sum_0^\infty P^n f$ converges uniformly, since we have the inequality

$$|P^n f(x) - \int f dy| = |2^{-n} \sum_{k=0}^{2^n - 1} f(x + \frac{k}{2^n}) - \int f(y)\ dy| \leq C 2^{-\gamma n}.$$

Therefore we can define $g = \sum_0^\infty P^n f$. We have then $f = g - Pg$.

Write $\varphi := g - TPg$, $h := Pg$. We have $f = \varphi + Th - h$, $S_n f = S_n \varphi + T^n h - h$. The sums $(S_n f)$ behave like the sums $(S_n \varphi)$.

The function φ generates a reverse martingale increment, since

$$\mathbb{E}(\varphi | T^{-1}\mathcal{A}) = TP\varphi = TP(g - TPg) = TPg - TPg = 0.$$

By Theorem 1.1, this shows the existence of the variance and the CLT (non-degenerate if $\sigma^2(f) = \int \varphi^2 \, d\mu \neq 0$, i.e. if f is not a coboundary).

□

Fourier operator

Given f, we define the Fourier operator corresponding to f by:

$$P_t u = P(e^{itf} u), u \in L^1(\mu).$$

Iterating P_t, we get $P_t^n(u) = P(e^{itS_n f} u)$. In particular the characteristic function of $tS_n f$, i.e. the Fourier transform of the distribution of $tS_n f$ with respect to μ, is

$$\mathbb{E}(e^{itS_n f}) = \int e^{itS_n f} \, d\mu = \int P(e^{itS_n f}) \, d\mu = \int P_t^n 1 \, d\mu.$$

The asymptotic behavior of the process $(S_n f)$ is related to the spectral analysis of the family of operators (P_t), which can be used to obtain limiting distribution of the normalized sums $(S_n f)$. This method started with S.V. Nagaev (1957) and was in particular developed by R. Bowen, Y. Guivarc'h, E. Le Page, W. Parry, G. Keller, M. Viana, M. Pollicot, C. Liverani, L.S. Young, D. Dolgopyat, ...

Other models

The method we have just described can be extended to a large class of piecewise expanding maps of the interval.

Let us call **exact** a non-invertible dynamical system $(E, \mathcal{A}, \mu, T)$ such that the σ-algebra $\mathcal{A}_\infty = \bigcap_n T^{-n}\mathcal{A}$ is trivial.

If the system is exact, for any $f \in L^1(\mu)$ we have

$$T^n P^n f = \mathbb{E}(f | T^{-n}\mathcal{A}) \to \mathbb{E}(f | \mathcal{A}_\infty) = \int f \, d\mu.$$

If $f \in L^2(\mu)$ is such that $\mu(f) = 0$, then $\lim_n P^n f = 0$. If we can prove a rate of decorrelation for a function f and obtain that $\sum_n \|P^n f\|_2 < \infty$, then the martingale method will apply.

There is a large class of piecewise expanding maps of the interval for which we have such a rate for regular functions and this gives the CLT.

Moreover, it is often possible to obtain a spectral gap for the operator P acting on spaces of regular functions, like in the example $x \to 2x \bmod 1$. This gives exponential decay of correlation, a CLT and, via "perturbative" methods in spectral theory for the operators P_t, stronger results for the ergodic sums, like the local limit theorem.

Invertible hyperbolic maps

The map $x \to 2x \bmod 1$ is a simple example of a non-invertible dilating transformation. To have an example of a chaotic invertible map, we have to consider systems in dimension > 1.

Example: *Automorphism of the torus*

The simplest one is the hyperbolic automorphism of the torus $\mathbb{T}^2$ defined by: $(x, y) \to (2x + y, x + y \mod 1)$. This map is a prototype of *hyperbolic* automorphism (or Anosov map).

The concept-key to study the stochastic properties of such a system is that of a coding of the system (using a Markovian partition when it exists). This makes it possible to represent it as a subshift and to use the operator method like in the dilating case. But for partially hyperbolic systems, where there is no Markovian partitions, we can still try to use martingale methods ([CoLe01], [Leb02], [Dol04]).

To have a chance to get an asymptotic distribution which is not the normal law, we need a slower rate of decay of correlations. In the last section we will see that this is the case for instance for intermittent maps.

2.2 Systems related to rotations

Let us now give examples of a completely different behavior with less stochastic systems.

Rotations

For a dynamical system defined by a rotation, we cannot expect a good statistical behavior of the sums for regular functions. Let T_α be the rotation: $x \to T_\alpha x = x + \alpha \mod 1$, $\alpha \notin \mathbb{Q}$.

If f is regular enough and α badly approximated by the rational numbers, we get using Fourier series that f is a coboundary (we can solve the functional equation in h: $f = h - T_\alpha h$). In that case, for any sequence (a_n) such that $\lim_n a_n \uparrow +\infty$, the limit $a_n^{-1} S_n f$ is degenerate.

Nevertheless, it should be remarked that even for an irrational rotation, there exist (non-regular) functions f such that $\|S_n f\|_2^{-1} S_n f \xrightarrow{d} \mathcal{N}(0, 1)$ (cf. Burton and Denker [BuDe87]).

We consider now more stochastic examples constructed over rotations.

Skew-products over a rotation

On $\mathbb{T}^2$ (with the Lebesgue measure denoted by λ), let R be the *skew-product* defined by:

$$(x, y) \to R(x, y) = (x + \alpha, y + x \mod 1).$$

For $\alpha = 0$, the (non-ergodic) transformation R gives an example of a stationary orthogonal sequence for which there is no normalization giving convergence towards a non-degenerate law. Taking f: $f(x, y) = e^{2\pi i y}$, we have $T^n f(x, y) = e^{2\pi i (y + nx)}$, $\langle T^n f, f\rangle = 0, \forall n \neq 0$,

$$S_n f(x, y) = e^{2\pi i y} \frac{e^{2\pi i n x} - 1}{e^{2\pi i x} - 1}, \quad \|\frac{S_n f(x, y)}{\sqrt{n}}\|_2 = 1.$$

For every sequence (a_n) tending to $+\infty$, $\lim_n \frac{S_n f(x)}{a_n} = 0$, for $x \neq 0 \bmod 1$.

Let us take now functions of the y-coordinate with discontinuities. Let $f = 1_{[a,b]}$ be the characteristic function of an interval on the circle.

In 1960, H. Kesten ([Kes60]) proved the following result, which gives an example of convergence towards the Cauchy law:

Theorem 2.2 *If $b-a \in \mathbb{Q}$, we have, for some ρ depending on $b-a$:*

$$\lambda(\{(x,y) : (\log N)^{-1} \sum_{n=1}^{N} [f(y+nx) - (b-a)] \leq t\}) \to \frac{1}{\pi} \int_{-\infty}^{\rho t} \frac{ds}{1+s^2}.$$

Question: what happens if $b-a \notin \mathbb{Q}$?

Weyl sums

For $\theta \notin \mathbb{Q}$ we consider the system:

$$T : (x,y) \to (x+\theta, y+2x+\theta).$$

We have:

$$T^n(x,y) = (x+n\theta, y+2nx+n^2\theta).$$

For $f(x,y) = e^{2\pi i y}$, the ergodic sums are:

$$S(n,\theta,x) = e^{2\pi i y} \sum_{k=0}^{n-1} e^{2\pi i(2kx+k^2\theta)}.$$

These sums were studied by Hardy and Littlewood in relation to the identity satisfied by the function theta of Jacobi

$$\Theta(z) = \frac{1}{\sqrt{z}} \Theta(\frac{1}{z}), \text{ where } \Theta(z) = \sum_{k=-\infty}^{+\infty} e^{-\pi k^2 z}, \Re e(z) > 0.$$

Write $S(n,\theta,x) = \sum_{k=0}^{n-1} e^{2\pi i(2kx+k^2\theta)}$.

Let τ be the Gauss transformation: $x \to \{\frac{1}{x}\}$ and $D_\theta : x \to 1 - \frac{x}{\theta}$. In 1914 Hardy and Littlewood ([HaLi14]) obtained: there is a constant C such that

$$|S(n,\theta,x)| \leq \theta^{-\frac{1}{2}}(|S(n\theta, \tau\theta, D_\theta x)| + C).$$

(In the preceding formula, the sums are extended to non integer values of n.)

Iterating this formula, one obtains for $p \geq 1$:

$$\begin{aligned} |S(n,\theta,x)| &\leq (\theta\tau\theta...\tau^p\theta)^{-\frac{1}{2}} |S(n\theta\tau\theta...\tau^p\theta, \tau^p\theta, D_{\tau^{p-1}\theta}...D_{\tau\theta}D_\theta x)| \\ &\quad + C[1 + \theta^{-\frac{1}{2}} + (\theta\tau\theta)^{-\frac{1}{2}} + (\theta\tau\theta...\tau^p\theta))^{-\frac{1}{2}}]. \end{aligned}$$

Let $\beta(r) := \theta\tau\theta...\tau^r\theta$ and let us choose r such that: $\beta(r) \leq \frac{1}{n} < \beta(r-1)$.

The preceding inequality gives:

$$|S(n,\theta,x)| \leq O(n\beta(r)^{\frac{1}{2}} + \beta(r)^{-\frac{1}{2}}).$$

For θ with bounded partial quotients, we have a geometrical growth of the sequence (q_n) and the bound:

$$|S(n,\theta,x)| = O(\sqrt{n}).$$

An ergodic stationary orthogonal sequence is thus obtained for which the normalized ergodic sums $S_n(x)/\sqrt{n}$ remain bounded and the CLT doesn't hold.

Question: what is the behavior in distribution of $(\frac{1}{\sqrt{n}}S_n(x))$?

3 Application of the CLT to recurrence of stationary walks

The CLT can be applied to establish a property of geometrical nature apart from the probabilistic context, namely recurrence.

Recurrence

Let $(E, \mathcal{A}, \mu, T)$ be a dynamical system, Φ a measurable function from E to $\mathbb{R}^d$.

As for the random walks defined by sums of independent random variables, we consider the "stationary walk" associated to Φ, denoted by (E, μ, T, Φ), i.e. the sequence

$$S_n\Phi(x) = \sum_{j=0}^{n-1} \Phi(T^j x), n \in \mathbb{N},\ x \in E.$$

We define on $E \times \mathbb{R}^d$ a transformation $T_\Phi : (x,y) \to (Tx, y + \Phi(x)),\ x \in E, y \in \mathbb{R}^d$, which leaves invariant the measure $\mu \times \lambda$ product of μ by the Lebesgue measure λ on $\mathbb{R}^d$.

The position of a point (x,y) under the iterated action of T_Φ is related to the stationary walk (E, μ, T, Φ) by the relation:

$$T_\Phi^n(x,y) = (T^n x,\ y + S_n\Phi(x)).$$

The walk (E, μ, T, Φ) is **recurrent** if, for μ-a.e. x, the sums $S_n\Phi(x)$ return infinitely often in any neighborhood of 0. It is **transient** if, for μ-a.e. x, $\lim_n \|S_n\Phi(x)\| = +\infty$.

For the usual random walk (sum of iid integrable centered random variables), recall that we have recurrence for $dim\ \leq 2$ and transience for $dim\ \geq 3$.

For a random walk over an ergodic dynamical system $(E, \mathcal{A}, \mu, T)$, it can be shown that the walk is either recurrent or transient.

In dimension 1, we have recurrence if Φ is integrable with $\int \Phi \, d\mu = 0$. On the other hand, the recurrence is not always valid in dimension $d \geq 2$. There are stationary walks in dimension 2, with Φ integrable with $\mu(\Phi) = 0$, which are transient. Therefore, we need stronger hypotheses for recurrence. The CLT is a sufficient condition in dimension 2, due to the following result (J.P. Conze in the context of martingale methods, K. Schmidt for the general case).

Theorem 3.1 *If the stationary random walk (E, μ, T, Φ) satisfies the CLT, then it is recurrent.*

An example

We consider the function $\varphi(x) = (\varphi_1(x), \varphi_2(x)) = (\cos(x), \sin(x))$ and the system defined on $[0,1] \times \mathbb{R}^2$ by

$$(x, y_1, y_2) \xrightarrow{T_\varphi} (2x \bmod 1, y_1 + \cos(x), y_2 + \sin(x)).$$

The product measure $dx \times dy_1 \times dy_2$ on $[0,1] \times \mathbb{R}^2$ is an (infinite) invariant measure. By iteration of the transformation T_φ we get

$$T_\varphi^n(x, y_1, y_2) = (2^n x \bmod 1, y_1 + S_n\varphi_1(x), y_2 + S_n\varphi_2(x)).$$

By the martingale result and Theorem 3.1, the walk is recurrent. Moreover, we have ergodicity of T_φ (cf. Y. Guivarc'h [Gui84]), which implies that, for a.e. x, the sums $S_n\varphi(x) = \sum_{k=0}^{n-1} e^{2\pi i 2^k x}$ visit infinitely often any non-void open set in the plane.

Recurrence of the billiards flow in the plane

Let us consider now the billiards (Sinaï's billiards or Lorentz process) with convex dispersive obstacles in the plane with periodically distributed obstacles.

The domain of the billiards is made of strictly convex obstacles with disjoint closures, and boundaries of class C^{r+1} of curvature > 0. We assume that the obstacles are distributed $\mathbb{Z}^2$-periodically in the plane. The toric model is the quotient of this distribution modulo $\mathbb{Z}^2$.

The *billiards flow* in the plane is defined on the space $\mathcal{T}^1 Q$ of configurations (position, speed vector) according the usual rule: a particle (the ball geometrically reduced to a point) moves with constant speed in straight line between two obstacles and follows the laws of the reflection of Descartes at the moment of the shock against the edge of an obstacle.

To reduced the flow to a transformation, one uses a "natural" section of Poincaré: the restriction to configurations corresponding to the boundary of the obstacles, considering only the moments of shock. One obtains thus the *billiards transformation.*

Since the Seventies, a series of works by Ya. G. Sinaï, L.A. Bunimovich, N.I. Chernov, then also by A. Krámli, N. Simányi, D. Szász, R. Markarian, L-S. Young, were devoted to the stochastic properties of this model of billiards.

As in Theorem 3.1, we can apply martingale-like methods and prove recurrence for the two dimensional billiards with "finite horizon". For the billiards with infinite horizon, D. Szász and T. Varju recently proved that the recurrence still holds.

4 Recent examples of nonstandard asymptotic distributions in dynamical systems

4.1 Example of skew-products with nonstandard rates

The first non-standard examples (with *slow slow decay of correlations*) we are going to present are based on the construction of skew- products.

We take a system $(X, \mathcal{X}, \mu, T)$ as a "basis" and a second space $(Y, \mathcal{Y}, \nu)$ on which acts a one parameter group or semi-group (A^t) of measure preserving automorphisms.

Let ϕ be a map from X to $\mathbb{R}$ (or $\mathbb{Z}$). The system (skew- product) is defined on the product space $X \times Y$ by:

$$(x, y) \to R(x, y) = (Tx, A^{\phi(x)}y).$$

For functions on $X \times Y$ of the form: $F(x, y) = f(y)$, the decorrelation is given by:

$$\langle R^n F, F\rangle = \int\int f(A^{S_n\phi(x)}y)\overline{f(y)}d\nu(y)\ d\mu(x) = \int \langle A^{S_n\phi(x)}f, f\rangle\ d\mu(x).$$

In particular, if ϕ has its values in $\mathbb{Z}$, we get:

$$\langle R^n F, F\rangle = \sum_k \langle A^k f, f\rangle\ \mu(\{S_n\phi = k\}).$$

Example 1

We take $(X, \mu, T) = (\Omega, \mathbb{P}, \theta)$, where $\Omega = \{\pm 1\}^{\mathbb{Z}}$, $\mathbb{P} = (\frac{1}{2}, \frac{1}{2})^{\otimes \mathbb{Z}}$, $\theta =$ the shift.

Let φ be X_0. The skew-product T_φ is the "(T, T^{-1})-transformation":

$$(\omega, \omega') \to R(\omega, \omega') = (\theta\omega, \theta^{X_0(\omega)}\omega').$$

If we take for f the function $f(\omega, \omega') = Y_0(\omega') = \omega'_0$, we get:

$$R^n F(\omega, \omega') = Y_{S_n(\omega)}(\omega').$$

From the local theorem (estimation of the probability that the elementary random walk is in 0 at time n), we get:

$$\langle R^{2n} F, F\rangle = \mathbb{P}(S_{2n} = 0) \simeq \frac{1}{\sqrt{\pi n}}.$$

This example is a prototype of those for which Kesten and Spitzer (1979) [KeSp79] proved that, after normalization by $n^{3/4}$, the ergodic sums converge in distribution towards a mixture of gaussian laws.

Example 2

We consider again the transformation $T : (x, y) \to (2x + y, x + y \text{ mod } 1)$ and a function f on $\mathbb{T}^2$.

We take an other system: let (g_t) be the geodesic flow acting on a compact quotient G/Γ of $G = Sl(2, \mathbb{R})$. The skew-product is defined by the transformation:

$$R : (x, y) \in \mathbb{T}^2 \times G/\Gamma \longrightarrow (Tx, g_{f(x)}y) \in \mathbb{T}^2 \times G/\Gamma.$$

Let $S_n f(x) = \sum_0^{n-1} f(T^j x)$ be the ergodic sums for the action of T. We have:

$$R^n(x, y) = (T^n x, g_{S_n f(x)}y).$$

We assume that $\int f d\mu = 0$. The model can be viewed as a smooth version of the (T, T^{-1})-transformation.

Let ϕ and ψ be two regular functions defined on $\mathbb{T}^2 \times G/\Gamma$ and of mean value 0. We have to estimate the quantity

$$\langle \phi \circ R^n, \psi \rangle = \int_{\mathbb{T}^2 \times G/\Gamma} \phi(T^n x, g_{S_n f(x)}y)\ \psi(x, y)\ dy\, dx.$$

Decorrelation, Variance

Using a version of the local theorem for the sums $S_n f$ due to Y. Guivarc'h and J. Hardy [GuiHa88], an estimation of the variance can be given. Assume that the function $\int_{\mathbb{T}^2} \phi(x, y)\ dx$ is not a coboundary for the action of the geodesic flow. Then for some constant $C \neq 0$ the estimate is:

$$\langle \phi \circ R^n, \phi \rangle \sim \frac{C}{\sqrt{n}}$$

and $\mathbb{E}\left(\left[\sum_{k=0}^{n-1} \phi \circ R^k\right]^2\right)$ is of order $n^{3/2}$.

Theorem 4.1 *(S. Le Borgne [Leb06]) Let us take $f(x) = \sin(2\pi x_1)$ (where $x = (x_1, x_2)$ is the first coordinate of $x \in \mathbb{T}^2$).*

Let ϕ be a regular function of 0 integral such that $\int_{\mathbb{T}^1} \phi(x, y)\ dx$ is not a coboundary for the action of the geodesic flow. Then we have convergence in distribution (for some constant C)

$$\frac{C}{n^{3/4}} \sum_{k=0}^{n-1} \phi \circ R^k \xrightarrow{\mathcal{L}} \int_{\mathbb{R}} L_1(-t) dW_-(t) + \int_{\mathbb{R}_+} L_1(t) dW_+(t),$$

where W, W_+, W_- are three independent Brownian motions and L_1 is the local time of W.

4.2 Non-uniformly dilating maps with a neutral fixed point and stable laws

In the study of non-uniformly dilating transformations with neutral point, the following family of transformations was used as a simple model, which presents a phenomenon of "intermittency" and was discussed by Pommeau and Manneville ([PoMa80]). It is defined on the interval $[0, 1]$ by

$$T_\alpha x = x(1 + 2^\alpha x^\alpha), \text{ if } 0 \leq x \leq 1/2, = 2x - 1, \text{ if } 1/2 < x \leq 1.$$

For $0 < \alpha < 1$, there is an absolutely continuous invariant measure (ACIM) whose density has a singularity at the neutral point. The trajectories spend a long time near the fixed neutral point, then follow a chaotic behavior, etc. When the parameter α is close to 0, the trajectories are more chaotic.

This model was in particular studied by Thaler, Liverani, Saussol and Vaienti, Hu, Young, Campanino, Isola, Conze and Raugi for a variant, and recently by Zweimuller and Gouëzel. O. Sarig (2002) developed a method to obtain lower bounds of the decorrelation in families of dynamical systems with neutral points.

The optimal estimate of the decorrelation for Lipschitz functions f is of order $n^{1-1/\alpha}$ and was obtained by L.S. Young. The decorrelation is summable for $\alpha < 1/2$ and the CLT for the ergodic sums $S_n f$ is valid.

S. Gouëzel obtained recently the behavior of the decorrelation, supplementing and specifying the results of O. Sarig, who had shown that, for the map T_α, the speed given by L.S. Young is optimal for f Hölderian and null on $[0, 1/2]$ for $\alpha < 1/2$.

Stable laws

The stable laws (distributions on $\mathbb{R}$) depend on 4 parameters (3 parameters when centered). The main parameter is the index p, a number in $[1, 2]$.

For $p < 2$, these distributions are characterized by the property of "heavy tails": if Z is a random variable whose distribution is a stable law of index p, there are two constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, such that

$$\mathbb{P}(Z > x) = (c_1 + 0(1))\, x^{-p}, \ \mathbb{P}(Z < -x) = (c_2 + 0(1))\, x^{-p}.$$

The Cauchy's law with density $\pi^{-1}\frac{1}{1+x^2}$ is a stable law of index 1. The normal laws are the stable laws of index 2.

If the distribution of a random variable $Z = Z_{p,c,\beta}$ is a centered stable law of parameters (p, c, β), (where p is the index), its characteristic function is given by

$$\mathbb{E}(e^{itZ_{p,c,\beta}}) = e^{-c|t|^p(1-i\beta sgn(t) tan(p\frac{\pi}{2}))}, 1 < p \leq 2, c > 0, |\beta| \leq 1.$$

Results of S. Gouëzel (2004) [Gou04])

For ($\alpha \in]0, 1[$), S. Gouëzel gives the exact expression of the principal term of the decorrelation. He can also give a precise estimate of the complementary term. The method applies to other classes of functions, for example Lipschitz functions of null integral and null at 0, satisfying $|f(x)| \leq Cx^{\gamma}$, for which he obtains a CLT when $\gamma = 1$.

For the transformations T_α, S. Gouëzel carries out the general study of the asymptotic behavior of the ergodic sums $S_n f$ for an Hölderian function f of null integral with respect to the ACIM of the dynamical system. He obtains
- for $\alpha = 1/2$, a CLT (with a normalization in $\sqrt{n\mathrm{Log}n}$ if $f(0) \neq 0$),
- for $1/2 < \alpha < 1$, if $f(0) \neq 0$, a convergence towards a stable law with normalization in n^α: $\frac{S_n f}{n^\alpha} \xrightarrow{d} Z_{1/\alpha, c, sgn(f(0))}$, for a constant c,
- for $1/2 < \alpha < 1$, if $f(0) = 0$ and $|f(x)| \leq Cx^\gamma$, for γ big enough, a CLT.

He can also prove accurate limit theorems: speed of convergence of Berry-Esseen type, local limit theorem. These deep and difficult results rest on precise estimates in the perturbative development of the operators of first return. The theorems are obtained within the framework of the locally Gibbs-Markov abstract models (L.-S. Young [You98]).

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