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Continuum model for steady, fully developed saltation above a horizontal particle bed

J. T. Jenkins,[1] I. Cantat,[2] and A. Valance[2]
[1]*Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853, USA*
[2]*Institut de Physique de Rennes, UMR CNRS 6251, Université de Rennes1, 35042 Rennes, France*

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We propose a continuum model for steady, fully developed saltation above a horizontal particle bed that provides local, analytical expressions for the particle pressure and shear stress. This analytical approach contrasts with discrete numerical simulations in which the trajectories of individual particles are computed as they interact with gravity, the wind, and the bed. The continuum model has the advantage that it can easily be extended to nonuniform and unsteady situations. We employ it to predict the fields of concentration, particle velocity, and wind velocity in steady, fully developed saltation above a particle bed over a range of wind speeds. The predicted profiles are in good agreement with those measured in wind tunnel experiments.

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I. INTRODUCTION

When a turbulent wind blowing over bed of sand becomes sufficiently strong, a grain may be lifted from the bed by a strong, localized turbulent eddy. The drag of the air then accelerates it, and it collides with the bed with increased momentum. Impacting grains rebound and eject other grains that may also be accelerated by the wind until a sufficient number of grains are participating in the process to diminish the wind near the bed and create a steady balance in the exchanges of momentum between the grains and the wind and the grains and the bed.

The result is a steady cloud of grains with diameters between 100 and 500 microns that jump (Latin: *saltare*) over the bed. This saltation is the primary mode of the initial sand movement [1]. Stronger winds can involve so many grains that collisions above the bed become important [2,3] and, as the strength of the wind increases, direct suspension by the turbulent velocity fluctuations occurs [4].

Analytical models of saltation [5–10] are typically nonlocal. They focus on the entire trajectories of single particles and attempt to determine the drag of the particle on the average turbulent wind and the difference in the flow momentum of a particle in its upward and downward motions at each height of its trajectory. In a steady, fully developed flow, the distributions of the velocities of the upward and downward moving particles are also steady. These velocity distributions are linked at the bed by the splash function [11–19] that provides a statistical characterization of the relationship between a single incoming grain and the products of the collision. Knowledge of the change in flow momentum at each height and the velocity distributions permits the calculation of steady profiles of average concentration, average wind and average particle velocity. Existing predictions of these profiles assume that the particle shear stress at the bed does not vary with the strength of the wind at the bed [5–9] or assume an explicit form for its variation with distance from the bed [10]. The latter assumption is consistent with the focus of the wind velocity profile seen in experiments [20–23] and numerical simulations [15,24,25], while the former is not.

In this work, we avoid such assumptions. We first obtain a local relation between the shear stress in the particle phase and the particle shear rate that is motivated by averaging the equations that govern the trajectories of single particles at each height of the trajectory. We then use this relation, the expression for the particle pressure, and the usual mixing length model for the turbulent shear flow, modified by the drag of the particles, in the equations of balances of horizontal and vertical momentum of the particles and horizontal momentum for the wind. We complete the problem by employing conditions [26] on the average exchange of particle mass and momentum at the surface of the bed that are based on measurements of the average and a first velocity moment of the splash function [18,19]. When we include the vertical force on the particles associated with their turbulent suspension the resulting profiles are in excellent agreement with those measured in wind tunnel experiments [26].

The analytical approach described above should be contrasted with discrete numerical simulations (e.g., [25]) that also have the capacity to reproduce the profiles of particle and wind velocities and particle concentration measured in the laboratory and in the field. In our view, the advantage of the local, analytical formulation is that it can be extended to nonuniform and unsteady situations. Such extensions would permit the determination, in the context of such a theory, of the saturation lengths and times that are important to the understanding of dune formation and motion [8].

II. SINGLE PARTICLE TRAJECTORIES

We consider first a single particle of mass m made of a material with a mass density ρ^s that is ejected from a horizontal bed and interacts with gravity and a shear flow of a turbulent gas with mass density ρ^f and viscosity μ^f. The gravitational acceleration is g, the average horizontal velocity of the wind is U, and the horizontal and vertical components of the particle velocity are ξ_x and ξ_y. The velocities are functions of the upward vertical coordinate y. A sketch of the situation is shown in Fig. 1. Lengths are made dimensionless by the diameter d of the particle, velocities are made dimensionless by $(gd)^{1/2}$, and stresses are made dimensionless by $\rho^s gd$. The dimensionless, nonlinear drag coefficient is denoted by D and primes label the upward velocity components in a trajectory.

The components of the horizontal particle velocity in the upward and downward parts of a trajectory are governed by

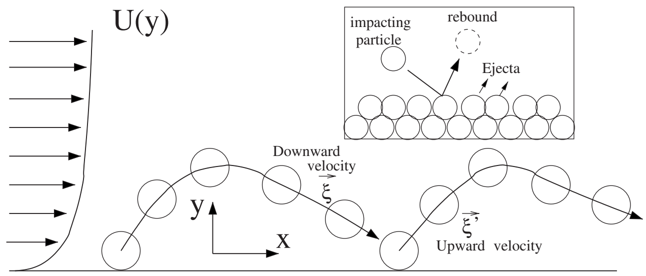

FIG. 1. A sketch of saltation.

$$\xi_y' \frac{d\xi_x'}{dy} = D(U - \xi_x'), \tag{1}$$

and

$$\xi_y' \frac{d\xi_x}{dy} = -D(U - \xi_x), \tag{2}$$

respectively. In order to obtain information regarding the variation of the particle shear stress s with height above the bed, we multiply these by the product of the particle volume fraction ν and the upward vertical velocity, sum them, and average over the distribution of particle velocities at each height,

$$\nu\overline{\xi_y'^2 \frac{d}{dy}(\xi_x' + \xi_x)} = \nu D\overline{\xi_y'(\xi_x - \xi_x')}. \tag{3}$$

We assume that the flow is two-dimensional and that the velocity fluctuations are isotropic. Then, the definitions of the average particle velocity u, the particle pressure p, and the particle shear stress s are

$$2u \equiv \overline{(\xi_x' + \xi_x)}, \quad p \equiv \nu\overline{\xi_y'^2} = \nu T, \quad \text{and} \quad 2s \equiv \nu\overline{\xi_y'(\xi_x' - \xi_x)}, \tag{4}$$

where T is the granular temperature. With these, we take the product of the averages in Eq. (3) to be the average of the products and write

$$p\frac{du}{dy} = \alpha D s, \tag{5}$$

where α is a constant that, roughly, takes into account the neglected correlations. Equation (5) links the particle shear stress to the derivative in the average particle velocity.

III. CONTINUUM THEORY

We assume that the granular temperature is constant throughout the flow and equal to its value T_0 at the bed. Then, the vertical component of the balance of force on the particles requires that the vertical derivative of the particle pressure of Eq. (4) equals their dimensionless weight,

$$\frac{d\nu}{dy} = -\frac{\nu}{T_0}. \tag{6}$$

The corresponding balance of particle horizontal momentum equates the derivative of the particle shear stress to the average particle drag,

$$\frac{ds}{dy} = -\nu D(U - u), \tag{7}$$

in which we employ the dimensionless form of an expression for the steady drag on a particle [28]: $D=(0.3|U-u|+18/R)/\sigma$, where $\sigma=\rho^s/\rho^f$ and $R=\rho^f d(gd)^{1/2}/\mu^f$ is a Reynolds number based on the fall velocity of the particles. Equation (5) then provides a first order equation for the determination of the average particle velocity. It is the key contribution of this paper.

Finally, we use the fact that the sum of the particle and gas shear stresses is constant through the flow and use the distance from the bed as the mixing length in the relation between the shear rate and the shear stress of the wind,

$$\frac{dU}{dy} = \frac{(S^* - s)\sigma}{1/R + [(S^* - s)\sigma]^{1/2}\kappa y}, \tag{8}$$

where S^* is the dimensionless shear stress in the particle-free gas, $\kappa=0.41$ is von Karman's constant, and the inverse of the Reynolds number is the dimensionless molecular viscosity. The dimensionless friction velocity u^* is related to S^*, the Shields parameter, by $u^*=\sqrt{S^*/\sigma}$.

We may incorporate turbulent suspension in the vertical momentum balance by assuming that the correlation between fluctuations in particle concentration and wind velocity is proportional to the gradient in particle concentration (e.g., [27]). The influence of this assumption on Eq. (6) is obtained by replacing T_0 by $T_0+\beta D\mu^T$ where $\mu^T \equiv [(S^*-s)\sigma]^{1/2}$ is the turbulent viscosity and β is the ratio of the turbulent diffusion of mass to that of momentum.

IV. BOUNDARY CONDITIONS

As discussed in detail by Creyssels, *et al.* [26], the boundary conditions at the bed result from a consideration of the average balance of mass and momentum in collision of particles with the bed.

Experiments [18] and simulations [19] on single spheres shot into a bed of like spheres indicate that the average magnitude and average vertical component of the rebound velocity of an incident particle are given in terms of their incident values and the angle θ between the bed and the incident trajectory by

$$\overline{\xi'} = e\xi \quad \text{and} \quad \overline{\xi_y'} = e_y|\xi_y|; \tag{9}$$

where $e=0.87-0.72 \sin\theta$ and $e_y=0.30/\sin\theta-0.15$; while the number N of ejected particles, including the rebound, is

$$N(\xi) = \begin{cases} 1 + N_0(1-e^2)(\xi/\xi_0 - 1) & \text{if } \xi > \xi_0 \\ 1 & \text{if } 1 \le \xi \le \xi_0 \\ 0 & \text{if } \xi \le 1 \end{cases} \tag{10}$$

where ξ_0 is the threshold velocity below which there is no ejection. The measurements indicate that $N_0=13$ and $\xi_0=40$. The numerical simulations [19] show that e and e_y are independent of the diameter of the spheres, but depend upon their coefficients of restitution and friction.

Equations (9) and (10) may be used with the simple and familiar three-parameter velocity distribution function

$$f(\xi) = \frac{n_0}{2\pi T_0}\exp\left[\frac{-(\xi_x - u_0)^2 - \xi_y^2}{2T_0}\right], \tag{11}$$

where $n_0 = 6\nu_0/\pi$ and the subscript zero indicates quantities evaluated at the bed, to calculate [26] the mass flux,

$$\begin{aligned}\dot{m} &= \int_{\xi_y \le 0} (N-1)\xi_y f(\xi) d\xi \\ &= \frac{13}{2\pi}\frac{n_0 T_0^2}{u_0(40-u_0)^2}\left[0.24 + 0.63\left(\frac{\pi T_0}{20u_0}\right)^{1/2}\right] e^{-(\xi_0 - u_0)^2/2T_0} \\ &\quad - \frac{74\sqrt{2}}{\pi}\frac{n_0}{T_0} e^{-u_0^2/2T_0},\end{aligned} \tag{12}$$

and the momentum fluxes,

$$\begin{aligned}\dot{M}_x &= -\frac{\pi}{6}\int_{\xi_y \le 0} (\overline{\xi_x'} - \xi_x)\xi_y f(\xi) d\xi \\ &= \nu_0 T_0[0.35 + 0.07u_0/T_0^{1/2} - 0.33T_0^{1/2}/u_0],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\dot{M}_y &= -\frac{\pi}{6}\int_{\xi_y \le 0} (\overline{\xi_y'} - \xi_y)\xi_y f(\xi) d\xi \\ &= \nu_0 T_0[0.12u_0/T_0^{1/2} + T_0^{1/2}/u_0 - 0.08] + \nu_0 T_0/2.\end{aligned} \tag{14}$$

With $\dot{M}_y = p = \nu_0 T_0$, Eq. (14) gives $u_0/T_0^{1/2} = 4.6$. In steady-state saltation, the mass flux is zero, so when this determination is employed in the steady version of Eq. (12), $u_0 \approx 17.5$ and $T_0 \approx 20$. Then, Eq. (13) gives $s_0 \approx 0.6\nu_0 T_0$. Finally, at the bed, we take $U=0$.

We define the top of the flow, $y=H$, to be where $s=0$ and the local particle flux $Q \equiv \nu u$ is equal to 0.001. We implement this condition by adding a differential equation for the integral I of Q up to y,

$$\frac{dI}{dy} = \nu u \tag{15}$$

with $I(0)=0$ and $I(H)=0.001$.

V. RESULTS

As in the experiments of Creyssels, *et al.* [26], we consider sand grains in air; for these, $d = 242$ μm, $\rho^s/\rho^f = 2{,}200$ and $\mu^f/\rho^f = 0.15$. We take $\alpha = 20$ and solve the two-point boundary-value problem for ν, s, u, U, and I, using the built-in Matlab function bvp4c, for $\beta = 0$ and 2. Because there are seven boundary conditions and five dependent variables, we determine both the parameter ν_0 and the depth of the flow as part of the solution.

One reason for the large value of α is that the form of the drag coefficient that we employ does not include unsteady effects. For example, Mabrouk, *et al.* [29] show that the unsteady drag coefficient that they measure has roughly the same form as that we employ, but is larger by at least a factor of five over the range of velocity differences that we consider. In the absence of an appropriate analytical expression for an unsteady drag coefficient over this range velocity differences, we employ the coefficient in Eq. (5) both to correct the drag coefficient and to relate the averages of products of fluctuations to the product of their averages. Values of β between 1 and 3 have been employed by Amoudry, *et al.* [30] in a study of sediment transport.

In Figs. 2–4, we plot the results of the integrations for the particle velocity, wind velocity, and particle concentration against the measurements for Shields parameters equal to 0.035 and 0.098. The agreement is good for $\beta = 2$. The boundary values that were employed to obtain these profiles are based on the measured values of the splash parameters for spheres [18]; the good fits to the measured grain and wind velocities result from the relative high value of α, while that to the measured concentrations result from the nonzero value of β. In contrast, Creyssels, *et al.* [26] adjust the boundary values in a simple discrete numerical simulation and obtain a reasonable fit to the wind velocity, but not to the particle velocity or the concentration, using $\alpha = 1$ and $\beta = 0$.

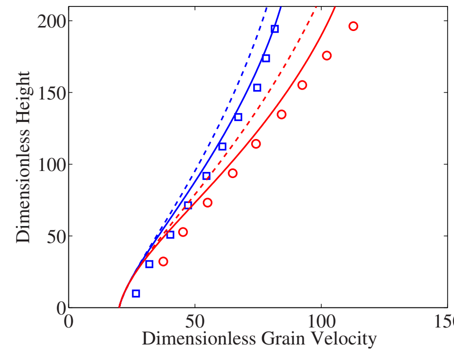

FIG. 2. (Color online) Dimensionless height versus dimensionless grain velocity: Experimental data of Creyssels *et al.* [26] for $S^* = 0.035$ (squares) and 0.098 (circles) and predictions for $\beta = 0$ (dashed lines) and 2 (solid lines) for $u_0 = 17.5$, $T = 20$, and $\alpha = 20$.

VI. CONCLUSION

Motivated by simple averaging of the equations of motion for a single particle moving under gravity through a turbulent

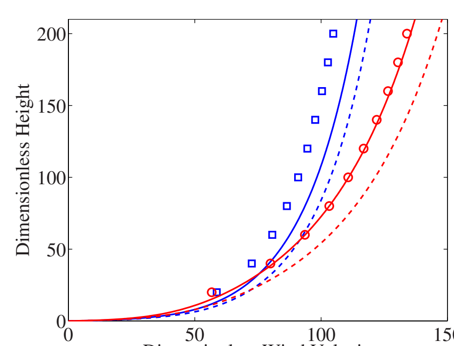

FIG. 3. (Color online) Dimensionless height versus dimensionless wind velocity: Experimental data of Creyssels, *et al.* [26] for $S* = 0.035$ (squares) and 0.098 (circles) and predictions for $\beta = 0$ (dashed lines) and 2 (solid lines) and the same values of the parameters as in Fig. 2.

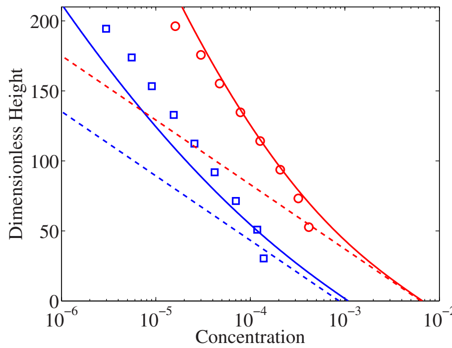

FIG. 4. (Color online) Dimensionless height versus particle concentration: Experimental data of Creyssels *et al.* [26] for $S^{*}=0.035$ (squares) and 0.098 (circles) and predictions for $\beta=0$ (dashed lines) and 2 (solid lines) and the same values of the parameters as in Fig. 2.

wind, we have proposed an expression for the local particle shear stress in terms of the particle pressure, drag coefficient, and vertical gradient of the mean particle velocity. This provided a closure to the continuum equations for steady, fully developed saltation above a horizontal particle bed that avoids any assumption about the particle shear stress at the bed.

We used this local constitutive equation with the balance equations for horizontal and vertical momentum of the particles, horizontal momentum of the gas, and boundary conditions at the bed to phrase a two-point boundary-value problem. The boundary conditions at the bed resulted from averaging low moments of a splash function, measured in experiments, over a Gaussian distribution of incoming particle velocities. When we included the vertical force on the particles associated with their turbulent suspension, we found that the fields of concentration and velocity generated in numerical solutions of the resulting boundary-value problem agreed well with those measured in wind tunnel experiments of steady, fully developed flows over a range of wind speeds.

The continuum model and the numerical solutions could be put on a firmer footing by laboratory experiments that involve turbulent shearing flows over a bed of sand. Three types of experiments are necessary for this. The first should characterize the turbulent drag on sand grains in unsteady motions with average velocity differences relevant to saltating particles; the second should employ dilute flows to measure the parameters that describe collisions with the bed for particles with the coefficients of restitution and sliding friction of rounded sand grains; and the third should measure the correlation between fluctuations in particle concentration and wind velocity. This information would eliminate the uncertainty regarding the particle drag coefficient, the bed collision parameters, and turbulent suspension in the present continuum model and provide a foundation for its extension to unsteady and developing flows.

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