7. Modulation and signal recovery

7.1 Introduction: modulation

- relevance to communications – transmission of an information-bearing signal
- recovery of very small signals buried in noise, via e.g. phase sensitive detection

Information-bearing signal denoted $m(t)$, message signal: transmission requires some type of manipulation, e.g. AM radio where $m(t)$ has its natural frequencies in the audio range

- not directly compatible with radio transmission frequencies
- must be modified in some way to be transmitted
- frequency range is shifted using *modulation*.
- modulation is defined as the process by which some characteristic of a carrier signal $x_c(t)$ is varied by modulating signal $m(t)$

A continuous wave carrier signal is a sinusoidal wave:

$$x_c(t) = A(t)\cos\left[\omega_c t + \phi(t)\right]$$

– $A(t)$ – instantaneous amplitude
– $\phi(t)$ – instantaneous phase angle
– $\omega_c = 2\pi f_c$ – carrier frequency

7. Modulation and signal recovery

7.1 Introduction: modulation (cont)

Amplitude modulation

- instantaneous amplitude $A(t)$ of carrier signal $x_c(t)$ linearly related to the message signal $m(t)$
- amplitude of the carrier signal is constant, the carrier amplitude, A_c.
- set $\phi(t) = 0$, so can write carrier signal as $x_c(t) = A_c \cos(\omega_c t)$

There are several types of amplitude modulation:

- standard or ordinary amplitude modulation
- double sideband modulation (DSB)
- single sideband modulation (SSB)
- vestigial sideband modulation (VSB)

7.2 Ordinary amplitude modulation

An ordinary AM signal can be created in three steps

- define carrier signal as above $x_c(t) = A_c \cos(\omega_c t)$
- multiply message signal $m(t)$ by $\cos(\omega_c t)$ to give $m(t)\cos(\omega_c t)$
- form sum of these two waves to produce the ordinary AM signal

$$x_{AM}(t) = m(t)\cos(\omega_c t) + A_c\cos(\omega_c t) = \left[A_c + m(t)\right]\cos(\omega_c t)$$

7. Modulation and signal recovery

7.2 Ordinary amplitude modulation (cont)

The envelope

- amplitude of the ordinary AM wave $x_{AM}(t)$
- given by $a(t) = A_c + m(t)$

Quality of transmission, use *modulation index* $\mu = |\max\{m(t)\}| / A_c$

- indicates amount of variation of modulated signal about normal value.

Two general cases:

$\mu \leq$ **1**: direct correspondence of envelope of $x_{AM}(t)$ with message signal. Wave can be demodulated, allowing recovery of original signal $m(t)$

μ **> 1**: indicates a problem, wave is overmodulated. The envelope of $x_{AM}(t)$ will not always directly correspond to $m(t)$. The signal suffers from envelope distortion.

Envelope sometimes written $a(t) = A_c\,[1 + k_a m(t)]$

- k_a is called the amplitude sensitivity
- percent modulation given by $100\, k_a\, |\max\{m(t)\}|$

7. Modulation and signal recovery

7.2 Ordinary amplitude modulation (cont)

Example:

Let $m(t) = \sin(\pi t/2)$ and $x_c(t) = 3\cos(20\pi t)$. Describe the AM wave generated, along with its envelope.

Avoiding envelope distortion

Require

- $\mu \leq 1$
- message bandwidth << carrier frequency

Modulation index can be expressed: $\mu = \dfrac{a_{\max} - a_{\min}}{a_{\max} + a_{\min}}$

Example:

Let $m(t) = A_m\cos(3\pi t/4)$ and $x_c(t) = 3\cos(20\pi t)$. Write the modulating signal in terms of the modulation index and then consider three cases of percent modulation: (a) 15%, (b) 40%, and (c) 125%

7. Modulation and signal recovery

7.2 Ordinary amplitude modulation (cont)

Power in AM waves

- carrier power given by $P_c = A_c^2 / 2$
- sideband power given by $P_S = \mu^2 A_c^2 / 4$
- the total power in the wave is $P_t = P_c + P_S$
- the efficiency of an ordinary AM wave is given by $\eta = (P_S / P_t) \times 100\%$

Can write the efficiency in terms of the modulation index: $\eta = \dfrac{\mu^2}{2+\mu^2} \times 100\%$

7.3 AM waves in the frequency domain

To describe spectrum of AM signal recall that FT of a cosine function is:

$\cos(\omega_0 t) \rightleftharpoons \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$

Signal in the time domain: $x_{AM}(t) = m(t)\cos(\omega_c t) + A_c\cos(\omega_c t)$

FT of second term: $FT\left[A_c\cos(\omega_c t)\right] = \pi A_c\left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right]$

The FT of the first term can be found from the modulation theorem:

If $m(t) \rightleftharpoons M(\omega)$ then $m(t)\cos(\omega_c t) \rightleftharpoons \dfrac{1}{2}M(\omega - \omega_c) + \dfrac{1}{2}M(\omega + \omega_c)$

Exercise: demonstrate the modulation theorem

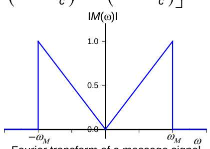
7. Modulation and signal recovery

7.3 AM waves in the frequency domain (cont)

$$FT\left[x_{AM}(t)\right] = FT\left[m(t)\cos(\omega_c t) + A_c\cos(\omega_c t)\right]$$
$$= \frac{1}{2}M(\omega - \omega_c) + \frac{1}{2}M(\omega + \omega_c) + \pi A_c\left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right]$$

FT of some message signal:

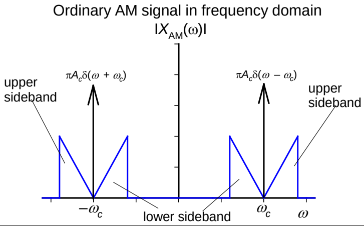
- spectral range is the baseband
- message signal known as baseband signal



Fourier transform of a message signal

Effect of AM in frequency domain shown below:

Ordinary AM signal in frequency domain



- 2 copies of message spectrum in signal
- one translated to $\omega = \omega_c$, the other to $\omega = -\omega_c$
- part of spectrum above ω_c known as upper sideband,
- part of spectrum below ω_c known as lower sideband
- carrier freq. ω_c much larger than bandwidth of message signal ω_M

7. Modulation and signal recovery

7.4 Generation and detection of ordinary AM waves

Generation

- using a square law modulator
- involves non-linear device such as a diode or a transistor

Given input signal $v_1(t)$ transfer characteristic is of the form

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

Input is sum message signal + carrier wave: $v_1(t) = m(t) + A_c \cos\omega_c t \Rightarrow$

$$v_1^2(t) = \left[m(t) + A_c \cos\omega_c t\right]^2 = m^2(t) + A_c^2 \cos^2\omega_c t + 2A_c m(t)\cos\omega_c t$$

So output signal is given by

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) = a_1\left[m(t) + A_c \cos\omega_c t\right] + a_2\left[m(t) + A_c\cos\omega_c t\right]^2$$
$$= a_1 A_c \cos\omega_c t + 2a_2 A_c m(t)\cos\omega_c t + \left\{a_1 m(t) + a_2 m^2(t) + a_2 A_c^2\cos^2\omega_c t\right\}$$

Remove unwanted terms {} by filtering to leave

$$a_1 A_c \cos\omega_c t + 2a_2 A_c m(t)\cos\omega_c t = a_1 A_c\left[1 + 2\frac{a_2}{a_1} m(t)\right]\cos\omega_c t$$

which has desired form of an AM wave.

7. Modulation and signal recovery

7.4 Generation and detection of ordinary AM waves (cont)

Detection

- using a square law detector

Given input signal $v_1(t)$ transfer characteristic is of the form

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

Input is the AM wave: $v_1(t) = x_{AM}(t) = \left[A_c + m(t)\right]\cos\omega_c t$

So output signal is given by

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) = a_1\left[A_c + m(t)\right]\cos\omega_c t + a_2\left[A_c + m(t)\right]^2\cos^2\omega_c t$$
$$= a_1\left[A_c + m(t)\right]\cos\omega_c t + a_2\left[A_c^2 + 2A_c m(t) + m^2(t)\right]\cos^2\omega_c t$$
$$= a_1\left[A_c + m(t)\right]\cos\omega_c t + a_2 A_c^2\cos^2\omega_c t$$
$$+ 2a_2 A_c m(t)\cos^2\omega_c t + a_2 m^2(t)\cos^2\omega_c t$$

Looks complicated.... but focus on the term $2a_2 A_c m(t)\cos^2\omega_c t$

$$2a_2 A_c m(t)\cos^2\omega_c t = 2a_2 A_c m(t)\left[\frac{1+\cos 2\omega_c t}{2}\right]$$
$$= a_2 A_c m(t) + a_2 A_c m(t)\cos 2\omega_c t$$

7. Modulation and signal recovery

7.4 Generation and detection of ordinary AM waves (cont)

Detection (cont)

- note term $a_2 A_c m(t)$, the message signal scaled by some constants
- appropriate filtering can remove other terms to leave message signal
- if $\omega_c >> \omega_M$ can be performed by application of low pass filter

Consider action of low pass filter excluding ω_c . We have

$$v_2(t) = a_1\left[A_c + m(t)\right]\cos\omega_c t + a_2 A_c^2\cos^2\omega_c t + a_2 A_c m(t)$$
$$+ a_2 A_c m(t)\cos 2\omega_c t + a_2 m^2(t)\cos^2\omega_c t$$
$$= a_1\left[A_c + m(t)\right]\cos\omega_c t + \left[a_2 A_c^2 + a_2 m^2(t)\right]\cos^2\omega_c t$$
$$+ a_2 A_c m(t)\cos 2\omega_c t + a_2 A_c m(t)$$
$$= a_1\left[A_c + m(t)\right]\cos\omega_c t + \left[a_2 A_c^2 + a_2 m^2(t)\right]\left[\frac{1+\cos 2\omega_c t}{2}\right]$$
$$+ a_2 A_c m(t)\cos 2\omega_c t + a_2 A_c m(t)$$

Applying low pass filter

$$\left[v_2(t)\right]_{\omega<\omega_c} = \frac{1}{2}a_2 A_c^2 + \frac{1}{2}a_2 m^2(t) + a_2 A_c m(t)$$

7. Modulation and signal recovery

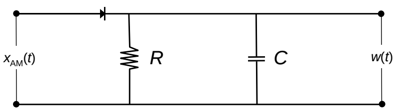
7.4 Generation and detection of ordinary AM waves (cont)

Detection (cont)

$$\left[v_2(t)\right]_{\omega<\omega_c} = \frac{1}{2}a_2 A_c^2 + \frac{1}{2}a_2 m^2(t) + a_2 A_c m(t)$$

- first term is just a constant and poses no problem
- third term is scaled message signal
- second term contains square of message signal and is source of unwanted distortion

Alternative is **envelope detector**, which can be constructed with a diode, resistor and capacitor:



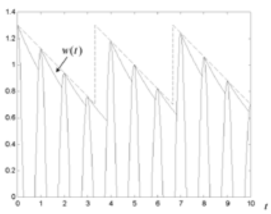
- diode half-wave rectifies signal
- when $x_{AM}(t)$ > capacitor voltage, capacitor voltage follows $x_{AM}(t)$
- when $x_{AM}(t)$ < capacitor voltage diode switches off and capacitor discharges through resistor with time constant $\tau = RC$

7. Modulation and signal recovery

7.4 Generation and detection of ordinary AM waves (cont)

Envelope detector (cont)

Resulting signal looks like this:



- envelope detector often followed by lowpass filter to remove components at frequencies around ω_c
- signal shown would be smoothed by this
- works best when modulation index is small, but this makes system inefficient as large part of power wasted on carrier
- for envelope detection to work, need sufficient power to be transmitted, requires the following condition to be fulfilled for all t:

$$A_c + m(t) > 0$$

Example

Design an envelope detector to demodulate the AM signal

$$x_{AM}(t) = \left[1 + 0.5\cos(200\pi t)\right]\cos\left(2\pi 10^6 t\right)$$

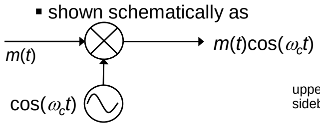
7. Modulation and signal recovery

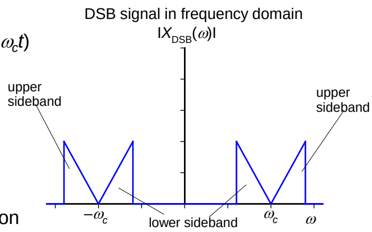
7.5 Double sideband modulation

- amplitude of AM signal proportional to message signal: $A(t) = am(t)$
- a constant, for simplicity, take a = 1, then $x_{DSB}(t) = m(t)\cos(\omega_c t)$

The generation of a DSB signal is conceptually straightforward:

- multiply the message signal by the carrier wave.
- shown schematically as





Frequency domain representation

- easy to find using modulation theorem
- same as for ordinary AM wave, except no Dirac delta functions
- called *suppressed carrier modulation* as absence of cosine term suppresses the appearance of the Dirac delta fns

7. Modulation and signal recovery

7.5 Double sideband modulation (cont)

Demodulation

- multiply by $\cos(\omega_c t)$ (local carrier), resulting in

$$x_{DSB}(t)\cos(\omega_c t) = m(t)\cos^2(\omega_c t) = m(t)\left[\frac{1+\cos(2\omega_c t)}{2}\right]$$

- bandwidth of message signal ω_m should be much less than that of the carrier signal ω_c, that is $\omega_m << \omega_c$
- so can apply a low pass filter that rejects the carrier frequency

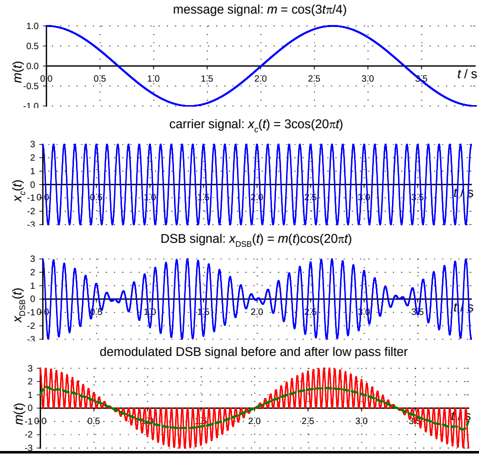
$$m(t)\left[\frac{1+\cos(2\omega_c t)}{2}\right] \xrightarrow{\text{filter } \omega_c} \frac{1}{2}m(t)$$

- called coherent detection as receiver must generate local wave that has same frequency ω_c as DSB signal, in phase with it
- this requirement can lead to practical difficulties

Example: demonstrate DSB modulation and demodulation where $m(t) = \cos(3\pi t/4)$ and $x_c(t) = 3\cos(20\pi t)$.
Consider also the case of a small frequency error in the local carrier

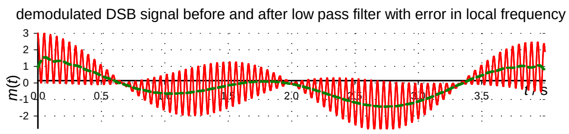
7. Modulation and signal recovery

7.5 Double sideband modulation : example



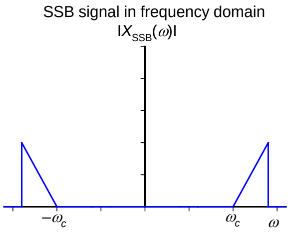
7. Modulation and signal recovery

7.5 Double sideband modulation : example (cont)



7.6 Single sideband modulation

- Both ordinary AM and DSB modulation produce two copies of the message signal in the frequency domain: wasted bandwidth
- only necessary to transmit one of them: single sideband modulation, SSB, transmit either upper or lower sideband



- generate SSB signal by frequency discrimination: create DSB signal, then eliminate unwanted frequencies using bandpass filter

7. Modulation and signal recovery

7.7 Angle modulation

General carrier wave: $x_c(t) = A_c \cos(\omega_c t + \phi) = A_c \cos(\theta)$

- $\theta(t) = \omega_c t + \phi(t)$ is instantaneous angle

Two types of angle modulation

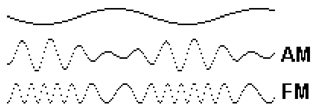
- phase modulation
- frequency modulation

Phase modulation

- $\theta(t)$ varied linearly with modulating signal: $\theta(t) = \omega_c t + k_p m(t)$
- k_p is phase deviation constant

Frequency modulation

- instantaneous frequency $\omega_i = \omega_c + \frac{d\phi}{dt} = \omega_c + k_f m(t)$
- k_f is frequency deviation constant
- $x_{FM}(t) = A_c \cos\left[\omega_c t + k_f \int_0^t m(\tau)d\tau\right]$



7. Modulation and signal recovery

7.8 Application to signal recovery: the lock-in amplifier

- used to detect and measure very small ac signals.
- can make accurate measurements of small signals even when these are obscured by noise sources a thousand times larger.

Essentially, a lock-in is a filter with an arbitrarily narrow bandwidth which is tuned to the frequency of the signal.

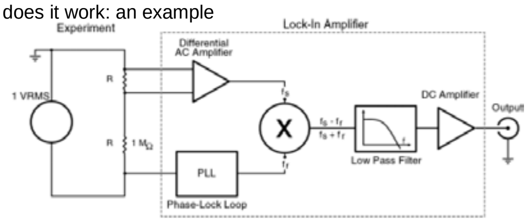
- will reject most unwanted noise to allow the signal to be measured.
- typical lock-in application may require a center frequency of 10 KHz and a bandwidth of 0.01 Hz.
- a lock-in also provides gain. For example, a 10 nanovolt signal can be amplified to produce a 10 V output--a gain of one billion.
- technique requires that the experiment be excited at a fixed frequency in a relatively quiet part of the noise spectrum.
- lock-in then detects the response from the experiment in a very narrow bandwidth at the excitation frequency.

Applications include low level light detection, Hall probe and strain gauge measurement, micro-ohm meters, electron spin and nuclear magnetic resonance studies

7. Modulation and signal recovery

7.8 Application to signal recovery: the lock-in amplifier (cont)

How does it work: an example



- reference source is 1 V_{rms} sine wave at frequency ω_r, current limited by 1 MΩ resistor to provide 1 μA ac excitation to 0.1Ω sample.
- two signals provided to lock-in: 1 V AC reference and amplified signal
- output of amplifier multiplied by the phase-locked loop (PLL) output in the Phase-Sensitive Detector (PSD), output given by:

$$V_{PSD} = \cos(\omega_r t + \phi)\cos(\omega_s t) = \tfrac{1}{2}\cos\left[(\omega_r + \omega_s)t + \phi\right] + \tfrac{1}{2}\cos\left[(\omega_r - \omega_s)t + \phi\right]$$

- sum frequency component attenuated by low pass filter, only difference frequency components within low pass filter's narrow bandwidth will pass through to the dc amplifier. High noise rejection.

8. Laplace and z- transforms

8.1 Laplace transform: introduction

The Laplace transform

- converts continuous time-domain signals into a function of a complex variable *s*

Very useful in the study of LTI systems:

- allows conversion of ODEs into algebraic equations
- converts convolution into simple multiplication

8.2 Laplace transform: definition

Laplace transform used to transform continuous function of time *t* into a function of *s*, which is in general a complex number.

- $s = \sigma + j\omega$ where $\sigma = \mathrm{Re}(s)$ and $\omega = \mathrm{Im}(s)$ are real variables

The LT $X(s)$ of a signal $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

or alternatively $X(s) = L\{x(t)\}$

- $L\{\ \}$ viewed as an operator

8. Laplace and z- transforms

8.3 Laplace transform: general procedure

More important to learn how to manipulate Laplace transforms rather than calculate them. The procedure can be summarised:

- given one or more input signals, look up their Laplace transforms in a table
- use the properties of the Laplace transform to accomplish various tasks algebraically, including solving differential equations or computing convolution of two signals
- function of *s* will be the result – manipulate this until in a form that can be readily transformed back to a function of time by inspection

8.4 Laplace transform: computation

Best shown by example

Example 8.4.1: Find the Laplace transform of $x(t) = u(t)$

8. Laplace and z- transforms

8.4 Laplace transform: computation (contd)

Example 8.4.2: Find the Laplace transform of $x(t) = e^{-at}u(t)$

yielding the Laplace transform pair

8. Laplace and z- transforms

8.5 Laplace transform: important properties

Linearity: $L\{\alpha x_1(t) + \beta x_2(t)\} = \alpha X_1(s) + \beta X_2(s)$

Time scaling: given $x(t)$ and that $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

what is the Laplace transform of the time-scaled function $x(at)$?

$$L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Time shifting: Supposing that $X(s) = L\{x(t)\}$ what is the Laplace transform of $x(t - t_0)$?

$$L\{x(t - t_0)\} = e^{-st_0} X(s)$$

8. Laplace and z- transforms

8.6 Laplace transform: differentiation

One of the most useful properties of the Laplace transform for solving ODEs.

$$L\left\{\frac{dx}{dt}\right\} = \int_0^{\infty} \frac{dx}{dt} e^{-st} dt$$

Integration by parts:

$$\int_0^{\infty} \frac{dx}{dt} e^{-st} dt = \left[x(t) e^{-st}\right]_0^{\infty} + s\int_0^{\infty} x(t) e^{-st} dt$$
$$= -x(0) + sX(s)$$

Now consider differentiation with respect to *s*:

$$\frac{d}{ds}\left[X(s)\right] = \frac{d}{ds}\int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} \frac{d}{ds}\left[x(t) e^{-st}\right] dt$$
$$= \int_{-\infty}^{\infty} x(t) \frac{d}{ds}\left[e^{-st}\right] dt = -\int_{-\infty}^{\infty} x(t) t e^{-st} dt \Rightarrow$$
$$-\frac{dX}{ds} = \int_{-\infty}^{\infty} t x(t) e^{-st} dt \Rightarrow L\{t x(t)\} = -\frac{dX}{ds}$$

8. Laplace and z- transforms

8.6 Laplace transform: differentiation (contd)

Example 8.6.1: Find the Laplace transform of $x(t) = tu(t)$

Example 8.6.2: Find the solution of $dy/dt = A\cos t$ for $t \geq 0$, $y(0) = 1$ given that $L\{\cos(\beta t) u(t)\} = \frac{s}{s^2 + \beta^2}$ and $L\{\sin(\beta t) u(t)\} = \frac{\beta}{s^2 + \beta^2}$

8.7 Inverse Laplace transform

Inverse Laplace transform written as $x(t) = L^{-1}\{X(s)\}$

Formally, can be calculated using integral: $x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$

In practice, do not use, but rather manipulate expression into recognisable terms, often using partial fraction decomposition.

Shift in s can be useful: $\int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t} dt = X(s - s_0)$

Example 8.7.1 Find the inverse Laplace transform of $X(s) = \frac{s+3}{s^2 + 6s + 18}$

8. Laplace and z- transforms

8.8 Laplace transform: region of convergence (ROC)

Definition

The range of values of the complex variables s for which the Laplace transform converges is called the region of convergence (ROC).

Example 8.8.1 Find the ROC for the signal $x(t) = e^{-at}u(t)$ a real

The Laplace transform of $x(t) = -e^{-at}u(-t)$ a real can be shown to be

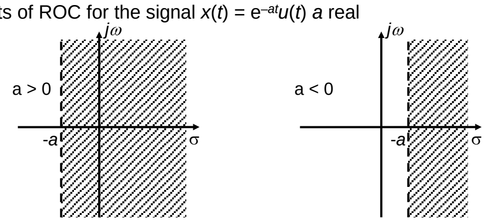
$$X(s) = \frac{1}{(s+a)} \quad \mathrm{Re}(s) < -a$$

Therefore in order for the Laplace transform to be unique the ROC must be specified as part of the transform.

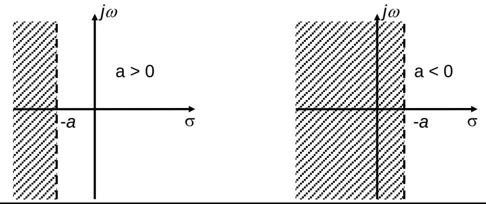
8. Laplace and z- transforms

8.8 Laplace transform: region of convergence (ROC) contd

Plots of ROC for the signal $x(t) = e^{-at}u(t)$ a real



Plots of ROC for the signal $x(t) = -e^{-at}u(-t)$ a real



8. Laplace and z- transforms

8.8 Laplace transform: region of convergence (ROC) contd

Poles and zeros of $X(s)$

Usually $X(s)$ is rational function of s:

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0 (s - z_1)(s - z_2)\cdots(s - z_m)}{b_0 (s - p_1)(s - p_2)\cdots(s - p_n)}$$

- roots of numerator polynomial z_k : zeros of $X(s)$, plotted as **o**
- roots of denominator polynomial p_k : poles of $X(s)$), plotted as **x**

Example 8.8.2

Plot the ROC, zeros and poles of $X(s) = \dfrac{2s+4}{s^2+4s+3} \quad \mathrm{Re}(s) > -1$

8. Laplace and z- transforms

8.8 Laplace transform: region of convergence (ROC) contd

Properties of ROC

1. ROC does not contain any poles
2. if $x(t)$ is *finite duration* signal then ROC is entire s-plane (except perhaps $s = 0$ or $s = \infty$)
3. if $x(t)$ is *right-sided* signal ($x(t) = 0$ for $t < t_1 < \infty$) then ROC is of form $\mathrm{Re}(s) > \sigma_{max}$ where σ_{max} equals the maximum real part of any of the poles of $X(s)$
4. if $x(t)$ is *left-sided* signal ($x(t) = 0$ for $t > t_2 > -\infty$) then ROC is of form $\mathrm{Re}(s) < \sigma_{min}$ where σ_{min} equals the minimum real part of any of the poles of $X(s)$
5. if $x(t)$ is *two-sided* signal ($x(t)$ is an infinite duration signal that is neither left-sided nor right-sided) then ROC is of form $\sigma_1 < \mathrm{Re}(s) < \sigma_2$ where σ_1 and σ_2 are the real parts of the two poles of $X(s)$.

8. Laplace and z- transforms

8.8 Laplace transform: region of convergence (ROC) contd

Causality

- a causal continuous time LTI system has h(t) = 0 t < 0
- $h(t)$ is right sided signal ⇒ ROC of $H(s)$ of form $\mathrm{Re}(s) > \sigma_{max}$
- ROC is region on s-plane to right of all system poles

Stability

- a continuous time LTI system is BIBO stable if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
- corresponding requirement on $H(s)$ is that ROC of $H(s)$ contains the $j\omega$ axis

Causal and stable systems

- if system is both causal and stable then all *poles* must lie in left half of s-plane
- as $\mathrm{Re}(s) > \sigma_{max}$ and $j\omega$ axis included in ROC such that $\sigma_{max} < 0$

8. Laplace and z- transforms

8.9 Laplace transform: Characterisation of LTI systems

For continuous time LTI system

- $y(t) = x(t)*h(t)$
- $Y(s) = X(s)H(s)$.
- $H(s)$ known as the *transfer function* or the *system function*.
- $H(s) = Y(s) / X(s)$
- transfer function reveals basic characteristics of system
- ignore $x(t)$ for $t < 0$: *relaxed* systems, all initial conditions set to zero.

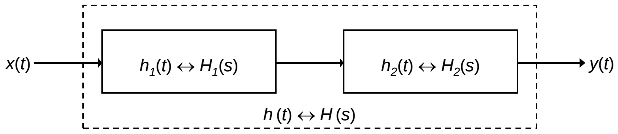
Example 8.9.1 Consider a relaxed LTI system for which $\dfrac{dy}{dt} = \dfrac{dx}{dt} - 3x(t)$

Assume the system is causal and find the impulse response $h(t)$.

8. Laplace and z- transforms

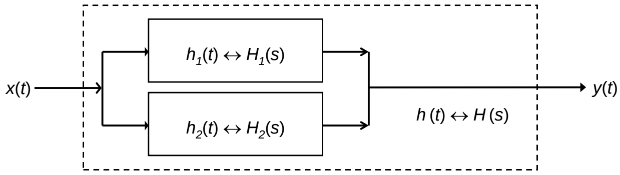
8.10 LTI systems interconnection

Two LTI systems in series:



time domain: $h(t) = h_1(t) * h_2(t)$ **s-domain: $H(s) = H_1(s)\, H_2(s)$**

Two LTI systems in parallel



time domain: $h(t) = h_1(t) + h_2(t)$ **s-domain: $H(s) = H_1(s) + H_2(s)$**

8. Laplace and z- transforms

8.10 LTI systems interconnection

Example 8.10.1 Two systems are arranged in series

$h_1(t) = e^{-2t}u(t)$ and $h_2(t) = e^{-4t}u(t)$

Find the impulse response of the entire system.

First, find Laplace transform of each function:

8. Laplace and z- transforms

8.11 The *z*-transform: introduction and definition

- discrete time equivalent to Laplace transform
- can be used to analyse discrete signals going to infinity
- simplifies analysis of DT signals by allowing us to convert finite difference equations into algebraic equations.

The z-transform of a discrete time signal $x[n]$ is given by $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

As z is a complex number, can be written in polar representation: $z = re^{j\Omega}$

If a signal $x[n]$ is zero when $n < 0$ then $X_I(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

In analogy to the FT, we define the *z-transform pair* $x[n] \leftrightarrow X[z]$

The discrete time Fourier transform (DTFT) is special case of the *z*-transform:

$$X\left(e^{j\omega}\right) = \left[X(z)\right]_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

In the *z*-plane, the DTFT is simply $X(z)$ evaluated on the unit circle.

8. Laplace and z- transforms

8.12 The *z*-transform: basic properties

Linearity: $y[n] = ax_1[n] + bx_2[n] \Rightarrow Y(z) = aX_1(z) + bX_2(z)$

Time shifting: $x[n-n_0] \leftrightarrow z^{-n_0} X(z)$

Time reversal: $x[-n] \leftrightarrow X\left(\frac{1}{z}\right)$

Differentiation: $nx[n] \leftrightarrow -z\frac{dX}{dz}$

Convolution: $x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z)$

Accumulation property: $\sum_{k=-\infty}^{\infty} x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z)$

Multiplication by z_0^n: $z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$

Example 8.12.1: find the *z*-transform of the unit impulse sequence $\delta[n]$

8. Laplace and z- transforms

8.13 The *z*-transform: region of convergence (ROC)

Infinite series: important to know when converges.

ROC, region of convergence for *z*-transform tells us this

Consider *z*-transform of $a^n u[n]$ (*a* real)

8.14 Inverse *z*-transform

Method of partial fraction expansion, using $a^n u[n] \leftrightarrow \frac{z}{z-a}$

Example 8.12.1: find the inverse *z*-transform of $X(z) = \frac{z}{(z-3)(z+4)}$

8. Laplace and z- transforms

8.15 The *z*-transform: power series expansion

Can write *z*-transform as a power series (a *Laurent* series):

$X(z) = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[-1]z^{-1} + x[2]z^{-2} + \cdots$

For sequences that are 0 for $n < 0$ then

$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$

Example 8.15.1: find *z*-transform of {1, 2, 2, 4, 5, 1} ($x[n] = 0$ for $n < 0$)

Example 8.15.2: find inverse *z*-transform of $X(z) = 1 + z^{-1} + 3z^{-2} - 2z^{-3}$

8.16 The *z*-transform: applied to LTI systems

Output of LTI system given by $y[n] = x[n] * h[n]$

Use convolution property of *z*-transform $Y(z) = X(z)\, H(z)$

$H(z) = Y(z) / X(z)$

Use to compute convolutions.

Example 8.16.1: given that $x[n]$ = {1, 2, 1, 2} and $h[n]$ = {1, 1, 1} , find response $y[n]$