6. Discrete Fourier transforms and sampling

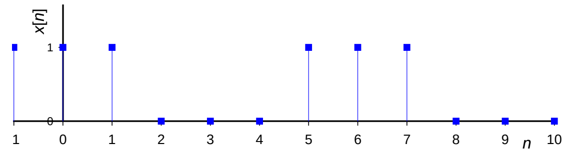
6.1 Discrete time Fourier series

- discrete time signal $x[n]$ with fundamental period N_0 : $x[n] = x[n + N_0]$.
- fundamental frequency $\Omega_0 = 2\pi / N_0$
- Fourier series representation of $x[n]$ is given by $x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$
- c_k – Fourier or spectral coefficients, given by $c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega_0 n}$
- if sum runs over any N_0 consecutive values of k: $x[n] = \sum_{k=\langle N_0\rangle} c_k e^{jk\Omega_0 n}$
- known as the *synthesis* equation.
- using same notation can express coefficients: $c_k = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]e^{-jk\Omega_0 n}$
- sometimes called the *analysis* equation.
- spectral coefficients and sequence $x[n]$ constitute Fourier series pair $x[n] \rightleftharpoons c_k$
- average value of $x[n]$ over a period is given by: $c_0 = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]$

6. Discrete Fourier transforms and sampling

6.1 Discrete time Fourier series (cont)

Example: find the spectral coefficients for the discrete time square wave shown below:



6.2 Properties of Discrete time Fourier series

For periodic discrete time signal $x[n] = x[n + N_0]$ spectral coeffients are also periodic: $c_k = c_{k+N_0}$

View members of discrete time sequence as Fourier coefficients of the c_k

$$c_k = c[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]e^{-jk\Omega_0 n} = \sum_{n=\langle N_0\rangle}\frac{x[n]}{N_0}e^{-jk\Omega_0 n}$$ Now let $m = -n$

$$c[k] = \sum_{m=\langle N_0\rangle}\frac{x[-m]}{N_0}e^{jk\Omega_0 m}$$ Now $k \to n$ and $m \to k$: $c[n] = \sum_{k=\langle N_0\rangle}\frac{x[-k]}{N_0}e^{jk\Omega_0 n}$

6. Discrete Fourier transforms and sampling

6.2 Properties of Discrete time Fourier series (cont)

This is just the discrete Fourier series representation for the $c[n]$. A demonstration of the *duality* property, which states

- if $x[n]$ and $c[k]$ form a Fourier series pair $x[n] \rightleftharpoons c[k]$
- then also have a Fourier series pair $c[n] \rightleftharpoons x[-k] / N_0$

Parseval's theorem for discrete Fourier series

Enables us to find the average power of a discrete time signal by summing the squared amplitudes of its harmonic components:

$$\frac{1}{N_0}\sum_{n=\langle N_0\rangle}|x[n]|^2 = \sum_{k=\langle N_0\rangle}|c[k]|^2$$

Example: demonstrate Parseval's theorem for the signal in 6.1

6. Discrete Fourier transforms and sampling

6.3 Fourier transform of a discrete time signal

FT of arbitrary non-periodic discrete time signal $x[n]$: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$

- FT is periodic in 2π, $X[\Omega] = X[\Omega + 2\pi]$
- product $X[\Omega]e^{j\Omega n}$ also periodic in 2π
- Inverse FT – integrate over interval 2π: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$
- FT of DT signal is linear: $ax_1[n] + bx_2[n] = aX_1[\Omega] + bX_2[\Omega]$
- time shift by n_0: $x[n - n_0] \rightleftharpoons e^{-j\Omega n_0}X(\Omega)$
- frequency shift by Ω_0: $e^{j\Omega n}x[n] \rightleftharpoons X(\Omega - \Omega_0)$
- using time shifting obtain: $x[n] - x[n-1] \rightleftharpoons (1 - e^{-j\Omega})X(\Omega)$
- accumulation property (where $|\Omega| \le 2\pi$):

$$\sum_{k=-\infty}^{\infty} x[k] \rightleftharpoons \pi X(0)\delta(\Omega) + \frac{1}{(1-e^{-j\Omega})}X(\Omega)$$

6. Discrete Fourier transforms and sampling

6.4 Discrete Fourier transform and sampling

Here we consider sampling of a continuous time signal $x(t)$ that is of finite duration.

- sample the signal at intervals of T_s called the sampling period
- total of N samples of the original signal, then we will have the sampled values $x(t)$, $x(T_s)$, $x(2T_s)$, ..., $x((N-1)T_s)$
- defines values of discrete time signal $x[n]$.

The DFT of $x[n]$ is denoted by $X[k]$ and is given by

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

The inverse discrete FT is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn}$$

6. Discrete Fourier transforms and sampling

6.4 Discrete Fourier transform and sampling (cont)

Example

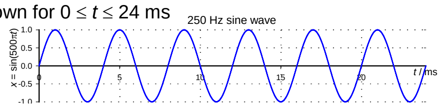
Given that $X[k] = \{0, -3 - 3j , -2, -3 + 3j \}$, use the inverse DFT to find $x[n]$

6. Discrete Fourier transforms and sampling

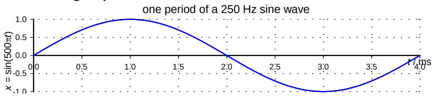
6.5 Sampling

Consider a sine wave with a frequency of f = 250 Hz.

- period $T = 1 / f = 1 / 250 \equiv 4$ ms
- continuous time signal $x(t) = \sin(2\pi f t) = \sin\{2\pi(250)t\}$
- shown for $0 \le t \le 24$ ms



- and for a single period of 4 ms

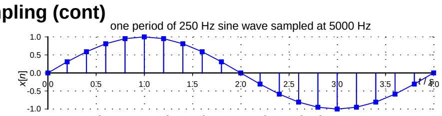


- sampling rate 5000 Hz, sampling interval $T_s = 1 / (5000 \text{ Hz}) \equiv 0.2$ ms
- out to 1 ms have the discrete time signal

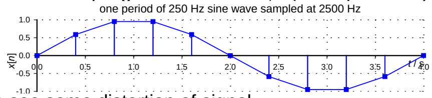
  $x[n]$ = {0.0000, 0.3090, 0.5878, 0.8090, 0.9511, 1.0000}

6. Discrete Fourier transforms and sampling

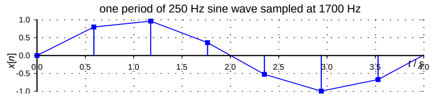
6.5 Sampling (cont)



- at 5000 Hz, good approximation to signal shape
- now, reduce sampling rate to 2500 Hz, $T_s = 1 / (2500 \text{ Hz}) \equiv 0.4$ ms



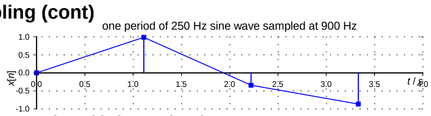
- start to see some distortion of signal
- now reduce sampling rate to 1700 Hz, $T_s = 1 / (1700 \text{ Hz}) \equiv 0.59$ ms



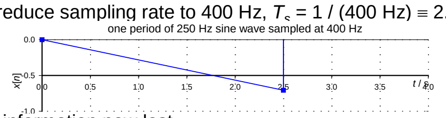
- further distortion evident
- sample signal now at 900 Hz, $T_s = 1 / (900 \text{ Hz}) \equiv 1.11$ ms

6. Discrete Fourier transforms and sampling

6.5 Sampling (cont)



- at 900 Hz, a lot of information lost
- now, reduce sampling rate to 400 Hz, $T_s = 1 / (400 \text{ Hz}) \equiv 2.5$ ms



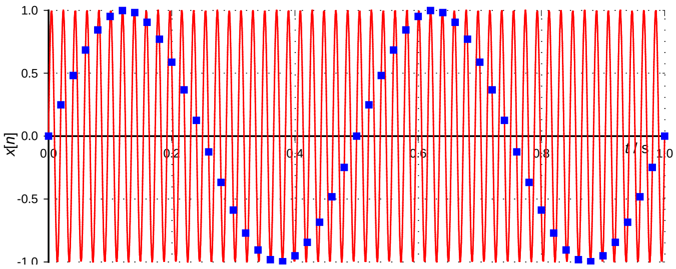
- much information now lost

Summarising:

- if signal changes rapidly in time, sampling interval T_s must be small enough to capture variations
- high frequency variation implies high frequency components in signal, requires high sampling rate
- when sampling rate not high enough / sampling interval too long to capture signal variation, we say that **aliasing** has occurred

6. Discrete Fourier transforms and sampling

Extreme example of aliasing



52Hz signal sampled at 50 Hz

6. Discrete Fourier transforms and sampling

6.5 Sampling (cont)

Nyquist sampling theorem

To sample a signal correctly, sampling rate (ω_s rad/sec) should be at least twice the highest frequency component (ω_h) present in the signal: $\omega_s \ge 2\omega_h$

For signals band width limited to $[-\omega / 2, \omega / 2]$

- the critical sampling interval $T_s = 2\pi / \omega$,
- $\omega_c = \omega$ is the *Nyquist critical frequency*
- Nyquist critical frequency is highest frequency that can pick up
- for a sine wave, this corresponds to a minimum of two samples per period
- an arbitrary band-width limited signal $x(t)$ is completely determined by its samples $x[n]$ taken at the Nyquist critical frequency:

$$x(t) = T_s \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\omega_c (t - nT_s)\right]}{\pi (t - nT_s)}$$

On the other hand, if sample a continuous function that is not bandwidth limited to less than the Nyquist critical frequency

6. Discrete Fourier transforms and sampling

6.5 Sampling (cont)

Nyquist sampling theorem (cont)

- all of power spectral density lying outside range $(-\omega_c / 2) < \omega < (\omega_c / 2)$ is incorrectly moved into that range: aliasing

Reconstruction of sampled signals

For example, reconstruction of sound from digital recording.

A band-limited signal sampled at frequency $\omega_s = 2\pi / T_s$ gives discrete time signal $x[n] = x(nT_s)$ from which we would like to recover the original continuous time signal.

- Ideally, we would do this by constructing a train of impulses from the $x[n]$ and then filter this signal with an ideal lowpass filter

In real life, two possibilities:

Zero-order hold, interpolates signal samples with a constant line segment over a sampling period for each sample

- frequency response is a poor approximation to ideal lowpass filter's

First-order hold

- triangular impulse response,
- gives a linear interpolation between each sample