5. Energy spectral density and correlation

5.1 Cross correlation

A way to measure the similarity between two energy signals

- measures the properties of an unknown signal by comparing it to a known signal.
- compare $x_1(t)$ to a time-delayed version of $x_2(t)$
- cross correlation function given by $R_{12}(\tau)=\int_{-\infty}^{\infty} x_1(t)^* x_2(t+\tau)dt$ (* denoted complex conjugate)
- reminder: functions orthogonal if $\int_{-\infty}^{\infty} x_1(t) x_2(t)dt = 0$

Example

Show that $x_1(t) = \sin t$, $x_2(t) = \cos t$ are orthogonal and calculate their cross correlation function for $\tau = \pi$

Consider over one period:

$$\int_{-\pi}^{\pi} \sin(t)\cos(t)dt = \left[\frac{\sin^2(t)}{2}\right]_{-\pi}^{\pi} = \frac{\sin^2(\pi)}{2} - \frac{\sin^2(-\pi)}{2} = 0$$

This will always be zero as $\sin^2 x = \sin^2(-x)$. The functions are orthogonal.

5. Energy spectral density and correlation

5.1 Cross correlation (cont) $R_{12}(\tau) = \int_{-\pi}^{\pi} \sin(t)\cos(t+\tau)dt$

However, let us now calculate the cross-correlation over one period

$$R_{12}(\tau=\pi/2) = \int_{-\pi}^{\pi} \sin(t)\cos(t+\pi/2)dt = \int_{-\pi}^{\pi} -\sin^2(t)dt = \frac{1}{2}\int_{-\pi}^{\pi} -(1+\cos(2t))dt$$

$$= \left[-\frac{t}{2}+\frac{\sin(2t)}{4}\right]_{-\pi}^{\pi} = -\frac{\pi}{2}-\frac{\pi}{2}+\frac{\sin(2\pi)}{4}-\frac{\sin(-2\pi)}{4} = -\pi$$

Further properties of cross correlation

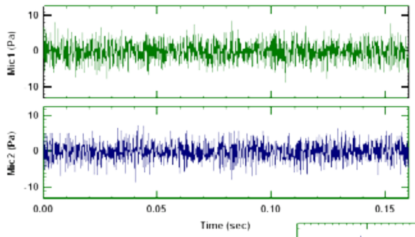
- sometimes written $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)^* x_2(t+\tau)dt = x_1(t) \star x_2(t)$
- related to convolution: $x_1(t) \star x_2(t) = x_1(-t)^* * x_2(t)$
- if either is an even function, then $x_1(t) \star x_2(t) = x_1(t) * x_2(t)$
- $(x_1(t) \star x_2(t)) \star (x_1(t) \star x_2(t)) = (x_1(t) \star x_1(t)) \star (x_2(t) \star x_2(t))$
- definition of discrete cross correlation: a series given by
  $r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]^* y[n+l] \qquad l = 0, \pm 1, \pm 2, \ldots$
- cross correlation theorem (often used with FFT to compute c-cs):
  $x_1(t) \star x_2(t) \rightleftharpoons X_1(f)^* X_2(f)$

5. Energy spectral density and correlation

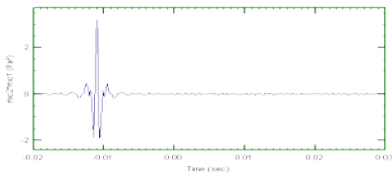
5.1 Cross correlation (cont)

Application: distinguishing signal from apparent noise

- two microphones at different distances from source gave signals:



- cross-correlating the two signals gives a peak at –11 ms, which corresponds to the time delay between the two microphones:



5. Energy spectral density and correlation

5.2 Autocorrelation

For a complex function $x_1(t)$ autocorrelation defined by

$$R_{11}(\tau) = x_1 \star x_1 = x_1(-\tau)^* * x_1(\tau) = \int_{-\infty}^{\infty} x_1(t)^* x_1(t+\tau)dt$$

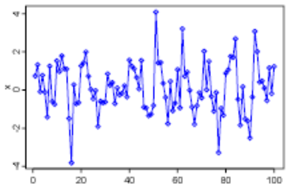
Normalised energy content obtained by setting $\tau = 0$:

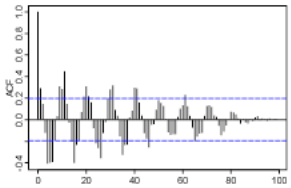
$$E = R_{11}(0) = \int_{-\infty}^{\infty} x_1(t)^* x_1(t)dt = \int_{-\infty}^{\infty} |x_1(t)|^2 dt$$

To find autocorrelation of a power signal compute time average:

$$\overline{R}_{11}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_1(t)^* x_1(t+\tau)dt \qquad \overline{R}_{11}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)^* x_1(t+\tau)dt \text{ (periodic sig)}$$

Example: A plot showing 100 random numbers with a "hidden" sine function, and an autocorrelation of the series on the right.





5. Energy spectral density and correlation

5.2 Autocorrelation (cont)

Applications

One application of autocorrelation is the measurement of optical spectra and the measurement of very-short-duration light pulses produced by lasers, both using optical autocorrelators.

In optics, normalized autocorrelations and cross-correlations give the degree of coherence of an electromagnetic field.

In signal processing, autocorrelation can give information about repeating events like musical beats or pulsar frequencies, though it cannot tell the position in time of the beat.

Further properties

$R_{11}(\tau) = x_1 \star x_1$ is a maximum at the origin

Autocorrelation is a Hermitian function as $R_{11}(-\tau) = R_{11}^*(\tau)$

Discrete version (real numbers)

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n+l] = r_{xx}[-l] \qquad l = 0, \pm 1, \pm 2, \ldots$$

5. Energy spectral density and correlation

5.3 Energy spectral density

- measures distribution of signal energy E over frequency
- found by taking the FT of the autocorrelation function
- denoted by $S_{11}(\omega)$ $S_{11}(\omega) = \int_{-\infty}^{\infty} R_{11}(\tau)e^{-j\omega\tau}d\tau$
- if signal $x(t)$ is real then $S_{11}(\omega) = |X(\omega)|^2$
- can compute energy in signal
  $E = R_{11}(0) = \int_{-\infty}^{\infty} |x_1(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_1(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{11}(\omega)d\omega$
- explains why call $S_{11}(\omega)$ energy spectral density

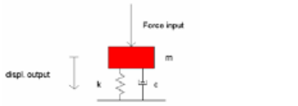
5.4 Power spectral density

- defined in a similar way to energy spectral density
- computed as the FT of the time-average autocorrelation:
  $\overline{S}_{11}(\omega) = FT\left[\overline{R}_{11}(\tau)\right] = \int_{-\infty}^{\infty} \overline{R}_{11}(\tau)e^{-j\omega\tau}d\tau$
- can find power in signal using this: $P = \overline{R}_{11}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{S}_{11}(\omega)d\omega$
- unit of PSD is (unit of measured quantity)^2/Hz.

5. Energy spectral density and correlation

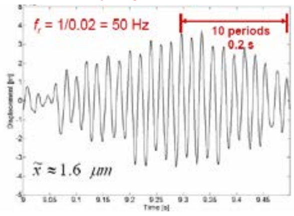
5.5 Applications of energy/power spectral density

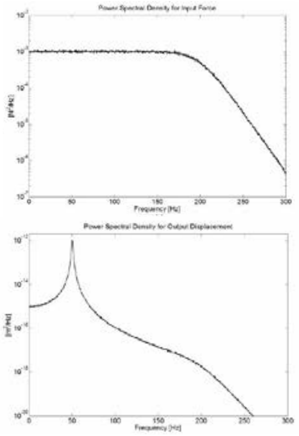
Vibration analysis



Mass	m	10 [kg]
Damping coefficient	c	100 [Ns/m]
Spring coefficient	k	1000000[N/m]

Resonance frequency 50.3 Hz

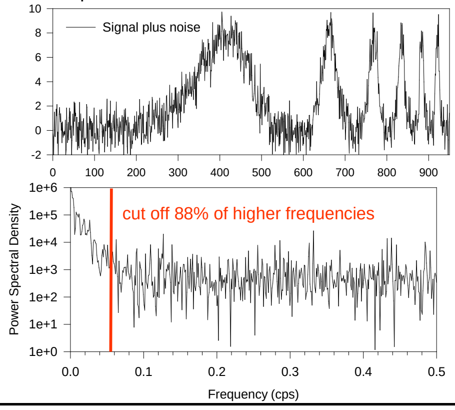




5. Energy spectral density and correlation

5.5 Applications of energy/power spectral density

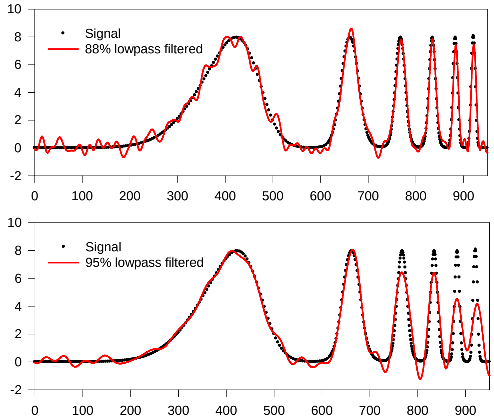
Simple Filtering : obtain PSD spectrum and use to design filter to block unwanted frequencies



5. Energy spectral density and correlation

5.5 Applications of energy/power spectral density

Simple filtering (cont)

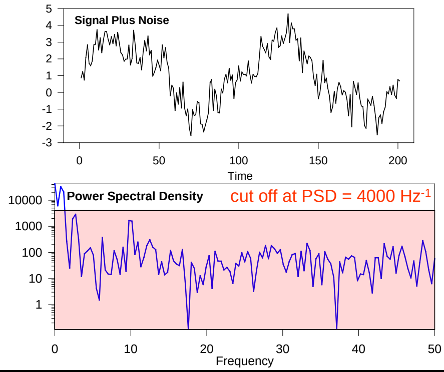


5. Energy spectral density and correlation

5.5 Applications of energy/power spectral density

Gain filter smoothing

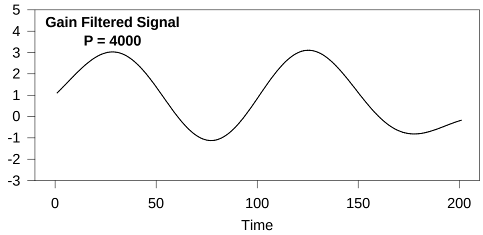
Gain filtering filters data by removing frequency components with power spectral density magnitude less than a specified value



5. Energy spectral density and correlation

5.5 Applications of energy/power spectral density

Gain filter smoothing (cont)



5. Energy spectral density and correlation

5.5 Applications of energy/power spectral density

Optical spectral density

The spectrum of a light source is a measure of the power carried by each frequency in a light source.

- optical power spectral densities, defined as the optical power per optical frequency (or wavelength) interval, e.g. specified in mW/THz or mW/nm
- care must be taken converting between frequency and wavelength, as the conversion factor depends upon frequency, e.g. for Planck black body radiation curves below: