4. Fourier Analysis and Applications

4.1 Introduction

A signal can be viewed from two different standpoints:

- the frequency domain
- the time domain

Any signal can be fully described in either of these domains

- go between the two by using a tool called the Fourier transform.

Why the frequency domain ?

- may be simpler to analyse signal in frequency domain

Fourier techniques have many applications

- optics: diffraction, interference
- audio: synthesis
- communications: filtering
- spectroscopy and dynamics: use of ultrafast lasers
- physics experiments: filtering noise, deconvolution

4. Fourier Analysis and Applications

4.2 Fourier series

Periodic signal $x(t)$ can be represented by a Fourier series expansion:

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T_0}\right) + b_n\sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

where T_0 is the fundamental period of the signal

The cos and sin functions are used as basis functions

- obey orthogonality relations

$$\int_{-T_0/2}^{T_0/2}\cos\left(\frac{2\pi mt}{T_0}\right)\sin\left(\frac{2\pi nt}{T_0}\right)dt = 0$$

$$\int_{-T_0/2}^{T_0/2}\cos\left(\frac{2\pi mt}{T_0}\right)\cos\left(\frac{2\pi nt}{T_0}\right)dt = \begin{cases} T_0/2 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

$$\int_{-T_0/2}^{T_0/2}\sin\left(\frac{2\pi mt}{T_0}\right)\sin\left(\frac{2\pi nt}{T_0}\right)dt = \begin{cases} T_0/2 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

Integrate basis functions over single period enables determination of mean value of the signal, a_0

4. Fourier Analysis and Applications

4.2 Fourier series (cont)

Mean value of a periodic signal, a_0

$$\int_{-T_0/2}^{T_0/2}\cos\left(\frac{2\pi nt}{T_0}\right)dt = \left[\frac{T_0}{2\pi n}\sin\left(\frac{2\pi nt}{T_0}\right)\right]_{-T_0/2}^{T_0/2} = \frac{T_0}{2\pi n}\left[2\sin(\pi n)\right] = 0$$

Similar result for sin function.

So, to obtain a_0, integrate Fourier series expansion over one period:

$$\int_{-T_0/2}^{T_0/2}x(t)dt = a_0\int_{-T_0/2}^{T_0/2}dt + 2\int_{-T_0/2}^{T_0/2}\left\{\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T_0}\right) + b_n\sin\left(\frac{2\pi nt}{T_0}\right)\right]\right\}dt$$

Bring integral inside sum: terms vanish

$$2\int_{-T_0/2}^{T_0/2}\left\{\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T_0}\right) + b_n\sin\left(\frac{2\pi nt}{T_0}\right)\right]\right\}dt$$

$$= 2\left\{\sum_{n=1}^{\infty}\left[a_n\int_{-T_0/2}^{T_0/2}\left\{\cos\left(\frac{2\pi nt}{T_0}\right)\right\}dt + b_n\int_{-T_0/2}^{T_0/2}\left\{\sin\left(\frac{2\pi nt}{T_0}\right)\right\}dt\right]\right\} = 0$$

Thus we are left with $\int_{-T_0/2}^{T_0/2}x(t)dt = a_0\int_{-T_0/2}^{T_0/2}dt = a_0\left[t\right]_{-T_0/2}^{T_0/2} = a_0T_0$

and so obtain a_0: $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)dt$

4. Fourier Analysis and Applications

4.2 Fourier series (cont)

Other coefficients

Multiply both sides of Fourier series expansion by $\cos\left(\frac{2\pi nt}{T_0}\right)$ and then integrate over one period:

$$a_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt$$

Similarly, multiplying by $\sin(2\pi nt / T_0)$ we obtain the remaining coeffts:

$$b_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt$$

Dirichlet conditions

Conditions to tell whether or not a periodic signal $x(t)$ can be represented by a Fourier series:

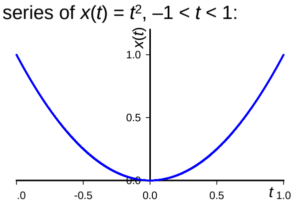
- $x(t)$ is single valued over the fundamental period
- $x(t)$ has finite number of discontinuities, minima and maxima over the fundamental period
- $\int_{-T_0/2}^{T_0/2}\left|x(t)\right|dt < \infty$ that is, $x(t)$ is *absolutely integrable*

4. Fourier Analysis and Applications

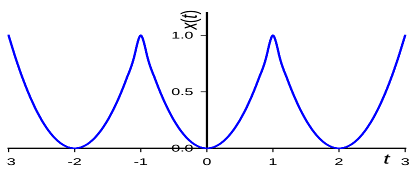
4.2 Fourier series (cont)

Example

Find the Fourier series of $x(t) = t^2$, $-1 < t < 1$:



Assume that function has been duplicated up and down the real line.



4. Fourier Analysis and Applications

4.3 Complex Fourier series

Using Euler's formulae: $\cos x = \frac{e^{jx} + e^{-jx}}{2}$ and $\sin x = \frac{e^{jx} - e^{-jx}}{2j}$

If write $x = 2\pi t / T_0$ then can write Fourier series expansion as

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T_0}\right) + b_n\sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

Now define: $c_0 = a_0$; $c_n = a_n - jb_n$; $c_{-n} = a_n + jb_n \Rightarrow$

$x(t) = c_0 + \sum_{n=1}^{\infty}\left[c_ne^{jnx} + c_{-n}e^{-jnx}\right]$ and recall $x = 2\pi t / T_0$ then obtain

$x(t) = \sum_{n=-\infty}^{\infty}\left[c_n\exp\left(j\frac{2\pi nt}{T_0}\right)\right]$ complex exponential Fourier series

4. Fourier Analysis and Applications

4.3 Complex Fourier series (cont)

The coefficients are given by $c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi nt/T_0}dt$

Can write these complex coefficients as $c_n = |c_n|e^{j\phi_n}$

where $\phi_n = \arg(c_n)$ is the phase.

- amplitude spectrum – plot of $|c_n|$ against frequency
- phase spectrum – plot of ϕ_n against frequency

4.4 Power in periodic signals

Recall that average power of a periodic signal over one period is

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|x(t)|^2 dt$$

It can be shown (Parseval's theorem) that if represent $x(t)$ by complex exponential Fourier series, then can write power as:

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|x(t)|^2 dt = \sum_{n=-\infty}^{\infty}|c_n|^2$$

4. Fourier Analysis and Applications

Example

Find complex exponential Fourier representation of $x(t) = 2\sin(t)\cos(t)$.

4. Fourier Analysis and Applications

4.5 Fourier transform

Conventionally we denote :

- signal in time domain $x(t)$
- signal in frequency domain $X(f)$
- can also write $X(\omega)$ where $\omega = 2\pi f$

The Fourier transform and inverse Fourier transform enable us to pass back and forth between the time and frequency domains:

$$x(t) \underset{\text{inverse Fourier transform}}{\overset{\text{Fourier transform}}{\rightleftarrows}} X(f)$$

The Fourier transform of a signal $x(t)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

and the inverse Fourier transform of $X(f)$ by

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

To be able to find FT of given signal $x(t)$ Dirichlet conditions are sufficient, but not strictly necessary, for example in the case of the unit impulse function:

4. Fourier Analysis and Applications

4.5 Fourier transform (cont)

Example

Find the Fourier transforms of (a) $\delta(t)$ and (b) $\delta(t-a)$.

Fourier transform pairs

Shorthand notation to denote signal in time domain and its Fourier transform – a Fourier transform pair:

$$x(t) \rightleftharpoons X(f)$$

So for the impulse functions in the last example we can write

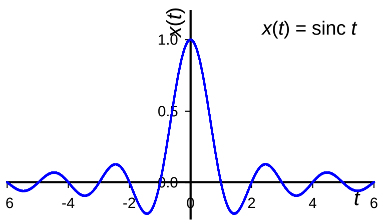
4. Fourier Analysis and Applications

4.5 Fourier transform (cont)

The sinc function

The sinc function is common in signal processing and is defined by

$$\text{sinc}\ t = \frac{\sin \pi t}{\pi t}$$



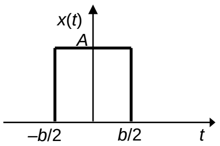
Exercise: show that sinc $t \to 1$ at origin (hint: Taylor expansion)

4. Fourier Analysis and Applications

4.5 Fourier transform (cont)

Further example

Find the Fourier transform of a square pulse :



4. Fourier Analysis and Applications

4.6 Properties of the Fourier transform

Time shifting

- if signal shifted in time domain by a, FT multiplied by $e^{-j2\pi af}$:
- FT[$x(t-a)$] = $e^{-j2\pi af}X(f)$

Frequency shifting

- multiplication of a signal in the time domain by $e^{j2\pi f_0 t} \equiv$ shift by f_0 in the frequency domain
- $e^{j2\pi f_0 t}x(t) \rightleftharpoons X(f-f_0)$

Time scaling

- compression in time of a signal $x(t)$ causes a broadening of frequency of $X(f)$
- broadening in time of a signal $x(t)$ causes a compression of frequency of $X(f)$
- for $x(t) \rightleftharpoons X(f)$, then $x(at) \rightleftharpoons \frac{1}{|a|}X\left(\frac{f}{a}\right)$
- leads to constant time-bandwidth product, see later

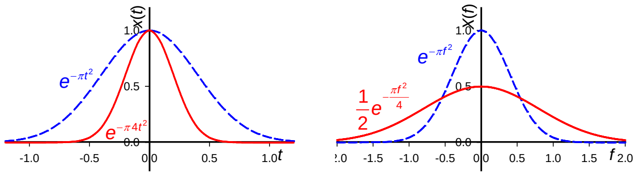
4. Fourier Analysis and Applications

4.6 Properties of the Fourier transform (cont)

Example

Investigate time scaling by finding the FT of $x(t) = e^{-\pi 4t^2}$

Time scaling factor here is two as $x(t) = e^{-\pi 4t^2} = e^{-\pi(2t)^2}$



Using time scaling property we know that since

$$x(t) = e^{-\pi t^2} \xrightarrow{FT} X(f) = e^{-\pi f^2} \Rightarrow x(2t) = e^{-\pi(2t)^2} \xrightarrow{FT} \frac{1}{2}X\left(\frac{f}{2}\right) = \frac{1}{2}e^{-\frac{\pi f^2}{4}}$$

Compression in time has resulted in expansion in frequency

4. Fourier Analysis and Applications

4.6 Properties of the Fourier transform (cont)

Superposition principle

- FT is linear $ax_1(t)+bx_2(t) \rightleftharpoons aX_1(f)+bX_2(f)$

Duality

- useful when computing FTs
- $x(t) \rightleftharpoons X(f) \quad \Rightarrow \quad X(t) \rightleftharpoons x(-f)$

Differentiation and integration

Inverse FT: $x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$

$$\frac{dx}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

$$\frac{dx}{dt} = \int_{-\infty}^{\infty}\frac{d}{dt}X(f)\exp(j2\pi ft)df = j2\pi f\int_{-\infty}^{\infty}X(f)\exp(j2\pi ft)df$$

Thus if $x(t) \rightleftharpoons X(f) \quad \Rightarrow \quad \frac{dx}{dt} \rightleftharpoons j2\pi fX(f)$

If $X(0)=0 \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}X(f)$

- differentiation in t domain ↔ multiplication by $j2\pi f$ in freq. domain
- integration in t domain ↔ division by $j2\pi f$ in freq. domain

4. Fourier Analysis and Applications

4.6 Properties of the Fourier transform (cont)

Convolution

- multiplying two functions together in the time domain results in convolution in the frequency domain
- $x_1(t)x_2(t) \rightleftharpoons \int_{-\infty}^{\infty} X_1(\sigma)X_2(f-\sigma)d\sigma$
- convolution in the time domain translates into multiplication in the frequency domain!
- $\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)d\tau \rightleftharpoons X_1(f)X_2(f)$ **The Convolution Theorem**

Even and odd parts of a function

Suppose

- $x(t)$ real signal decomposed into even / odd parts: $x(t) = x_e(t) + x_o(t)$
- and $X(f) = A(f) + jB(f)$

$\Rightarrow x_e(t) \rightleftharpoons A(f) \qquad x_o(t) \rightleftharpoons jB(f) \qquad X(-f) = X^*(f)$

- real part of the FT and even part of signal constitute a FT pair
- imaginary part of FT and odd part of signal also constitute a FT pair

4. Fourier Analysis and Applications

4.7 Spectrum plots

In general, FT of a signal $x(t)$ is a complex function, so can write $X(f)$ in polar representation, that is:

$X(f) = |X(f)|e^{j\phi}$

where $|X(f)|$ is the amplitude of $X(f)$ and $\phi = \arg(X(f))$ is the phase

- plot of $|X(f)|$ is known as the amplitude spectrum of the signal
- plot of $\phi = \arg(X(f))$ is known as the phase spectrum of the signal

Exercise

Suppose the FT of some signal is

$$X(f) = \frac{1}{2+jf}$$

Plot the amplitude and phase spectra.

4. Fourier Analysis and Applications

4.8 Parseval's theorem

- the energy content of a signal is equivalent to the energy spectral density of the signal, found by integrating $|X(f)|^2$
- $\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}|X(f)|^2 df = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$
- alternative statement: the total average power of a periodic signal is equal to the sum of the average powers in all of its harmonic components (cf Fourier series representation)
- $P_k = \frac{1}{T}\int_0^T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_0^T |a_k|^2 dt = |a_k|^2$
- $P_k = P_{-k}$ so the total power of the kth harmonic components of the signal (i.e. the total power at frequency $k\omega_0$) is $2P_k$
- total average signal power is given in the frequency domain by the Parseval theorem therefore as
- $P = \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2$

4. Fourier Analysis and Applications

4.9 Frequency response

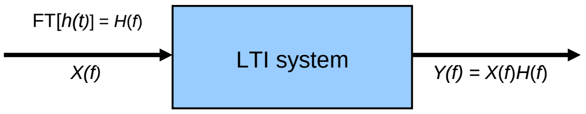
$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)d\tau \rightleftharpoons X_1(f)X_2(f)$ **The Convolution Theorem**

Recall: for continuous LTI system with impulse response $h(t)$, the response $y(t)$ to any input signal $x(t)$ given by convolution:

- $y(t) = x(t)*h(t)$.

Using the convolution theorem we can write

- $Y(f) = X(f)H(f)$
- $H(f)$ is the *frequency response* or transfer function of the system



In general $H(f)$ is complex: $H(f) = |H(f)|e^{j\theta_H}$

- $|H(f)|$ – amplitude response
- $\theta_H(f) = \arg[H(f)]$ – phase response

4. Fourier Analysis and Applications

4.9 Frequency response (cont)

FT of output signal:

$y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df$

$Y(f) = |Y(f)|e^{j\theta_Y}$ $X(f) = |X(f)|e^{j\theta_X}$ $H(f) = |H(f)|e^{j\theta_H}$

$Y(f) = X(f)H(f)$

Amplitude of output signal related to *product* of amplitudes of input signal and frequency response:

- $|Y(f)| = |X(f)||H(f)|$

Phase of output signal related to *sum* of phases of input signal and frequency response:

- $\theta_Y(f) = \theta_X(f) + \theta_H(f)$

4. Fourier Analysis and Applications

4.10 Distortion

Distortionless transmission:

- transmission through an LTI system preserves signal shape
- even if amplitude changed or if there is a time delay.

For arbitrary time delay a distortionless transmission described by

- $y(t) = Kx(t-a)$
- where K is the gain constant.

$\text{FT}[x(t-a)] = e^{-j2\pi fa}X(f)$

Now recall $Y(f) = X(f)H(f)$

$\Rightarrow$ frequency response is $H(f) = Ke^{-j2\pi fa}$

- amplitude constant $|H(f)| = K$
- phase $\theta_H(f) = -2\pi fa$

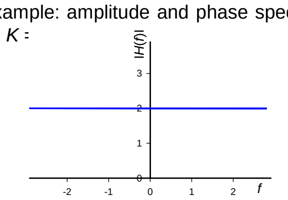
For frequency response of system with distortionless transmission :

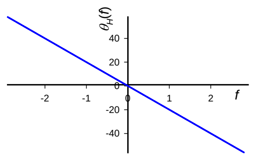
- the amplitude is constant for all frequencies and is equal to the gain constant
- the phase varies linearly with frequency

4. Fourier Analysis and Applications

4.10 Distortion (cont)

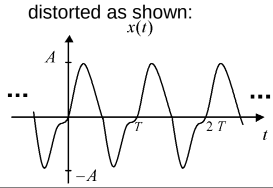
Example: amplitude and phase spectra for distortionless $H(f)$ with gain K :





Total harmonic distortion, THD

Suppose signal was supposed to be pure sine wave amplitude A, but distorted as shown:



Distorted sine wave, e.g. in overloaded line voltage

4. Fourier Analysis and Applications

4.10 Distortion (cont); Total harmonic distortion, THD (cont)

Fourier series representation of periodic signal:

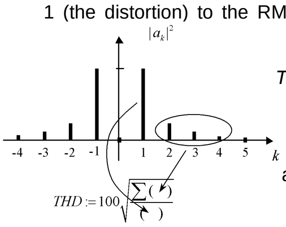
$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2\pi kt}{T_0}\right) + b_k\sin\left(\frac{2\pi kt}{T_0}\right)\right]$$

Start by defining RMS (root mean square) value of a periodic signal:

$X_{RMS} = \sqrt{\frac{1}{T}\int_0^T |x(t)|^2 dt}$ from Parseval theorem $\Rightarrow$ $X_{RMS} = \sqrt{\sum_{k=-\infty}^{\infty}|a_k|^2}$

For real signal with $a_0 = 0$: $X_{RMS} = \sqrt{\sum_{k=1}^{\infty}2|a_k|^2}$

Can define THD as ratio of RMS value for all higher harmonics with $k > 1$ (the distortion) to the RMS value of the fundamental (pure sine



$$THD = 100\frac{\sqrt{\sum_{k=2}^{\infty}2|a_k|^2}}{\sqrt{2|a_1|^2}}\% = 100\sqrt{\frac{\sum_{k=2}^{\infty}|a_k|^2}{|a_1|^2}}\%$$

audio definition: $THD_a = 100\sqrt{\frac{\sum_{k=2}^{\infty}|a_k|^2}{\sum_{k=1}^{\infty}|a_k|^2}}\%$

4. Fourier Analysis and Applications

4.11 Filters

A filter allows a specified range of frequencies to pass through and suppresses all other frequencies.

- band: a given range of frequencies
- passband : band of frequencies allowed through filter
- stopband: band of frequencies suppressed by the filter
- can describe filters by frequency response function $H(f)$

We can group filters into four categories:

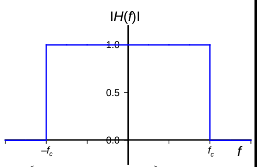
- **Low pass filter**: allows low frequencies to pass through to output, rejects high frequencies
- **High pass filter**: suppresses low frequencies, allows high frequencies to pass through
- **Band pass**: allows a band of frequencies through, rejects frequencies that fall below of above the band
- **Band stop**: opposite of band pass filter, rejects frequencies that fall in a particular range

Ideal filters: transition from stop to pass band is immediate

4. Fourier Analysis and Applications

4.11 Filters (cont)

Ideal low pass filter



- defined by cutoff frequency, f_c.
- allows frequencies to pass if $|f| < f_c$.
- amplitude described by $|H(f)| = \begin{cases} 1 & |f| < f_c \\ 0 & |f| \geq f_c \end{cases}$
- complete frequency response function $H(f) = \begin{cases} e^{-j2\pi t_0 f} & |f| < f_c \\ 0 & |f| \geq f_c \end{cases}$
- bandwidth of filter defined by cutoff frequency, f_c.

Recall rectangular pulse and sinc function constitute FT pair:

$rect\left(\frac{t}{T}\right) \rightleftharpoons T\mathrm{sinc}(Tf)$ where $-T/2 \leq t \leq T/2$

- use duality property to find inv. FT of rectangle in frequency domain:

$rect\left(\frac{f}{2f_c}\right) \rightleftharpoons 2f_c \mathrm{sinc}(2f_c t)$

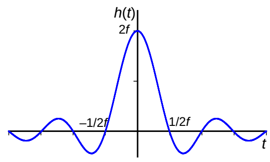
- gives impulse response function for ideal low pass filter, a sinc function

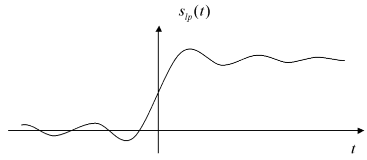
4. Fourier Analysis and Applications

4.11 Filters (cont)

Ideal low pass filter (cont)

Impulse response for ideal low pass filter (l) step response (r)





Drawbacks to ideal low pass filter

- it is non-causal as has elements to impulse response at $t < 0$
- 'ripples' (=small oscillations) in time domain impulse response may have undesirable consequences
- a practical low pass filter is the Butterworth filter

4. Fourier Analysis and Applications

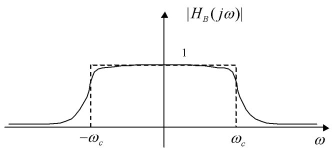
4.11 Filters (cont)

Butterworth filter

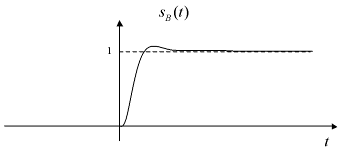
Butterworth filter has a frequency response (for Nth order – diagram shows magnitude of frequency response for second order):

$$|H_B(j\omega)| = 1\Big/\left(1+\left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}$$

Transition band is frequency band around ω_c where the magnitude rolls off (=drops)



- the higher the order, the narrower the transition band
- the step response shows far less oscillation than the ideal filter, even for $N = 2$:



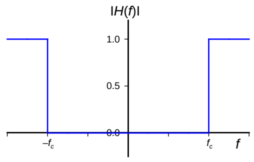
4. Fourier Analysis and Applications

4.11 Filters (cont)

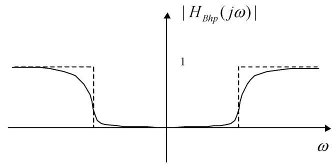
Ideal high pass filter

- block frequencies $|f| < f_c$, allow frequencies $|f| > f_c$ to pass through
- amplitude spectrum:
- formally we write:

$$|H(f)| = \begin{cases} 0 & |f| \leq f_c \\ 1 & |f| > f_c \end{cases}$$



- realistic version may be accomplished using inverted second order Butterworth filter

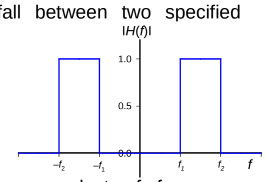


4. Fourier Analysis and Applications

4.11 Filters (cont)

Ideal band pass filter

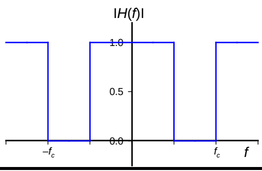
- allows frequencies to pass if they fall between two specified frequencies f_1 and f_2 (where $f_1 < f_2$)
- $|H(f)|$ defined by $|H(f)| = \begin{cases} 1 & f_1 \leq |f| \leq f_2 \\ 0 & \text{otherwise} \end{cases}$
- amplitude spectrum:



- frequency response can be written as product of frequency responses of overlapping lowpass and highpass filters

Ideal band stop filter

- rejects a band of frequencies
- $|H(f)|$ defined by $|H(f)| = \begin{cases} 0 & f_1 < |f| < f_2 \\ 1 & \text{otherwise} \end{cases}$
- amplitude spectrum:



4. Fourier Analysis and Applications

4.11 Filters (cont)

System bandwidth

Bandwidth of ideal bandpass filter is difference between the two cutoff frequencies used to define the pass band (ω_1 and ω_2 with $\omega_1 < \omega_2$).

- bandwidth B is given by $B = \omega_1 - \omega_2$
- midband frequency is $\omega_0 = (\omega_1 + \omega_2)/2$
- if $B << \omega_0$ we say that the filter is a narrowband filter

For non-ideal filters owing to the transition band it is not possible to define the bandwidth so simply.

- instead, define 3-dB bandwidth
- find frequency ω_B at which $|H(\omega)| = \frac{|H(\omega_0)|}{\sqrt{2}}$
- if frequency at which this condition is met ω_B then 3-dB bandwidth given by $\omega_B - \omega_0$

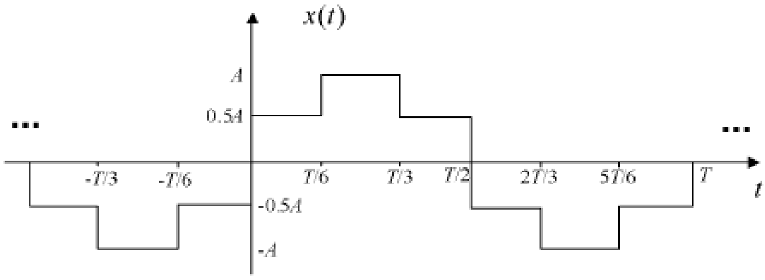
Fourier analysis and applications

Fourier series

1. Find the complex exponential Fourier series representation of $x(t) = \frac{4}{\pi}(\sin t + \sin 3t)$.

2. Digital Sine Wave Generator (long!)

A programmable digital signal generator generates a sinusoidal waveform by filtering the staircase approximation to a sine wave shown in the figure:



(a) Find the complex Fourier series coefficients c_k of the periodic signal $x(t)$. Show that the even harmonics vanish. Express $x(t)$ as a Fourier series.

(b) Write $x(t)$ using the real form of the Fourier series.

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right] \equiv a_0 + 2\sum_{k=1}^{\infty}\left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right]$$

(c) Design an ideal lowpass filter that will produce the perfect sinusoidal waveform $y(t) = \sin\left(\frac{2\pi}{T}t\right)$ at its output with $x(t)$ as its input. Sketch its frequency response and specify its gain K and cutoff frequency ω_c.

Fourier transforms

3. Sketch the following signal and find its Fourier transform: $x(t) = (1 - e^{-|t|})[u(t + 1) - u(t - 1)]$. Show that $X(j\omega)$ is real and even.

4. Show that the Fourier transform of a Gaussian pulse in the time domain $x(t) = e^{-\pi t^2}$ is a Gaussian pulse in the frequency domain.

Filters

5. Consider the periodic function of the example in section 4 of the lecture notes where $x(t) = t^2$, $-1 < t < 1$ was extended on the real line. Suppose this signal is passed through an ideal low pass filter described by

$|H(\omega)| = \begin{cases} 1 & |\omega| \le 3\pi \\ 0 & |\omega| > 3\pi \end{cases}$. Find the output signal $y(t)$ for this filter. Consider what happens if the cutoff frequency is reduced to $\omega_c = 2\pi$.

Hint: in the lectures, we found that the Fourier series expansion of the input signal was given by

$x(t) = \frac{1}{3} - \frac{4}{\pi^2}\cos \pi t + \frac{1}{\pi^2}\cos 2\pi t - \frac{4}{9\pi^2}\cos 3\pi t + \ldots$. If you can, plot the original function, and the various unfiltered and filtered Fourier series representations to compare them.