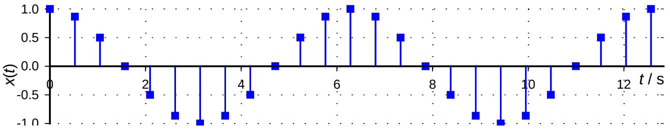
3. Discrete time signals

3.1 Introduction

In almost all physics experiments, analogue signal will be sampled and digitized, transforming continuous signal into discrete time signal.

periodic discrete time signal: $x[n] = \cos(nT_s)$ $T_s = \pi / 6$



- sampling interval: T_s
- discrete time signal $x[n] = x(nT_s)$ (n is integer) – a sequence $\{x_n\}$

For two discrete time signals $\{x_n\}$, $\{y_n\}$

- $\{z_n\} = \{x_n\} + \{y_n\} \Rightarrow z[n] = x[n] + y[n]$
- $\{z_n\} = \{x_n\}\{y_n\} \Rightarrow z[n] = x[n]\, y[n]$
- $\{z_n\} = \alpha\{x_n\} \Rightarrow z[n] = \alpha x[n]$

To plot, draw point and draw vertical line from time axis as above

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3.2 Energy and power in discrete signals

Normalised energy content: $E = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$

Normalised average power: $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2$

- $0 < E < \infty$, $P = 0 \Rightarrow x[n]$ is an energy signal
- $0 < P < \infty$, $E = \infty \Rightarrow x[n]$ is a power signal

Example: Find the energy content and average power of the following signal and determine whether it is an energy signal or a power signal:

$$\{x_n\} = \left(\frac{1}{2}\right)^n \quad n \ge 0$$

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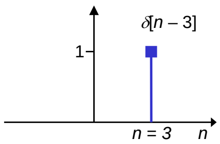
3.3 Unit impulse sequence

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases}$$

can shift the unit impulse sequence by integer k

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \ne k \end{cases}$$

In the figure we show $\delta[n-3]$



Satisfies the sampling property so $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

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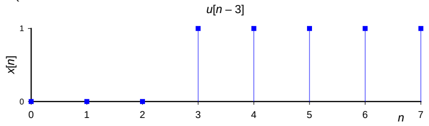
3.4 Unit step sequence

$$u[n] = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}$$

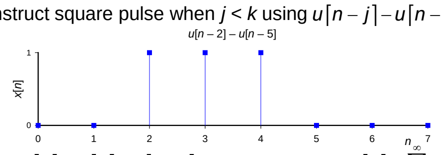
a sequence of unit pulses starting at zero

$$u[n-k] = \begin{cases} 1 & n \ge k \\ 0 & n < k \end{cases}$$

shifted as illustrated for $k = 3$:



Can construct square pulse when $j < k$ using $u[n-j] - u[n-k]$ e.g.



Can define $\delta[n] = u[n] - u[n-1]$ and, conversely, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

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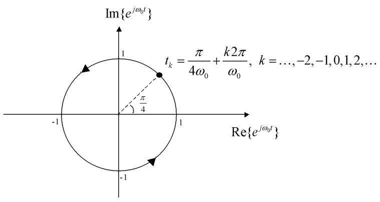
3.5 Periodic discrete time signals

Continuous-time sinusoid: $x(t) = \sin(\omega t)$, $-\infty < t < \infty$

- periodic for any frequency ω
- increasing ω results in faster oscillations

Continuous-time phasor: $x(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, $-\infty < t < \infty$

- unit-length vector in complex plane that rotates as t increases
- anticlockwise for $\omega > 0$, clockwise for $\omega < 0$
- periodic for all values of ω
- fundamental period $T = 2\pi / |\omega|$ decreases as $|\omega|$ increases
- projections of this vector on real and imaginary axes yield $\cos(\omega t)$ and $\sin(\omega t)$
- representation:



3. Discrete time signals

3.5 Periodic discrete time signals (cont)

Discrete time phasor: $x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$, $-\infty < n < \infty$

- only periodic if ω is a rational multiple of 2π
- that is for integer m and positive integer N, $\omega = (m / N)\, 2\pi$
- fundamental period N_0 : least N, such that $x[n + N] = x[n]$ for all n
- fundamental frequency $\omega_0 = 2\pi / N_0$ rad / sample
- may not necessarily rotate "faster" and N decrease as frequency increases
- changing frequency by 2π does not change signal
- follows from $e^{j(\omega \pm 2\pi)n} = e^{j\omega n} e^{\pm j 2\pi n} = e^{j\omega n}$
- see http://www.jhu.edu/~signals/dtphasor/index.htm for demo

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3.6 Properties of discrete time signals

As before: linearity, memory, causality, stability and time invariance.

Linearity: check that $\hat{T}\{ax_1[n]+bx_2[n]\}=a\hat{T}\{x_1[n]\}+b\hat{T}\{x_2[n]\}$ is satisfied.

Example: Determine whether $y[n] = x[n]\, u[n-1]$ is **linear**

Example: Determine whether $y[n] = x[n-2]$ is **memoryless** and **causal**

Stability: if $|x[n]| \le k_1 \;\forall n \;\Rightarrow\; |y[n]| \le k_2$ then system is BIBO stable.

Example: is $y[n] = 2nx[n]$ **stable** and **time invariant**?

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3.7 Discrete linear time-invariant systems

- impulse response $h[n] = \mathbf{T}\{\delta[n]\}$
- system is time invariant so can write $h[n-k] = \mathbf{T}\{\delta[n-k]\}$
- remember $x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- so response to arbitrary input $x[n]$ is given by: $y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- this is discrete version of convolution $y[n] = x[n] * h[n]$
- commutative, associative and distributive properties apply
- since convolution is commutative we can also write $y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$

Procedure for discrete time convolution:

- compute signals $x[k]$ and $h[n-k]$ as functions of k
- multiply them at each k
- sum all these values to yield output signal
- alternatively, use $h[k]$ and $x[n-k]$

Best seen by example (TD).

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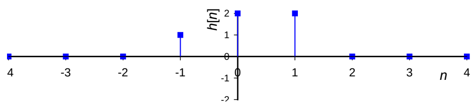
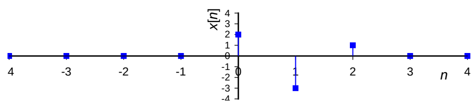
3.7 Discrete linear time-invariant systems (cont)

Additional example

Compute $y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$

for the following impulse response and input signal:





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3.7 Discrete linear time-invariant systems (cont)

Memory

- memoryless discrete LTI system has form $y[n] = Kx[n]$ where K is gain constant
- impulse response takes form $h[n] = K\delta[n]$
- system is memoryless if $h[n] = 0$ when $n \neq 0$ otherwise system has memory

Causality

- memoryless system is causal
- more generally a discrete LTI system is causal if $h[n] = 0$ for $n < 0$

Stability

- discrete LTI system is BIBO stable if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Step response

- step response for a discrete LTI system given by $s[n] = h[n] * u[n]$

2. Linear time-invariant systems

2.12 Convolution (contd)

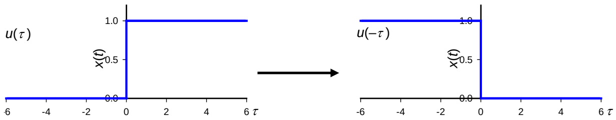
Example: given $x(t) = u(t)$ and $h(t) = \cos(\pi t)\, u(t)$, find the response $y(t)$

Note first that $h(t) = 0$ for $t < 0$ so system is causal, so can apply

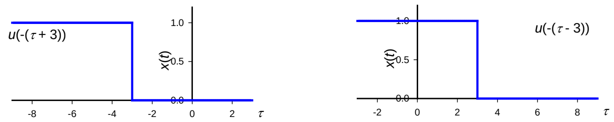
$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau=\int_0^{\infty}\cos(\pi\tau)u(t-\tau)d\tau$$

Rely on graphical approach to find limits of integration:

Step 1: reflect $x(\tau) = u(\tau)$ about vertical axis to give $u(-\tau)$



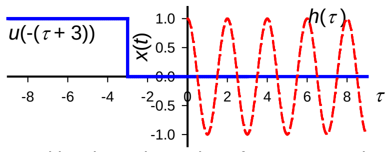
Step 2: shift to left by |t| for $t < 0$, to right by |t| for $t > 0$ (*e.g.* here $t = 3$)



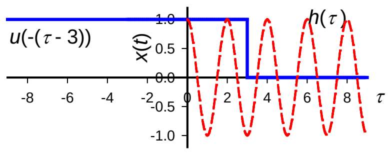
2. Linear time-invariant systems

2.12 Convolution (contd)

For negative times, no overlap as $h(t) = \cos(\pi t)\, u(t)$ is zero for $t < 0$



For positive times the region of nonzero overlap is $0 \le \tau \le t$



Therefore system response is:

$$y(t)=\int_0^t \cos(\pi\tau)d\tau=\left[\frac{1}{\pi}\sin(\pi\tau)\right]_0^t=\frac{1}{\pi}\sin(\pi t)$$