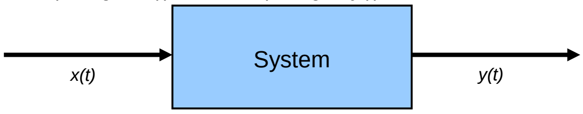
2. Linear time-invariant systems

2.1 Introduction to systems

system: a mathematical model that represents the transformation of some input signal *x*(*t*) into an output signal *y*(*t*)



Represent system by a *transformation* or an *operator* $\hat{T}$ and then the action of a system on a signal: $y(t) = \hat{T}\{x(t)\}$

Examples: A resistor transforms current signal into voltage: $\hat{T} \equiv R$

$x(t) \equiv i(t)$, $y(t) \equiv v(t)$, and the system relationship is $y(t) = R\,x(t)$

Slightly more complex for a capacitor $i(t) = C\dfrac{dv}{dt}$: $\hat{T} = C\dfrac{d}{dt}$

A **continuous time system**: both *x*(*t*) and *y*(*t*) are continuous time signals.

Also possible to have discrete time systems.

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2.2 Memory

memoryless system: output depends only on input *at same time*, e.g. $y(t) = \alpha x(t)$ where $\alpha \in \mathbb{R}$

system with **memory**: output depends on the values of input at previous times

Example: determine if the following systems are memoryless

a) $y(t) = \sin t \cos t$

b) $y(t) = \int_{-\infty}^{t/3} x(\tau)\, d\tau$ for some general input function $x(t)$

c) $y(t) = \int_{t_0}^{t} x(\tau)\, d\tau$, consider $x(t) = te^{-t}$

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2.3 Causal and non-causal systems

Causal system: output *y*(*t*) depends only on the input at present or earlier times

- output does not anticipate future values of the input
- any real time-dependant system is causal (laws of Physics!)

Example of **non-causal system**: $y(t) = Cx(t + a)$ where $a \in \mathbb{R}$

2.4 Linear systems

Suppose operator acts on two input signals to produce output signals:

$\hat{T}\{x_1(t)\} = y_1(t)$ and $\hat{T}\{x_2(t)\} = y_2(t)$

Transformation is linear if for two constants *a, b*

$\hat{T}\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t)$

Linear system: system represented by a linear transformation.

To determine if a system is linear:

- consider 2 i/o relationships $y_1(t)$, $y_2(t)$ and form sum $ay_1(t) + by_2(t)$
- construct $T\{ax_1(t) + bx_2(t)\}$ – if equal to $ay_1(t) + by_2(t)$ for scalars *a, b* then system is linear

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2.3 Linear systems (contd)

Example: determine if the following system is linear: $y(t) = \dfrac{d^2x}{dt^2}$

2.5 Time invariance

If time-shift of input signal: $x(t) \to x(t \pm \tau)$ causes same time-shift in output signal, system is *time-invariant*.

- if linear system, then called *linear time-invariant system* or LTI

Can write $y_\tau(t) = \hat{T}\{x(t-\tau)\}$

If $y_\tau(t) = y(t-\tau)$ then system is time-invariant

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2.6 System stability

Signal *x*(*t*) is *bounded* if can find constant α such that for all *t,* $|x(t)| \le \alpha$

- If output signal *y*(*t*) is also bounded
- i.e. given $y(t) = T\{x(t)\}$ and some constant β, $|y(t)| \le \beta$ then
- system is *bounded-input, bounded-output stable* or BIBO

Example (refer to TD for more)

Determine if the following system is memoryless, causal, stable, and time-invariant: $y(t) = \sin[x(t)]$

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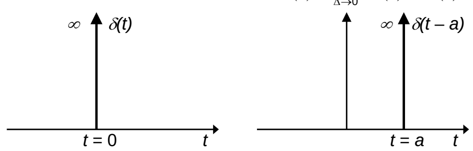
2.7 Unit impulse function

Dirac delta function, defined as $\delta(t) = \lim_{\Delta\to 0} \delta_\Delta(t) : \delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$



Time shift: e.g. $t \to t + a$

Area under curve = 1: $\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\infty}^{\infty} \delta(t \pm a)dt = 1$

Sampling property: use to pick out value at given time:

$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$ $\int_{-\infty}^{\infty} \phi(t)\delta(t-a)dt = \phi(a)$

Further useful properties of unit impulse function:

$\delta(at) = \dfrac{1}{|a|}\delta(t)$ $\delta(-t) = \delta(t)$

Any continuous time signal can be written: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

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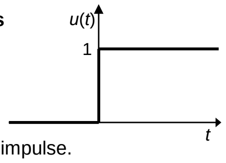
2.8 Unit step function

Defined as $u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$

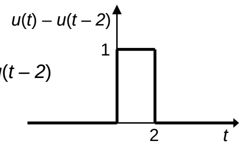


Time shifting in same way as unit impulse.

Change limits of integration:

$\int_{-\infty}^{\infty}u(t)x(t)dt=\int_{0}^{\infty}x(t)dt$ or $\int_{-\infty}^{\infty}u(t-3)x(t)dt=\int_{3}^{\infty}x(t)dt$

Square pulse: e.g. $u(t)-u(t-2)$



Simplification of integrals: e.g.

$\int_{-\infty}^{\infty}\left[u(t-1)-u(t-2)\right]\cos t\ dt=\int_{2}^{1}\cos t\ dt=\sin 2-\sin 1$

Unit impulse and unit step functions related:

$\delta(t)=\frac{du(t)}{dt}$ and $u(t)=\int_{-\infty}^{t}\delta(t)\ dt$

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2.9 Impulse response of an LTI

Impulse response $h(t)=\hat{T}\{\delta(t)\}$

Use to determine system response to arbitrary input (*linear* system)

$y(t)=\hat{T}\{x(t)\}=\hat{T}\left\{\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau\right\}=\int_{-\infty}^{\infty}x(\tau)\hat{T}\{\delta(t-\tau)\}d\tau$

Time invariant system $\Rightarrow y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$

convolution of input signal with impulse response $h(t)$

2.10 System step response

System step response $s(t)=\hat{T}\{u(t)\}$

Determine by *convolution*: $s(t)=u(t)*h(t)=\int_{-\infty}^{\infty}u(\tau)h(t-\tau)d\tau$

Convolution commutative (see on) so can write

$s(t)=h(t)*u(t)=\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}h(\tau)d\tau$

If know step response of system can find impulse response : $h(t)=\frac{ds}{dt}$

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2.11 Impulse response and system properties

Use of impulse response to determine system properties

Memory: a *memoryless* system has $y(t) = Kx(t)$ (K is the gain constant)

impulse response $h(t) = K\delta(t)$

A system *has memory* if $h(t) \neq 0$ for $t \neq 0$

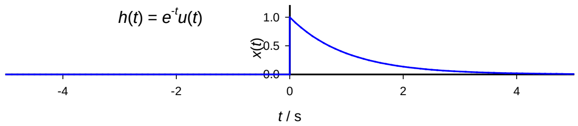
Causality: for causal *system*, $h(t) = 0$ when $t < 0$. Can write

$y(t)=\int_{0}^{\infty}h(\tau)x(t-\tau)d\tau$

Causal *signal*: $x(t) = 0$ for $t < 0$; Anticausal signal: $x(t) = 0$ for $t > 0$

Stability: an LTI system is stable if $\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$

Example: An LTI system has impulse response function $h(t) = e^{-t}u(t)$. Is the system memoryless? Causal? Stable?



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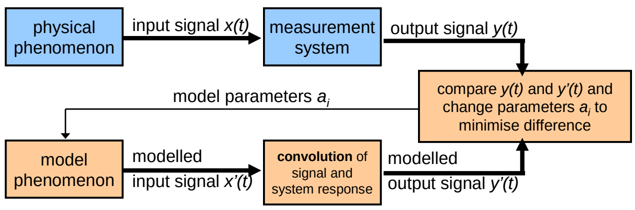
2.12 Convolution

Very important for data analysis

- enables simulation of effect of instrument function on signal
- this can be used to fit data to physical model by 'forward convolution'



- *deconvolution* can be used (see later) to recover original signal

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2.12 Convolution (contd)

Convolution of two functions $f(t)$ and $g(t)$: $f(t)*g(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau$

The convolution operation is

- commutative: $f(t) * g(t) = g(t) * f(t)$
- associative: $[f(t) * g(t)] * w(t) = f(t) * [g(t) * w(t)]$
- distributive: $f(t) * [g(t) + w(t)] = f(t) * g(t) + f(t) * w(t)$
- commutative with respect to multiplication by a scalar:
  $[\alpha f(t)] * g(t) = f(t) * [\alpha g(t)] = \alpha[f(t) * g(t)]$

Finally, convolution of any signal with a unit impulse leaves the signal unchanged: $f(t) * \delta(t) = f(t)$

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2.12 Convolution (contd)

Calculation of the convolution integral

First, obtain signal $h(t - \tau)$ as function of τ, then multiply by $x(\tau)$ to obtain another function $g(\tau)$ then integrate $g(\tau)$ to get $y(t)$

Step 1: sketch the time-reversed impulse response $h(-\tau)$

Step 2: shift this new function to the right by t (time delay) for $t > 0$ to obtain $h(-(\tau - t)) = h(t - \tau)$, or to the left by $|t|$ (time advance) for $t < 0$ to obtain $h(-(\tau + |t|)) = h(t - \tau)$

Note: convolution is commutative so sometimes easier to work with $h(\tau)$ and $x(t - \tau)$ instead of $x(\tau)$ and $h(t - \tau)$