1. Let $x(t) = A \cos \omega t$ where A is a positive real constant. Find

a. the signal energy over one period

b. the average power of the signal

a. period is given by $T_0 = \frac{2\pi}{\omega}$ and so energy over one period is:

$$E_0 = \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt = \int_{-T_0/2}^{T_0/2} |A\cos\omega t|^2 \, dt = A^2 \int_{-T_0/2}^{T_0/2} \cos^2(\omega t)\, dt$$

$$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$$

$$E_0 = \frac{A^2}{2}\int_{-T_0/2}^{T_0/2}\left[1+\cos(2\omega t)\right]dt = \frac{A^2}{2}\int_{-T_0/2}^{T_0/2} dt + \frac{A^2}{2}\int_{-T_0/2}^{T_0/2}\cos(2\omega t)\,dt = \frac{A^2}{2}T_0 + 0 = \frac{A^2}{2}T_0$$

b. $P_0 = \frac{E_0}{T_0} = \frac{A^2}{2}T_0 \Big/ T_0 = \frac{A^2}{2}$

2. Determine if each of the following systems are memoryless, causal, stable, and time-invariant :

a. $y(t) = x(t-6)$

b. $y(t) = dx/dt$

c. $y(t) = \int_{-\infty}^{t/3} x(s)\, ds$

a. $y(t) = x(t-6)$

Not memoryless as depends on values other than at present time.

Causal, as only depends on previous values of t.

Stable: if $|x(t)| \le \alpha, |y(t)| \le \alpha$

Time invariant: Time-shifted output is $y(t-\tau) = x(t-\tau-6)$

Time shifted input $x_\tau(t) = x(t-\tau)$

Obviously, $x_\tau(t) = \hat{T}\{x_\tau(t)\} = \hat{T}\{x(t-\tau)\} = x(t-\tau-6) = y(t-\tau)$

So, the system is time invariant.

b. The system $y(t) = dx/dt$

Is not memoryless as derivative cannot be determined from a single point

Is causal: output does not anticipate future values of input

Is not necessarily stable consider $x(t) = (2-t)^{2/3}$ defined for $0 \le t \le 2$

$\frac{dx}{dt} = \frac{2t}{3(2-t)^{1/3}}$. As $t \to 2$ $x(t) \to 0$ but $y(t) \to \infty$

Time invariance ?

Time-shifted output is $y(t-\tau) = \frac{dx(t-\tau)}{d(t-\tau)}$

Transformation of time shifted input, where for clarity we let $u = t - \tau$

$$\hat{T}\{x(t-\tau)\} = \frac{d}{dt}\left[x(t-\tau)\right] = \frac{du}{du}\frac{dx(t-\tau)}{dt} = \frac{du}{dt}\frac{dx(t-\tau)}{du}$$

But we have $\frac{du}{dt} = \frac{d(t-\tau)}{dt} = 1$

So, $\hat{T}\{x(t-\tau)\} = \frac{dx(t-\tau)}{d(t-\tau)} = y(t-\tau)$

and the system is time invariant.

c. $y(t) = \int_{-\infty}^{t/3} x(s)\,ds$

Not memoryless, depends on all previous times

Causal, as only depends on times up to *t*/3, < *t*.

Stability: take as example input signal given by unit step function *x*(*t*) = *u*(*t*). Then integral becomes

$$y(t) = \int_{-\infty}^{t/3} x(s)\,ds = \int_{-\infty}^{t/3} u(s)\,ds = \int_{0}^{t/3} ds = \frac{t}{3}$$

This output grows without bound, even though *x*(*t*) = 1 for all positive times. So the system is **not stable**.

Time invariance:

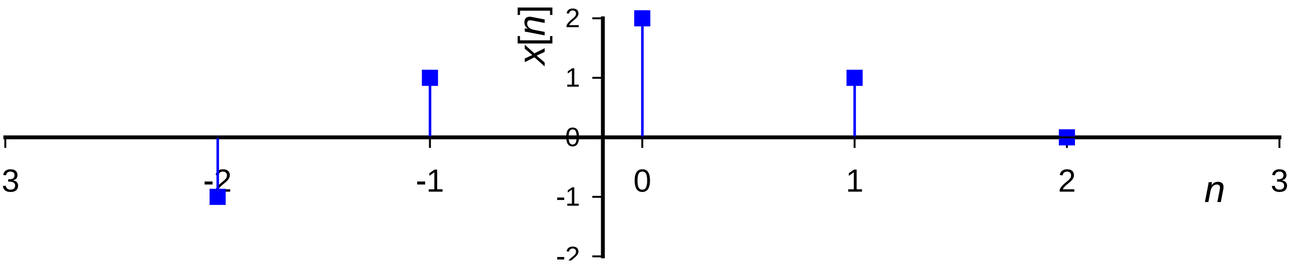
$$y(t-\tau) = \int_{-\infty}^{(t-\tau)/3} x(s)\,ds$$

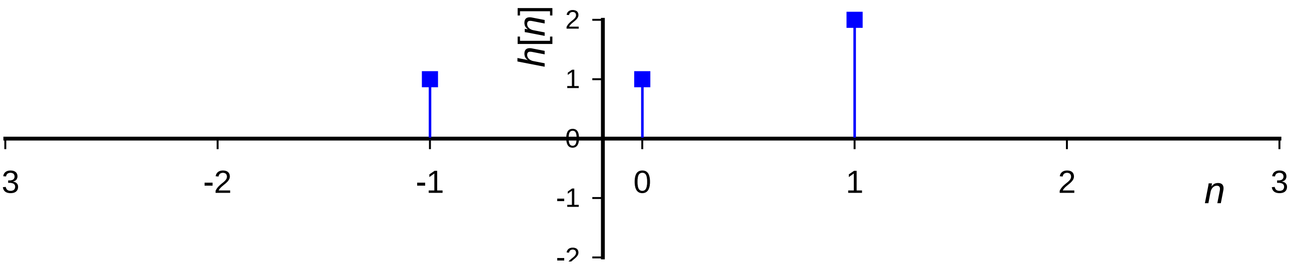
Let $u = (t-\tau)$. For an input $x(t-\tau)$ system produces output as follows:

$$\hat{T}\{x(t-\tau)\} = \int_{-\infty}^{t/3} x(s-\tau)\,ds = \int_{-\infty}^{t/3-\tau} x(u)\,du = \int_{-\infty}^{(t-3\tau)/3} x(u)\,du = y(t-3\tau) \neq y(t-\tau)$$

Thus the system is **not time-invariant**.

3. Find *y*[*n*] = *x*[*n*] ∗ *h*[*n*] for the signals shown below



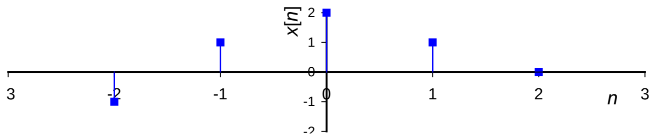


Let us do this simply and graphically. Remember that

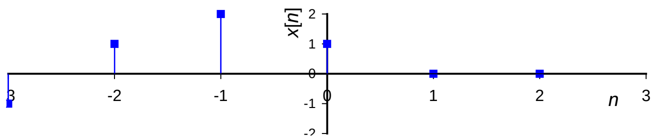
$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

We will pick the first of the two, as there are fewer terms:

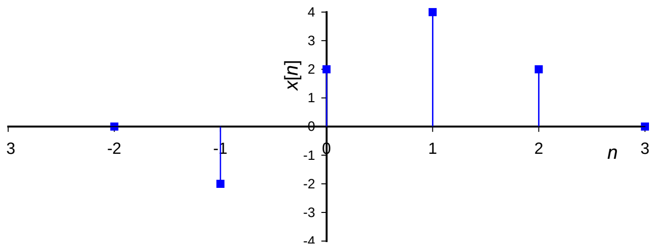
k = 0, $h[0]\,x[n] = x[n]$:



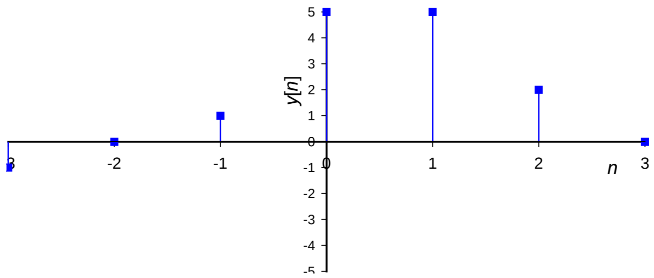
k = –1, $h[-1]\,x[n+1] = x[n+1]$:



k = 1, $h[1]\,x[n-1] = 2x[n-1]$:



Sum :



{–1 0 1 5 5 2 0}

4. Compute the autocorrelation function for $x(t)=e^{-at}u(t)$. Assume that $\tau > 0$. Find the energy content of the above signal by considering the energy spectral density.

(Note that $FT\left[e^{-a\tau}u(\tau)\right]=\dfrac{1}{a+j\omega}$ and $\int\dfrac{1}{1+x^2}dx=\arctan x$)

$R_{11}(\tau)=\int_{-\infty}^{\infty} x_1(t)^* x_1(t+\tau)\,dt \equiv \int_{-\infty}^{\infty} x_1(t)^* x_1(t-\tau)\,dt$ (autocorrelation is even function)

$R_{11}(\tau)=\int_{-\infty}^{\infty}\left[e^{-at}u(t)\right]e^{-a(t-\tau)}u(t-\tau)\,dt$

To simplify integral, recall $u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$. Since τ > 0 the other unit step fn is $u(t-\tau)=\begin{cases}1 & t\geq \tau\\ 0 & t<\tau\end{cases}$.

Thus, product us zero for $t<\tau$, that is $u(t)u(t-\tau)=\begin{cases}1 & t\geq \tau\\ 0 & t<\tau\end{cases}$. Thus can rewrite the autocorrelation fn:

$$R_{11}(\tau)=\int_{\tau}^{\infty} e^{-at}e^{-a(t-\tau)}dt = e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt = -\frac{e^{a\tau}}{2a}\left[e^{-2at}\right]_{\tau}^{\infty} = -\frac{e^{a\tau}}{2a}e^{-2a\tau} = \frac{e^{-a\tau}}{2a}$$

We have $R_{11}(\tau)=\frac{e^{-a\tau}}{2a}u(\tau)$. The unit step function is included to remind us that $\tau > 0$.

The energy spectral density is given by

$S_{11}(\omega)=\left|X(\omega)\right|^2=\left|FT\left[x(t)\right]\right|^2=\left|\frac{1}{a+j\omega}\right|^2=\left(\frac{1}{a+j\omega}\right)\left(\frac{1}{a-j\omega}\right)=\frac{1}{a^2+\omega^2}$. Now, we seek the energy content given by

$E=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{11}(\omega)\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}\,d\omega=\frac{1}{2\pi a^2}\int_{-\infty}^{\infty}\frac{1}{1+(\omega/a)^2}\,d\omega=\frac{1}{2\pi a}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$. Recall that

$\int\frac{1}{1+x^2}\,dx=\arctan x$ and so $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx=\left[\arctan x\right]_{-\infty}^{\infty}=\pi \Rightarrow E=\frac{1}{2\pi a}\pi=\frac{1}{2a}$.

Note that this agrees with the standard evaluation of the energy content:

$$E=\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt=\int_{-\infty}^{\infty}\left|e^{-at}u(t)\right|^2 dt=\int_{0}^{\infty} e^{-2at}dt=\frac{-1}{2a}\left[e^{-2at}\right]_{0}^{\infty}=\frac{1}{2a}$$

thus demonstrating Parseval's theorem.

5. Find the discrete Fourier transform of the signal $x[n]$ = {2, 0, –1, 3}

For the first term we have

$$X[0] = \sum_{n=0}^{3} x[n] = 2+0-1+3=4$$

and for the subsequent terms:

$$X[1] = \sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = x[0]+x[1]e^{-j\frac{\pi}{2}}+x[2]e^{-j\pi}+x[3]e^{-j\frac{3\pi}{2}} = 2+0+1+3j=3+3j$$

$$X[2] = \sum_{n=0}^{3} x[n]e^{-j\pi n} = x[0]+x[1]e^{-j\pi}+x[2]e^{-j2\pi}+x[3]e^{-j3\pi} = 2+0-1-3=-2$$

$$X[3] = \sum_{n=0}^{3} x[n]e^{-j\frac{3\pi}{2}n} = x[0]+x[1]e^{-j\frac{3\pi}{2}}+x[2]e^{-j3\pi}+x[3]e^{-j\frac{9\pi}{2}} = 2+0+1-3j=3-3j$$

So, the DFT is given by $X[n]=\{4,3+3j,-2,3-3j\}$.

6. Consider the demodulation of a DSB signal $x(t)=m(t)\cos(\omega_c t)\cos(\omega_c t+\pi/2)$. What is the effect of the phase error on the output?

$$x(t)=m(t)\cos(\omega_c t)\cos(\omega_c t+\pi/2)=-m(t)\cos(\omega_c t)\sin(\omega_c t)=-\frac{1}{2}m(t)\sin(2\omega_c t)$$

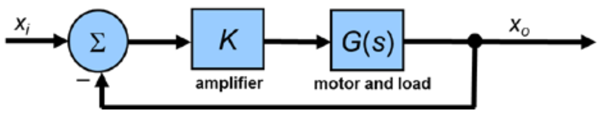
Demodulation involves low pass filtering which removes all components with $\omega \geq \omega_c$ and thus the output $y(t) = 0$.

7. Describe the working of the *lock-in amplifier*, and explain how it can amplify very weak low frequency signals in the presence of typical background noise sources.

See course notes.

8.

A feedback motor control system may be represented by the block diagram shown on the right.



a) Explain how this system acts to make the output position x_o follow the requested input position x_i.

Let input to amplifier be z. $z = x_i - x_o$

z is amplified and applied to motor, which moves load until $x_i = x_o$ at which point $z = 0$ and the motor stops. If x_i is changed, the system will move until again $x_i = x_o$.

b) Show that the system transfer function $H(s)=KG(s)/[1+KG(s)]$.

Let $x_i(t)\leftrightarrow X_i(s)$ $\quad x_o(t)\leftrightarrow X_o(s)$ $\quad z(t)\leftrightarrow Z(s)$

$$e(t)=x_i(t)-x_o(t)\Rightarrow Z(s)=X_i(s)-X_o(s)$$

$$X_o(s)=Z(s)KG(s)=[X_i(s)-X_o(s)]KG(s)\Rightarrow X_o(s)[1+KG(s)]=X_i(s)KG(s)$$

$$H(s)=\frac{X_o(s)}{X_i(s)}=\frac{KG(s)}{1+KG(s)}$$

c) If $G(s)=1/s(s+4)$ and $K = 3$ determine the time dependent output to a step function input $x_i = u(t)$.

$$G(s)=\frac{1}{s(s+4)} \quad \Rightarrow H(s)=\frac{\frac{K}{s(s+4)}}{1+\frac{K}{s(s+4)}}=\frac{\frac{K}{s(s+4)}}{\frac{s(s+4)+K}{s(s+4)}}=\frac{K}{s(s+4)+K}=\frac{K}{s^2+4s+K}$$

$$x(t)=u(t)\Rightarrow X(s)=\frac{1}{s}$$

$$K=3: Y(s)=X(s)H(s)=\frac{1}{s}H(s)=\frac{3}{s(s^2+4s+3)}=\frac{3}{s(s+3)(s+1)}=\frac{1}{s}-\frac{\frac{3}{2}}{(s+1)}+\frac{\frac{1}{2}}{(s+3)}$$

$$y(t)=x_o(t)=\left(1-\tfrac{3}{2}e^{-t}+\tfrac{1}{2}e^{-3t}\right)u(t)$$

d) Is the system under-damped or over-damped? For what value of K would it be critically damped?

Critical value of K when $"b^2-4ac"=0$, i.e. when $4^2-4K=0\Rightarrow K=4$

So, system is OVERdamped.

NB $\dfrac{3}{s(s+3)(s+1)}=\dfrac{1}{s}-\dfrac{\frac{3}{2}}{(s+1)}+\dfrac{\frac{1}{2}}{(s+3)}$