

# Morales-Ramis Theorems *via* Malgrange pseudogroup

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*À Bernard Malgrange pour son quatre-vingtième anniversaire.*

## Résumé

Dans cet article nous montrons que les équations variationnelles le long d'une solution d'une équation différentielle intégrable par quadrature ont un groupe de Galois différentielle virtuellement résoluble. Dans le cas particulier des systèmes hamiltoniens intégrables au sens de Liouville la preuve redonne le théorème de Morales-Ramis-Simó. La preuve consiste à montrer que le groupe de Galois de l'équation variationnelle est un quotient d'un sous groupe d'un groupe d'isotropie du pseudogroupe de Malgrange de l'équation non linéaire. On relie ensuite les propriétés de ce groupe d'isotropie en un point spécial à celles du groupe d'isotropie au point générique en utilisant le théorème d'approximation d'Artin.

## Abstract

In this article we give an obstruction to integrability by quadrature of an ordinary differential equation on the differential Galois group of variational equations of any order along a particular solution. In Hamiltonian situation the condition on the Galois group gives Morales-Ramis-Simó Theorem. The main tools used are Malgrange pseudogroup of a vector field and Artin approximation Theorem.

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## Introduction

Morales-Ramis Theorems give conditions for integrability in sense of Liouville of a Hamiltonian system in term of the differential Galois group of the linearized system along a particular solution. First Theorem of this kind was obtained by Ziglin [28] in term of monodromy of the variational equation. Later this condition was translated in term of differential Galois group. Following previous work of Churchill, Rod and Singer [9, 10], Morales and Simó [19] and Morales and Ramis [20], Morales, Ramis and Simó prove in [22] that the variational equations of any order of an integrable Hamiltonian system have virtually abelian (= almost commutative) differential Galois group. Extensions of this Theorem to other kind of integrability (in Bogoyavlensky’s sense [4], in Jacobi’s sense [25], in noncommutative sense [14] or for discret dynamical systems [8]) give same abelianity condition.

In this article we prove the following generalization of these Theorems in algebraic setting for integrability by quadrature.

**Definition 3.4.** *Let  $E(t, y, y', \dots, y^{(n)}) \in \mathbb{C}[t, y', \dots, y^{(n)}]$  be an ordre  $n$  differential equation given by an irreducible polynomial.*

*A Liouvillian solution is a solution  $f$  in a differential extension  $K_N$  of  $K_1 = \mathbb{C}(t)$  build by successive elementary extensions  $K_{i-1} \subset K_i$ ,  $1 \leq i \leq N$ , of the form  $K_i = K_{i-1}(u_i)$  with  $u_i$  algebraic over  $K_{i-1}$  or  $u'_i \in K_{i-1}$  or  $\frac{u'_i}{u_i} \in K_{i-1}$ .*

*The equation  $E$  is said to be integrable by quadrature if there is a Liouvillian solution  $f$  with  $\text{transc.deg.}\mathbb{C}(t, f, f', \dots, f^{(n-1)})/\mathbb{C}(t) = n$ .*

**Theorem 3.6.** *If a rational differential equation is integrable by quadrature then the Galois group of its variational equation of order  $q$  along an algebraic solution is virtually solvable.*

*If  $N$  elementary extensions are needed to build the general solution then the  $N$ th derived Lie algebra of the Galois group is null.*

The key arguments are the use of Malgrange pseudogroup (Galois  $D$ -groupoid of [15]) of a vector field and the use of Artin approximation Theorem [2] (see 1.7 p 7) to replace Ziglin Lemma as it is done in [22]. They are organized as follow. First we prove that the Galois group of the variational equation is a quotient of a subgroup of the isotropy group of Malgrange pseudogroup at a generic point of the particular solution. Virtual solvability of the isotropy group at generic point of Malgrange pseudogroup of an equation integrable by quadrature is not very difficult to prove but a generic point on a curve is not generic. To prove virtual solvability of the isotropy group at a non generic point we use Artin approximation Theorem.

In a first part, definitions and basic theorems about algebraic “Lie pseudogroup” following [15, 24] are given. In a second part we give definition of Malgrange pseudogroup of a rational vector field and Galois group of a linear differential equation and relation between them by means of variational equations. The main Theorem is Theorem 2.3. From this Theorem we get usual Galoisian obstructions to integrability and exhibit new ones in the third part. Two examples of applications are given in the fourth part.

It would be interesting to insert in the setting decribed here results of Zung [29] and Ito [13].

Results of this article originate from discussions with B. Malgrange. I would like to thank him for his enthusiasm for sharing mathematical ideas and good mood.

## 1 Definitions

Definitions and missing proofs of this section can be found in [12, 15, 18, 24, 26].

### 1.1 Frame bundles

Let  $V$  be the affine space over  $\mathbb{C}$  with coordinates  $r_1, \dots, r_d$  and  $(\mathbb{C}^d, 0)$  be the germ of analytic space at 0 with coordinates  $x_1, \dots, x_d$ . An order  $q$  frame on  $V$  is a  $q$ -jet

$$j_q r = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq q}} r_i^\alpha \frac{x^\alpha}{\alpha!}$$

of germ of biholomorphism  $r : (\mathbb{C}^d, 0) \rightarrow V$ . The space of  $q$ -frames is denoted by  $R_q V$  and its coordinate ring is

$$\mathbb{C}[R_q V] = \mathbb{C} \left[ r_i^\alpha, \frac{1}{\det(r_i^{\epsilon(j)})} \mid 1 \leq i \leq d, \alpha \in \mathbb{N}^d, |\alpha| \leq q \right]$$

where  $\epsilon(j)$  is the multiindex  $(0, \dots, \underset{j^{th}}{1}, \dots, 0)$ . One gets projections  $\pi_q^{q+1} : R_{q+1}V \rightarrow R_qV$  from inclusions  $\mathbb{C}[R_qV] \subset \mathbb{C}[R_{q+1}V]$  and projections  $\pi^q : R_qV \rightarrow V$  from identifications  $r_i = r_i^{(0, \dots, 0)}$ . Elementary properties of this space can be found in [12] and [24, p285 with different notations]

The  $q$ -frames space is a principal bundle over  $V$  with structural group

$$\Gamma_q = \{j_q g \mid g : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}^d, 0) \text{ biholomorphism}\}$$

acting by ‘source composition’ :  $j_q g \cdot j_q r = j_q(r \circ g)$ .

Thanks to Faa di Bruno formulae, these groups and actions are algebraic. Because of projections  $\pi_q^{q+1}$  one can defined the formal frame bundle  $RV = \varprojlim R_qV$  with structural group  $\Gamma = \varprojlim \Gamma_q$ .

## 1.2 Groupoids

The algebraic variety

$$Aut_qV = \{j_q \varphi \mid \varphi : (V, a) \rightarrow (V, b) \text{ biholomorphism}\}$$

with coordinate ring

$$\mathbb{C}[Aut_qV] = \mathbb{C}\left[r_i, \varphi_j^\alpha, \frac{1}{\det(\varphi_j^{\epsilon(k)})} \mid 1 \leq i, j \leq d, \alpha \in \mathbb{N}^d, |\alpha| \leq q\right]$$

is a groupoid. The groupoid structure is given by the following morphisms

- sources and targets  $(s, t) : Aut_qV \rightarrow V \times V$ ,
- composition  $c : Aut_qV \times_{sVt} Aut_qV \rightarrow Aut_qV; c(j_q \varphi_1, j_q \varphi_2) = j_q(\varphi_1 \circ \varphi_2)$ ,
- inverse  $i : Aut_qV \rightarrow Aut_q; i(j_q \varphi) = j_q \varphi^{-1}$ ,
- identity  $id : V \rightarrow Aut_q; id(r) = j_q id_r$ ,

satisfying obvious commutative diagrams [18], [24, p 270]. This groupoid acts on  $R_qV$  by ‘target composition’ :

$$\begin{aligned} \circ : Aut_qV \times_{sV} R_qV &\rightarrow R_qV \\ j_q \varphi \circ j_q r &= j_q(\varphi \circ r). \end{aligned}$$

A subgroupoid  $\mathcal{G}_q$  of  $Aut_qV$  is a closed algebraic subvariety such that the induced morphisms give a groupoid structure on  $\mathcal{G}_q$  [24, definition 2.2].

A singular subgroupoid  $\mathcal{G}_q$  with singularities on  $S$  a closed subvariety of  $V$  is a closed subvariety of  $Aut_qV$  whose localisation with source and target out of  $S$  gives a subgroupoid of  $Aut_q(V - S)$  [15, definition 4.1.1].

A variant of Chevalley Theorem [11, theorem 8.1.], [24, proposition 2.3.6.] for this type of groupoids is the following.

**Theorem 1.1** *Let  $\mathcal{G}_q$  be a singular subgroupoid of  $Aut_q V$ . There are  $H_1, \dots, H_n$  in  $\mathbb{C}(R_q V)$  such that, out of its singular locus  $S$ ,*

$$\mathcal{G}_q = \{j_q \varphi \mid H_i(\cdot \circ j_q \varphi) = H_i(\cdot)\}.$$

Singular subgroupoids are essentially characterized by their field of rational invariants  $F_q \subset \mathbb{C}(R_q V)$ .

Groupoids  $Aut_q V$  have ‘Lie algebras’ (usually called Lie algebroids)

$$\mathfrak{aut}_q V = \{j_q Y \mid Y \text{ holomorphic vector field on } (V, a)\}.$$

We will not directly use the Lie algebroid structure of  $\mathfrak{aut}_q V$  but the fiberwise bracket

$$\begin{array}{ccc} \mathfrak{aut}_q V \times_V \mathfrak{aut}_q V & \rightarrow & \mathfrak{aut}_{q-1} V \\ (j_q Y_1, j_q Y_2) & \mapsto & j_{q-1}[Y_1, Y_2] \end{array}.$$

The Lie algebra of a groupoid  $\mathcal{G}_q$  will be described in next section.

### 1.3 Prolongations and invariants

Let  $\varphi : U_1 \rightarrow U_2$  be a biholomorphism between two open sets of  $V$ . It induces a biholomorphism

$$\begin{array}{ccc} R_q \varphi : R_q U_1 & \rightarrow & R_q U_2 \\ j_q r & \mapsto & j_q(\varphi \circ r) \end{array}$$

called the order  $q$  prolongation of  $\varphi$ .

Let  $X$  be a holomorphic vector field on an open set  $U$  of  $V$ . Prolongations of its flows define a local 1-parameter action on  $R_q U$ . The infinitesimal generator of this action is  $R_q X$  the prolongation of  $X$ .

These prolongations are defined by polynomial formulas and can be extended to formal biholomorphism  $\widehat{\varphi} : \widehat{V}, a \rightarrow \widehat{V}, b$  (and to formal vector fields on  $\widehat{V}, a$ ). The prolongation is  $R_q \widehat{\varphi} : (R_q \widehat{V}, R_q V_a) \rightarrow (R_q \widehat{V}, R_q V_b)$  a formal biholomorphism from a formal neighborhood of frames at  $a \in V$  to formal neighborhood of frames at  $b$ .

Cartan derivations are given by the action of  $\frac{\partial}{\partial x_i}$  on  $\mathbb{C}[R_q V]$ , the ring of PDE in  $d$  functions,  $r_1, \dots, r_d$  of  $d$  variables  $x_1, \dots, x_d$  in the neighborhood of 0 :

$$D_i : \begin{array}{ccc} \mathbb{C}[R_q V] & \rightarrow & \mathbb{C}[R_{q+1} V] \\ r_j^\alpha & \mapsto & r_j^{\alpha + \epsilon(i)} \end{array}.$$

The proof of the following lemma is left to the reader following [24, pp258–270].

#### Lemma 1.2

- *Let  $\varphi : U_1 \rightarrow U_2$  be a local biholomorphism on  $V$  and  $(\mathcal{R}_q \varphi)^* : \mathbb{C}[R_q U_2] \rightarrow \mathbb{C}[R_q U_1]$  the induced isomorphism of rings then*

$$D_i \circ (\mathcal{R}_q \varphi)^* = (\mathcal{R}_{q+1} \varphi)^* \circ D_i.$$

- Let  $X$  be a local holomorphic vector field  $U \subset V$  then

$$D_i \circ \mathcal{R}_q X = \mathcal{R}_{q+1} X \circ D_i.$$

- The order  $q$  prolongation of a vector field  $X = \sum_j a_j \frac{\partial}{\partial r_j}$  is

$$R_q X = \sum_{j,\alpha} D^\alpha a_j \frac{\partial}{\partial r_j^\alpha}.$$

**Example 1.3** Let  $V$  be the affine line over  $\mathbb{C}$  with coordinates ring  $\mathbb{C}[r]$  the order  $q$  frame bundle is  $R_q V = V \times \mathbb{C}^* \times \mathbb{C}^{q-1}$  with coordinates  $\mathbb{C}[r, r', \dots, r^{(q)}]$ .

If  $\varphi : U_1 \rightarrow U_2$  is a biholomorphism between open sets of  $V$  its third prolongation is  $R_3 \varphi : U_1 \times \mathbb{C}^* \times \mathbb{C}^2 \rightarrow U_2 \times \mathbb{C}^* \times \mathbb{C}^2$  and  $R_3 \varphi(r, r', r'', r''')$  is

$$(\varphi(r), \varphi'(r)r', \varphi''(r)r'^2 + \varphi'(r)r'', \varphi'''(r)r'^3 + 3\varphi''(r)r'r'' + \varphi'(r)r''')$$

**Example 1.4** Let  $V$  be the affine sapce of dimension  $d$  over  $\mathbb{C}$  with coordinates ring  $\mathbb{C}[r_1, \dots, r_d]$  the order 1 frame bundle is  $R_1 V = V \times GL_d(\mathbb{C})$  with coordinates ring  $\mathbb{C}\left[r_1, \dots, r_d, r_1^1, \dots, r_d^d, \frac{1}{\det(r_i^j)}\right]$ .

If  $X = \sum a_j(r) \frac{\partial}{\partial r_j}$  then

$$R_1 X = \sum a_j(r) \frac{\partial}{\partial r_j}(r) + \sum \frac{\partial a_j}{\partial r_i}(r) r_i^k \frac{\partial}{\partial r_j^k}.$$

When  $r(t)$  is a trajectory of  $X$  then the restriction of  $R_1 X$  above this trajectory is

$$\frac{\partial}{\partial t} + \sum \frac{\partial a_j}{\partial r_i}(r(t)) r_i^k \frac{\partial}{\partial r_j^k}$$

i.e. the first variational equation of  $X$  along  $r(t)$ .

Let  $\mathcal{G}_q$  be a singular subgroupoid with invariants field  $F_q$ . Its first prolongation  $\mathcal{G}_{q+1}$  is the singular subgroupoid defined by the subfield of  $\mathbb{C}(R_{q+1}V)$  generated by  $F_q$  and  $D_i F_q$  for all  $i$ . The field of rational functions of any order  $\mathbb{C}(RV) = \varinjlim \mathbb{C}(R_q V)$  with Cartan derivations is a differential field. The differential field  $F$  generated by  $F_q$  defines a subvariety  $\mathcal{G}$  of  $\text{Aut}V = \varprojlim \text{Aut}_q V$  by formulae of theorem 1.1 whose projection on  $\text{Aut}_q V$ ,  $(\mathcal{G})_q$  can be smaller than  $\mathcal{G}_q$ .

By a Theorem of B. Malgrange [15, theorem 4.4.1.] the subvariety of  $\text{Aut}V$  defined by invariance of a differential subfield  $F$  of  $\mathbb{C}(RV)$  defines for any  $q$  a singular subgroupoid with singularities on  $S$  independant of  $q$ . Let  $F$  be a differential subfield of  $\mathbb{C}(RV)$  and  $F_q = F \cap \mathbb{C}(R_q V)$ . Let us define

$$Iso(F) = \{\widehat{V}, a \rightarrow \widehat{V}, b \mid \forall q, \forall H \in F_q, H \circ R_q \varphi = H\}$$

whose ‘Lie algebra’ is

$$\mathfrak{iso}(F) = \{\text{formal vector field } Y \text{ on } \widehat{V}, a \mid \forall q, \forall H \in F_q, R_q Y \cdot H = 0\}$$

and  $Iso_q, \mathfrak{iso}_q$  the closure of their projections on order  $q$  jets spaces.

**Theorem 1.5** *The subspace  $Iso(F)$  of  $AutV$  is stable by composition and inversion. The subspace  $\mathfrak{iso}(F)$  of  $\mathfrak{aut}V$  is stable by Lie bracket.*

This is a set theoretical stability. The proalgebraic variety  $Iso(F)$  is singular subgroupoid and the singularities are unavoidable. But the Theorem says that the set of formal solutions is a set theoretical groupoid. Before giving the proof let show an example.

**Example 1.6**  *$V$  is the affine line  $\mathbb{A}^1$ , we look at formal diffeomorphisms preserving the 1-form  $\frac{dx}{r}$  (this gives an order 1 invariant). The differential equation satisfied by such transformations  $\varphi$  is  $E(r, \varphi) = r \frac{d\varphi}{dr} - \varphi = 0$ .  $E(r, \varphi_1 \circ \varphi_2)$  is a consequence of  $E(r, \varphi_2)$  and  $E(\varphi_2, \varphi_1)$  only after localisation of sources and targets out of 0. This is a singular groupoid with singularity at 0 but invertible solutions are  $\varphi(r) = \lambda r$  for  $\lambda \in \mathbb{C}^*$  and form a groupoid (even a group !).*

In many proofs of the third part and in the following proof of theorem 1.5 the following theorem is used

**Artin approximation Theorem 1.7**

*Consider an arbitrary system of analytic equations*

$$(E) \quad f(x, y) = 0$$

where  $f(x, y) = (f_1(x, y), \dots, f_m(x, y))$  are convergent series in the variables  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_N)$ . Suppose that  $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$  are formal power series without constant term which solve (E) For any integer  $q$  there exists a convergent series solution  $y(x) = (y_1(x), \dots, y_N(x))$  of (E) such that for all  $i$   $j_q(\bar{y}_i(x) - y_i(x)) = 0$ .

**Proof of theorem 1.5.** – By Malgrange Theorem (above mantionned), the Theorem is clear if sources and targets of two composable elements of  $Iso(F)$  are not in the singular locus  $S$ . By analytic continuation, it is also clear for convergent elements. This implies the theorem because of the following.

Let  $\varphi_1$  and  $\varphi_2$  be a composable couple of elements of  $Iso(F)$ . For any  $q$  and  $i = 1, 2$ ,  $j_q(\varphi_i)$  are formal sections of  $Iso_q(F)$  over  $V$ . By Artin approximation theorem [2] there are holomorphic nonholonomic sections  $\psi_i$  on neighborhoods  $U_1$  of  $a$  and  $U_2$  of  $\varphi_1(a)$  such that  $j_q\psi_1 = j_q(j_q(\varphi_1))$  at  $a$  and  $j_q\psi_2 = j_q(j_q(\varphi_2))$  at  $\varphi_1(a)$ . These new sections are no more jets of sections.

For each  $\tilde{a} \in U_1$ ,  $\psi_1(\tilde{a}) \in Aut_q V$  is an order  $q$  jet of biholomorphism with source  $\tilde{a}$  and target a point  $t(\psi_1(\tilde{a}))$  near  $\varphi_1(a)$ . Because de  $q$ -jet of  $t(\psi_1) : U_1 \rightarrow V$  at  $a$  equals the  $q$ -jet of  $\varphi_1$  at  $a$ ,  $U_1$  can be chosen small enough so that  $t(\psi_1)(U_1)$  is an open set included in  $U_2$ .

Then one can compose  $\psi_2 \circ \psi_1$  pointwise. Because  $Iso_q(F)$  is a groupoid out of some singular locus  $S$ ,  $\psi_2 \circ \psi_1$  is a section of  $Iso_q(F)$  out of  $S$ .  $Iso_q(F)$  is closed so  $\psi_2 \circ \psi_1$  is a section of  $Iso_q(F)$  and in particular  $\psi_2(a) \circ \psi_1(a) = j_q(\varphi_2 \circ \varphi_1)$  belongs to  $Iso_q(F)$ . This is true for all integer  $q$  thus  $\varphi_2 \circ \varphi_1 \in Iso(F)$ .

Same arguments are used to prove the second part of the Theorem.  $\square$

**Remark 1.8** *Following B. Malgrange, one can give another proof by using Ritt approximation Theorem [26] of formal solutions of PDE by convergent solutions (not define at the same point) instead of Artin approximation Theorem of these solutions by nonholonomic sections (define at the same point).*

## 2 Galois theories

### 2.1 ‘Nonlinear’ Galois theory

The definition of Malgrange pseudogroup of a rational vector field  $X$  is

$$MalX = Iso(Inv(X))$$

whose Lie algebra is

$$\mathfrak{mal}X = \mathfrak{iso}(Inv(X)).$$

For  $(a, b) \in V \times V$ ,  $MalX_{(a,b)}$  is the subspace of formal biholomorphisms with source  $a$  and target  $b$ . One gets the following corollary of theorem 1.5.

**Corollary 2.1** *Let  $a \in V$  the formal solution of Malgrange pseudogroup at  $a$*

$$MalX_{(a,a)} = \{\varphi : \widehat{(V, a)} \rightarrow \widehat{(V, a)} \mid \varphi \in MalX\}$$

*is a group with Lie algebra*

$$\mathfrak{mal}X_a = \{Y \text{ on } \widehat{V, a} \mid Y \in \mathfrak{mal}X\}.$$

These groups may be different depending on  $a$  belongs to the singular locus  $S$  of  $MalX$  or not. However, as we will see in the last section, they share lots of properties.

### 2.2 Linear Galois theory

#### 2.2.1 Principal version

Let  $\mathcal{C}$  be an algebraic curve over  $\mathbb{C}$ ,  $E \xrightarrow{\pi} \mathcal{C}$  a principal  $G$ -bundle, i.e.  $E \times_{\mathcal{C}} E \sim E \times G$  over  $E$  for the first projection and  $G$  is an algebraic linear group. For a  $\pi$ -projectable,  $G$ -invariant rational vector field  $Y$  on  $E$  with  $\pi_* Y \neq 0$ ,  $PY$  denotes a closed minimal  $Y$ -invariant subvariety of  $E$  dominating  $\mathcal{C}$  and  $GalY$  its stabilizer in  $G$ .

- Two such  $PY$  are isomorphic under action of  $G$  and called Picard-Vessiot varieties of  $Y$ . The field extension  $\mathbb{C}(\mathcal{C}) \subset \mathbb{C}(PY)$  is usually called the Picard-Vessiot extension for  $Y$ .
- The group  $GalY$  is well defined up to conjugation in  $G$ . It is the Galois group of  $Y$ .



- Common level set of all rational first integrals of  $Y$  in  $\mathbb{C}(E)$  dominating  $\mathcal{C}$  are Picard-Vessiot varieties.

Malgrange pseudogroup of such a  $Y$  is simple to describe. Let  $Z_1, \dots, Z_N$  be infinitesimal generators of the action of  $G$  then  $Y, Z_1, \dots, Z_N$  is a  $Y$ -invariant rational parallelism of  $E$  *i.e.* a basis of the  $\mathbb{C}(\mathcal{C})$  vector space of rational vector field on  $\mathcal{C}$  such that  $[Y, Z_i] = 0$ . Let  $\mathbb{C}(E)^Y$  be the field of rational first integrals of  $Y$ . One has

$$MalY = \{\varphi \mid \varphi^*Y = Y, \forall i \varphi^*Z_i = Z_i, \forall F \in \mathbb{C}(E)^Y F \circ \varphi = F\}.$$

The inclusion ‘ $\subset$ ’ is clear from the definition. To prove the other inclusion one remarks that  $Y$  and  $Z$ ’s give rise to lots of order 1 invariants. Because they form a basis,  $\mathbb{C}(R_1E)$  is generated over  $\mathbb{C}(E)$  by these invariants. This implies that  $\mathbb{C}(R_qE)$  is generated over  $\mathbb{C}(E)$  by derivatives of these invariants. Each new differential invariant for  $Y$  reduces modulo this first field of invariants to order 0 invariant *i.e.* to a rational first integral of  $Y$ .

Let  $MalY_a$  be the restriction of this pseudogroup to the fiber  $E_a$  at generic  $a \in \mathcal{C}$ . The fiber  $E_a$  is isomorphic to  $G$  and one can choose this isomorphism to send  $PY_a$  on  $GalY$ .

This isomorphism conjugates the action of  $G$  on  $E_a$  to the left translation on  $G$ . Because the action of  $MalY$  commutes to left translation on  $G$  each  $\varphi \in MalY_a$  is the restriction on some open set of right translation by a  $g_\varphi \in G$ . But  $MalY_a$  must preserve  $GalY$  so  $g_\varphi \in GalY$ . We have proved the following theorem.

**Theorem 2.2** *Under this isomorphism  $MalY_a$  equals  $GalY$  as pseudogroup generated by a subgroup of  $G$ .*

### 2.2.2 Non principal version

Galois group can be defined for more general bundles with special kind of connections. They are built from principal bundles by ‘fiber reduction’

Let  $E \rightarrow \mathcal{C}$  be a  $G$ -principal bundle and  $F$  be an algebraic variety with a left action of  $G$  with  $G$  a algebraic linear group. This group  $G$  acts on  $F \times E$  by  $g(p, e) = (pg^{-1}, ge)$ . The bundle  $P = (F \times E)/G$  has structural group  $G$  and fibers isomorphic to  $F$ . If  $Y$  is a  $\pi$ -projectable,  $G$ -invariant rational vector field on  $E$  with  $\pi_*Y \neq 0$  one defines  $\tilde{Y}$  on  $F \times E$  such that  $pr_{1*}\tilde{Y} = 0$  and  $pr_{2*}\tilde{Y} = Y$  and  $Y_P$  on  $P$  by projection.

Galois group of  $Y_P$  is by definition the Galois group of  $Y$  and one can prove that it is well defined. From [7] Malgrange pseudogroup of  $Y_P$  is isomorphic to Malgrange pseudogroup of  $\tilde{Y}$  which is itself isomorphic to the one of  $Y$ .

## 2.3 Variational equations

### 2.3.1 Principal variational equations

Let  $X$  be a rational vector field on  $V$  and  $\mathcal{C}$  be a algebraic  $X$ -invariant curve with  $X|_{\mathcal{C}} \neq 0$ . Its prolongations are rational vector fields  $R_qX$  on frames bundles

$R_q V$ . The restriction of the frames bundles over  $\mathcal{C}$  are  $\Gamma_q$ -principal bundle over  $\mathcal{C}$  with a projectable  $\Gamma_q$ -invariant vector field given by the restriction of  $R_q X$  over  $\mathcal{C}$ .

Because  $R_{q+1} X$  is  $\pi_q^{q+1}$ -projectable on  $R_q X$ , this is also true for Malgrange pseudogroup [7, lemme 4.6.] and Galois group. One has surjective morphisms

$$\text{Mal}(R_{q+1} X|_{\mathcal{C}}) \rightarrow \text{Mal}(R_q X|_{\mathcal{C}}) \quad \text{and} \quad \text{Gal}(R_{q+1} X|_{\mathcal{C}}) \rightarrow \text{Gal}(R_q X|_{\mathcal{C}}).$$

One sets

$$\text{Mal}(RX|_{\mathcal{C}}) = \varprojlim \text{Mal}(R_q X|_{\mathcal{C}}) \quad \text{and} \quad \text{Gal}(RX|_{\mathcal{C}}) = \varprojlim \text{Gal}(R_q X|_{\mathcal{C}}).$$

**Theorem 2.3** *Let  $a$  be a generic point on  $\mathcal{C}$ , one gets*

$$\text{Gal}(RX|_{\mathcal{C}}) \subset \text{Mal}X_{(a,a)}.$$

**Proof.** – Let  $\text{Mal}(RX|_{\mathcal{C}})_a$  be the restriction of  $\text{Mal}(RX|_{\mathcal{C}})$  on the fiber  $RV_a$ . The  $\Gamma$ -principal bundle  $RV|_{\mathcal{C}}$  is isomorphic to the subspace of  $\text{Aut}V_{(a,\mathcal{C})}$  with source  $a \in \mathcal{C}$  and target in  $\mathcal{C}$  where  $\Gamma = \varprojlim \Gamma_q$  is the group of formal biholomorphisms from  $(V, a)$  to  $(V, a)$ . Under this identification

- $RV_a$  is  $\Gamma$ ,
- $\text{Gal}(RX|_{\mathcal{C}})$  is a subgroup acting by left translation,
- $\text{Mal}(RX|_{\mathcal{C}})_a$  is isomorphic to  $\text{Gal}(RX|_{\mathcal{C}})$  but acts by right translation.

The closed subvariety  $\text{Mal}X_{(a,\mathcal{C})}$  with source  $a$  and target in  $\mathcal{C}$  of  $\text{Aut}V_{(a,\mathcal{C})}$  is

- $RX$ -invariant because  $\text{Mal}X$  is  $RX$ -invariant,
- dominates  $\mathcal{C}$  because it contains flows of  $X$  along  $\mathcal{C}$ .

This implies that  $\text{Gal}(RX|_{\mathcal{C}})$  preserves  $\text{Mal}X_{(a,a)}$  by left translation in  $\Gamma$  thus  $\text{Gal}(RX|_{\mathcal{C}}) \subset \text{Mal}X_{(a,a)}$ .  $\square$

Normal variation equations are defined on the subspaces  $N_q \subset R_q V|_{\mathcal{C}}$  of frames  $r$  such that  $r^* X = \frac{\partial}{\partial x_d}$ . This subspaces are  $\Gamma_1$ -principal bundle where  $\Gamma_1 \subset \Gamma$  is the subgroup of  $g \in \Gamma$  such that  $g_* \frac{\partial}{\partial x_d} = \frac{\partial}{\partial x_d}$ .

### 2.3.2 ‘Non principal’ variational equations

Let  $A^q V$  be the space of order  $q$  arcs on  $V : j_q \gamma$  with  $\gamma : (\mathbb{C}, 0) \rightarrow V$  holomorphic. This bundle is associated to the  $\Gamma_q$  principal bundle  $R_q V$  by the procedure of 2.2.2 with  $\Gamma_q$  acting on

$$A_q = \{j_q \gamma \mid \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^d, 0)\}$$

by target composition. Groupoid  $\text{Aut}_q V$  acts on  $V^q$  by composition. A rational vector field  $X$  on  $V$  acts on  $A^q V$  as a rational vector field  $A_q X$ . This vector field can be obtained from ‘fiber reduction’ given in 2.2.2. The restriction of this vector field over a  $X$ -invariant curve  $\mathcal{C}$  is the usual variational equation.

## 3 Corollaries

### 3.1 Abelianity

Abelianity of  $\mathfrak{mal}X$  implies Abelianity of the identity components of variational equations (see next section for a proof). In Hamiltonian context one gets the following Theorem, consequence of the ‘key’ Lemma [3].

**Theorem 3.1 (J.-P.Ramis [16])** *If  $X$  is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over  $\mathbb{C}$  by means of rational first integrals then  $\mathfrak{mal}X$  is Abelian.*

Together with Theorem 2.3 it implies Morale-Ramis-Simó theorem in algebraic context.

**Theorem 3.2 ([22])** *If  $X$  is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over  $\mathbb{C}$  by means of rational first integrals and  $\mathcal{C}$  be a algebraic  $X$ -invariant curve with  $X|_{\mathcal{C}} \neq 0$  then identity component of the Galois group of the order  $q$  variational equation is Abelian.*

### 3.2 Solvability

**Lemma 3.3** *If the  $N$ th derived algebra of  $\mathfrak{mal}X$  is null the same is true for any variational equations.*

**Proof.** – We have to prove that  $\mathfrak{mal}X_a$  satisfies this property for any  $a \in V$  as soon as it is satisfied at any  $a$  out of an hypersurface  $S$  on  $V$ . We follow the proof of Theorem 1.5.

Let  $Y_1, \dots, Y_{2N}$  be  $2^N$  formal vector fields at  $a$  solutions of  $\mathfrak{mal}X$ . By Artin Approximation theorem there are  $\tilde{Y}_1, \dots, \tilde{Y}_{2N}$  be holomorphic nonholonomic sections of  $\mathfrak{mal}_{N+q}X$  whose  $N+q$  jets at  $a$  are given by  $Y$ 's. The iterated fiberwise bracket in the  $N$ th derived algebra of  $\mathfrak{mal}X$  obtained from the  $\tilde{Y}$ 's is zero out of  $S$  thus everywhere. It is determined at  $a$  by the  $q$ -jet of the iterated Lie bracket of the formal vector fields. Because it is true for any  $q$  it proves the Lemma.  $\square$

Let say that differential equation over  $\mathbb{C}(t)$  is integrable by quadrature if the general solution belongs to a Liouvillian extension (with possibly new constants).

**Definition 3.4** *Let  $E(t, y, y', \dots, y^{(n)}) \in \mathbb{C}[t, y', \dots, y^{(n)}]$  be an ordre  $n$  differential equation given by an irreducible polynomial.*

*A Liouvillian solution is a solution  $f$  in a differential extension  $K_N$  of  $K_1 = \mathbb{C}(t)$  build by successive elementary extensions  $K_{i-1} \subset K_i$ ,  $1 \leq i \leq N$  of the form  $K_i = K_{i-1}(u_i)$  with  $u_i$  algebraic over  $K_{i-1}$  or  $u_i' \in K_{i-1}$  or  $\frac{u_i'}{u_i} \in K_{i-1}$ .*

*The equation  $E$  is said to be integrable by quadrature if there is a Liouvillian solution  $f$  with  $\text{transc.deg.}\mathbb{C}(t, f, f', \dots, f^{(n-1)})/\mathbb{C}(t) = n$ .*

**Remark 3.5** *It is important to allow new constant in order to get  $y'' = 0$  integrable by quadrature.*

**Theorem 3.6** *If a rational ordinary differential equation is integrable by quadrature then its variational equations along algebraic solutions have solvable identity component of their Galois groups.*

**Proof.** – We have to prove that if  $X$  on  $V$  is a vector field given by the equation on a phase space,  $\mathfrak{mal}X_a$  is solvable for a generic  $a \in V$ . Then apply Lemma 3.3 and Theorem 2.3 and the proof is done. Let  $\mathbb{C}(t) \subset K_2 \cdots \subset K_N$  be a Liouvillian tower such that  $E$  has a transcendence degree  $d - 1$  solution in  $K_N$  over  $\mathbb{C}(t)$ . Assume that all extensions are transcendental. Each  $K_i$  is the field of rational functions on some affine space  $\mathbb{A}^1 \times \mathbb{A}^i$  with a vector field  $X_i = \frac{\partial}{\partial t} + \sum_{j=1}^i u'_j \frac{\partial}{\partial u_j}$  projectable on  $X_{i-1}$ . Hypothesis of the Theorem are :

- there is a dominant map  $V \rightarrow \mathbb{A}^1$  and  $X$  is projectable on  $\frac{\partial}{\partial t}$ ,
- there is a dominant map from  $\mathbb{A}^1 \times \mathbb{A}^N$  to  $V$  over  $\mathbb{A}^1$  and  $X_N$  is projectable on  $X$ .

From [7, lemme 4.6.] if  $\mathfrak{mal}X_N$  is solvable so is  $\mathfrak{mal}X$ .

Because of the structure of the tower of extension one can find  $N$  1-forms,  $\theta_i = du_i - X_N u_i dt$ ,  $1 \leq i \leq N$ , constant on  $X_N$  satisfying  $d\theta_i = 0 \pmod{(\theta_1, \dots, \theta_{i-1})}$ . Because  $L_{X_N} \theta_i = 0 \pmod{(\theta_1, \dots, \theta_{i-1})}$  same equations are satisfied by vector field of  $\mathfrak{mal}X$ . Let  $x_1, \dots, x_N$  be local (analytic) coordinates such that  $dx_i = \theta_i \pmod{(\theta_1, \dots, \theta_{i-1})}$  then  $Y \in \mathfrak{mal}X$  can be written

$$Y = c_1 \frac{\partial}{\partial x_1} + c_2(x_1) \frac{\partial}{\partial x_2} + \cdots + c_N(x_1, \dots, x_{N-1}) \frac{\partial}{\partial x_N}.$$

The  $N$ th derived algebra of this type of Lie algebra of formal vector field is zero.  $\square$

For instance,  $X$  on  $V$  is Jacobi integrable if it has  $d - 2$  rational first integrals and an invariant rational  $d$ -form. Morales-Ramis type theorem for this kind of integrability was obtained by M. Przytylska in [25] in a particular case. Theorem 2.3 gives the general situation. Computation of vector fields in the Lie algebra of the Malgrange pseudogroup of such a vector field is left to the reader. In suitable local (analytic) coordinates one gets vector field of the form  $c_1(x_2, \dots, x_n) \frac{\partial}{\partial x_1} + c_2(x_3, \dots, x_n) \frac{\partial}{\partial x_2}$ . The first derived algebra is Abelian.

**Corollary 3.7** *If  $X$  is a Jacobi integrable rational vector field on an algebraic variety then identity components of Galois groups of variational equations are solvable and their first derived Lie algebras are Abelian.*

### 3.3 Finiteness

**Lemma 3.8** *Finiteness of the dimension of  $\mathfrak{mal}X$  implies that the dimensions of Galois groups of variational equations are uniformly bounded.*

**Proof.** – We have to prove that if  $\mathfrak{mal}X_a$  is finite dimensional at generic  $a$  in  $V$ , it is finite dimensional at any  $a$  in  $V$  with smaller dimension. Then Theorem 2.3 can be used to conclude.

Let  $N$  be the generic dimension of  $\mathfrak{mal}X$ . If  $X_1, \dots, X_{N+1}$  are  $N+1$  element of  $\mathfrak{mal}X_a$ , by Artin approximation Theorem, there are  $\tilde{X}_1, \dots, \tilde{X}_{N+1}$  holomorphic nonholonomic section of  $\mathfrak{mal}_q X$  whose  $q$ -jet at  $a$  is given by  $X$ 's. For each  $a \in V - S$  the vectors  $\tilde{X}_1(a), \dots, \tilde{X}_{N+1}(a)$  are colinear. By analytic continuation it is also true for any  $a \in V$  and any order  $q$ . This proves the lemma.  $\square$

From

**Theorem 3.9 (B. Malgrange [17])** *If  $X$  is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over  $\mathbb{C}$  by means of rational first integrals then  $\mathfrak{mal}X$  is finite dimensional.*

one gets

**Theorem 3.10** *In the situation of theorem 3.2 the sequence of dimensions*

$$(\dim \text{Gal}(R_q X|_{\mathcal{C}}))_{q \in \mathbb{N}}$$

*is bounded uniformly in  $\mathcal{C}$ .*

There is no uniform bound for all algebraic Hamiltonians but it depends on the geometry of the moment map. For instance if the moment map is the restriction on some open set of an algebraically isotrivial fibration in Abelian varieties then the bound should be the degree of freedom. If the fibration is not isotrivial, *e.g.* for algebraic complete integrability [1], the bound depend on the Gauss-Manin connexion of such a family.

## 4 Applications

### 4.1 Painlevé II equation

Irreducibility of these equations is proved in [23, 27] and implies that these equations are not integrable by quadrature. Here is another proof of this weaker assertion in a particular case. These computations have been done in [21] to apply usual Morales-Ramis Theorem.

Second Painlevé equation depends on a parameter

$$y'' = 2y^3 + xy + \alpha \quad \text{for } \alpha \in \mathbb{C}.$$

For  $\alpha = 0$ , it is the vector field

$$\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + (2y^3 + xy) \frac{\partial}{\partial y'}$$

with first order (non principal) variational equation along  $\mathcal{C} = \{y = y' = 0\}$

$$\frac{\partial}{\partial x} + r_3 \frac{\partial}{\partial r_2} + xr_2 \frac{\partial}{\partial r_3}$$

on  $T\mathbb{C}^3|_{\mathcal{C}}$  with induced coordinates  $(x, r_1, r_2, r_3)$ . The rank 2 subsystem on  $r_2, r_3$  is Airy equation with Galois group  $SL_2(\mathbb{C})$ . This group is not solvable so Painlevé II equation is not integrable by quadrature when  $\alpha = 0$ . From Okamoto (see [23, 27]), we know that two Painlevé II equations with parameter  $\alpha$  and  $\alpha + n$ ,  $n \in \mathbb{Z}$  are isomorphic by a birational change of coordinates on the phase space. Non integrability for  $\alpha = 0$  implies non integrability for any interger values of  $\alpha$ .

## 4.2 Lorenz system

Computations presented here originates from [6] where the non integrability in sense of Liouville of a Hamiltonian form of this system is proved by Morales-Ramis theorem. Lorenz system depends on 3 constants  $\sigma$ ,  $\rho$  and  $\beta$  :

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{cases} .$$

First assume  $\beta \neq 0$ . An invariant curve is the  $z$ -axis  $\mathcal{C}$  and  $X|_{\mathcal{C}} = -\beta z \frac{\partial}{\partial z}$ . Let us consider the following time dependent form of this equation on  $\mathbb{C}^3 \times \mathbb{C}$  with coordinates  $(x, y, z, t)$

$$\sigma(x - y) \frac{\partial}{\partial x} + (\rho x - xz - y) \frac{\partial}{\partial y} + (xy - \beta z) \frac{\partial}{\partial z} - \beta t \frac{\partial}{\partial t} .$$

an invariant curve is  $\mathcal{C} = \{x = y = z - t = 0\}$  and the first variational equation is the linear system

$$-\beta t \frac{d}{dt} A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ \rho - z & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix} A .$$

The  $2 \times 2$  subsystem given by the first block is equivalent to the second order equation

$$\beta^2 t^2 \frac{d^2 a}{dt^2} + (\beta^2 - \beta\sigma - \beta)t \frac{da}{dt} + \sigma(t - \rho + 1)a = 0 .$$

Applying [5] (or Maple 11), we get solution

$$t^{\frac{\sigma+1}{2\beta}} J \left( \frac{\sigma^2 - 2\sigma + 1 + a\sigma\rho}{\beta^2}, \frac{2\sigma^{1/2}}{\beta} t^{1/2} \right)$$

where  $J(\alpha, x)$  is any solution to Bessel equation  $x^2 \frac{d^2 J}{dx^2} + x \frac{dJ}{dx} + (x^2 - \alpha)J = 0$ . The Galois group of this equation is  $SL_2(\mathbb{C})$ . Lorenz system is not integrable

by quadratures.

If  $\sigma \neq 0$  then using the vector field

$$\sigma(x-y)\frac{\partial}{\partial x} + (\rho x - xz - y)\frac{\partial}{\partial y} + (xy - \beta z)\frac{\partial}{\partial z} - \sigma t\frac{\partial}{\partial t}.$$

an the invariant curve is  $\mathcal{C} = \{y = z - \rho = x - t = 0\}$  we get the variational equation

$$-\sigma t \frac{d}{dt} A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ 0 & -1 & t & 0 \\ 0 & t & -\beta & 0 \\ 0 & 0 & & -\sigma \end{bmatrix} A.$$

From the middle  $2 \times 2$  subsystem we get the second order equation

$$\sigma^2 t^2 \frac{d^2 a}{dt^2} - \sigma(t+1)t \frac{da}{dt} + ((1-\beta)t + \sigma)a = 0.$$

Solutions are

$$t^{\frac{1}{2\sigma}} \exp\left(\frac{t}{2\sigma}\right) W\left(\frac{1-2\beta}{2\sigma}, \frac{\sigma-1}{2\sigma}, \frac{t}{\sigma}\right)$$

where  $W(k, m, x)$  is any solution of Whittaker equation  $x^2 \frac{d^2 W}{dx^2} + x^2 \frac{dW}{dx} + (kx - m^2 + 1/4)W = 0$ . Galois group of this equation is  $SL_2(\mathbb{C})$ .

If  $\sigma = \beta = 0$  the Lorenz system is  $2 \times 2$  linear system with constant coefficients and a parameter  $x$ . Such systems can be explicitly solved by exponentials.

## References

- [1] M. ADLER, P. VAN MOERBEKE & P. VANHAECKE – Algebraic integrability, Painlevé geometry and Lie algebras. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 47. Berlin: Springer. xii+483pp. (2004)*
- [2] M. ARTIN – On the solutions of analytic equations. *Invent. Math. 5 277–291. (1968)*
- [3] M. AUDIN – Les systèmes hamiltoniens et leur intégrabilité. *Cours Spécialisés (Paris) 8. Société Mathématique de France. viii+170pp. (2001)*
- [4] M. AYOUL & N.T. ZUNG – Galoisian obstruction to non-Hamiltonian integrability. *arXiv:0901.4586 (2009)*
- [5] M. BRONSTEIN & S. LAFAILLE – Solutions of linear ordinary differential equations in terms of special functions. *Proceedings of ISSAC'2002, Lille, ACM Press, 23–28 (2002)*

- [6] M. CANALIS, J.-P. RAMIS, P. ROUCHON & J.-A. WEIL – Calculations on the Lorenz system: Variational equation, Bessel dynamics *MAPLE worksheet available on [http://perso.univ-rennes1.fr/~guy.casale/ANR/ANR\\_html/publications.html](http://perso.univ-rennes1.fr/~guy.casale/ANR/ANR_html/publications.html)* (2001)
- [7] G. CASALE – Une preuve galoisienne de l'irréductibilité au sens de Nishioka-Umemura de la 1ère équation de Painlevé. *Differential Equation and Singularities. 60th years of J.M.Aroca Astérisque 324* 83–100 (2009)
- [8] G. CASALE & J. ROQUES – Dynamics of rational symplectic mappings and difference Galois theory. *Int. Math. Res. Not.* (2008).
- [9] R.C. CHURCHILL & D.L. ROD – On the determination of Ziglin monodromy groups. *SIAM J. Math. Anal.* 22, no. 6, 1790–1802 (1991).
- [10] R.C. CHURCHILL, D.L. ROD & M.F. SINGER – Group-theoretic obstructions to integrability. *Ergodic Theory Dynam. Systems* 15, no. 1, 15–48 (1995).
- [11] P. GABRIEL – Construction de préschémas quotients. *Schémas en groupes (SGA 63-64) Fasc 2a exposé 5 Lecture Notes in Mathematics 151, 250–283, Springer-Verlag* (1970)
- [12] V. GUILLEMIN & S. STERNBERG – An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.* 70 16–47 (1964)
- [13] H. ITO – On the holonomy group associated with analytic continuations of solutions for integrable systems *Bol. Soc. Bras. Mat., vol21 no 1, 95–120* (1990)
- [14] A. MACIEJEWSKI & M. PRZYBYLSKA – Differential Galois obstructions for non-commutative integrability. *Physics Letters A, 372, no. 33, 5431–5435* (2008)
- [15] B. MALGRANGE – Le groupoïde Galois d'un feuilletage. *Ghys, Étienne (ed.) et al., Essays on geometry and related topics. Mémoires dédiés à André Haefliger. Vol. 2. Genève: L'Enseignement Mathématique. Monogr. Enseign. Math.* 38, 465–501 (2001).
- [16] B. MALGRANGE – On nonlinear differential Galois theory. *Dedicated to the memory of Jacques-Louis Lions. Chinese Ann. Math. Ser. B* 23, no. 2, 219–226 (2002).
- [17] B. MALGRANGE – personal discussions (2007)
- [18] K. MACKENZIE – Lie groupoids and Lie algebroids in differential geometry. *London Mathematical Society Lecture Note Series, 124. Cambridge University Press, Cambridge. xvi+327pp* (1987)



- [19] J.J. MORALES-RUIZ & C. SIMÓ – Picard-Vessiot theory and Ziglin’s theorem. *J. Differential Equations* 107, no. 1, 140–162 (1994)
- [20] J.J. MORALES-RUIZ & J.-P. RAMIS – Galoisian obstructions to integrability of Hamiltonian systems I & II. *Methods Appl. Anal.* 8, No. 1, 33–111 (2001)
- [21] J.J. MORALES-RUIZ – A remark about the Painlevé transcendents. *Delabaere, Éric (ed.) et al. Théories asymptotiques et équations de Painlevé. S.M.F. Séminaires et Congrès 14*, 229–235 (2006)
- [22] J.J. MORALES-RUIZ, J.-P. RAMIS & C. SIMÓ – Integrability of Hamiltonian systems and differential Galois groups of higher variational equations. *Ann. Sci. Éc. Norm. Supér. (4)* 40, No. 6, 845–884 (2007)
- [23] M. NOUMI & K. OKAMOTO – Irreducibility of the second and fourth Painlevé equation *Funkcial. Ekvac.* 40 139–163 (1997)
- [24] J.-F. POMMARET – Differential Galois theory. *Mathematics and its Applications*, 15. Gordon & Breach Science Publishers, New York. viii+759pp (1983)
- [25] M. PRZYBYLSKA – Differential Galois obstructions for integrability of homogeneous Newton equations. *J. Math. Phys.* 49, No. 2, 022701 (2008).
- [26] J.F. RITT – Differential algebra. *Dover Publications, Inc., New York*, viii+184pp (1966)
- [27] H. UMEMURA & H. WATANABE – Solutions of the second and fourth Painlevé equation, I. *Nagoya Math. J. Vol 148* 151–198 (1997)
- [28] S.L. ZIGLIN – Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics. I. *Funct. Anal. Appl.* 16, 181–189 (1983); translation from *Funkts. Anal. Prilozh.* 16, No.3, 30–41 (Russian) (1982).
- [29] N.T. ZUNG – Convergence versus integrability in Poincaré-Dulac normal form. *Math. Res. Lett.* 9, no. 2-3, 217–228 (2002)