PROXIMALITY AND EQUIDISTRIBUTION ON THE FURSTENBERG BOUNDARY

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Abstract. Let G be a connected semisimple Lie group with finite center and without compact factors, P a minimal parabolic subgroup of G, and Γ a lattice in G. We prove that every Γ-orbits in the Furstenberg boundary G/P is equidistributed for the averages over Riemannian balls. The proof is based on the proximality of the action of Γ on G/P.

1. Introduction

Let G be a connected semisimple Lie group with finite center and without compact factor, and Γ a lattice in G, that is, a discrete subgroup of G such that $\Gamma\backslash G$ has finite volume. In this article we investigate the distribution of orbits of Γ acting on the Furstenberg boundary of G. Recall that the Furstenberg boundary can be identified with the factor space G/P, where P is a minimal parabolic subgroup of G. It is known that every orbit of Γ in G/P is dense (see [Mo]). We show that orbits of Γ are equidistributed with respect to the averages over Riemannian balls.

Since we study the action of a nonamenable group on a space without a finite invariant measure, our result lies outside the scope of the classical ergodic theory. The published results about distribution of dense orbits of nonamenable groups are limited to a few special examples. Arnold and Krylov showed in [AK] that dense orbits of groups generated by two rotations acting on the 2-dimensional sphere are equidistributed. A similar problem was considered by Kazhdan in [Ka] where he studied the action of a group generated by two affine isometries on the plane $\mathbb{R}^2$. Distribution of dense orbits of a lattice in $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{R}^2$ was investigated by Ledrappier [L] and Nogueira [N].

Let X be the symmetric space of G equipped with a right invariant Riemannian metric d. Note that X can be identified with $L\backslash G$ for a maximal compact subgroup L of G.

Fix $x, \tilde{x} \in X$ and denote by K and $\tilde{K}$ the stabilizers of x and $\tilde{x}$ respectively. Let ν and $\tilde{\nu}$ be the probability Haar measures on K and $\tilde{K}$ and $m_{\tilde{x}}$ the harmonic measures at $\tilde{x}$ on G/P, that is, the unique $\tilde{K}$-invariant probability measure on G/P. For $S \subset G$

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and $T > 0$, define

$$\begin{aligned} S_T(\tilde{x}) &= \{s \in S : d(x, \tilde{x}s) < T\}, \\ S_T &= S_T(x). \end{aligned}$$

Our main result is the following theorem.

Theorem 1. *For every $f \in C(G/P)$, $\tilde{x} \in X$, and $y \in G/P$,*

$$\lim_{T\to\infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) = \int_{G/P} f dm_{\tilde{x}},$$

Moreover, the convergence is uniform for $y \in G/P$.

We remark that it was shown in [EM] (see also [DRS]) that

$$|\Gamma_T(\tilde{x})| \sim_{T\to\infty} \frac{\mathrm{Vol}(G_T(\tilde{x}))}{\mathrm{Vol}(\Gamma\backslash G)}, \tag{1}$$

and the exact asymptotics of the volume $\mathrm{Vol}(G_T(\tilde{x})) = \mathrm{Vol}(G_T)$ as $T \to \infty$ was computed in [Kn].

The first result in the direction of Theorem 1 was established in [Ma], where the case of the real hyperbolic spaces was considered. A similar problem for $G = \mathrm{SL}(n, \mathbb{R})$ was considered in [G], where the averages were defined with respect to a matrix norm. A different proof of Theorem 1 is given in [GO]. The approach presented here allows us to show that the convergence is uniform whereas in [GO], the uniformness doesn't seem obvious from the method of the proof. While the proof in [GO] uses equidistribution of solvable flows on $\Gamma\backslash G$, our proof is based on the strong proximality of the action of G on G/P (see Theorem 2 below). This result is of independent interest, and it might be useful for other applications.

Recall that an action of a group H on a compact metric space (Y, d) is called *proximal* if for every $u, v \in Y$ there exists a sequence $\{h_n\} \subset H$ such that $d(h_n u, h_n v) \to 0$ as $n \to \infty$. The fact that the action of G on G/P is proximal plays important role in the study of random walks on G (see, for example, [F]). It turns out that a typical sequence in G acts on G/P in proximal fashion.

Theorem 2 (**Strong proximality**)**.** *Let $\mathcal{O}$ be a neighborhood of the diagonal in $G/P \times G/P$ and $u, v \in G/P$. Then*

$$\lim_{T\to\infty} \frac{\mathrm{Vol}(\{g \in G_T(\tilde{x}) : (gu, gv) \notin \mathcal{O}\})}{\mathrm{Vol}(G_T(\tilde{x}))} = 0$$

and

$$\lim_{T\to\infty} \frac{|\{\gamma \in \Gamma_T(\tilde{x}) : (\gamma u, \gamma v) \notin \mathcal{O}\}|}{|\Gamma_T(\tilde{x})|} = 0$$

uniformly on u, v.

In the case of the real hyperbolic space, Theorem 2 was proved in [Ma] using geometric methods.

2. Proof of Theorem 2

2.1. **Cartan decomposition.** Let $G = K\exp(\mathfrak{p})$ be the Cartan decomposition of G and $A \subset \exp(\mathfrak{p})$ a split Cartan subgroup of G, that is, a maximal connected abelian subgroup in $\exp(\mathfrak{p})$. We fix a system of positive roots Σ^+ on $\mathfrak{a} = \mathrm{Lie}(A)$, and let

$$A^+ = \{a \in A : \alpha(\log a) \geq 0 \text{ for all } \alpha \in \Sigma^+\}$$

denote the closed positive Weyl chamber in A. Then $G = KA^+K$, and a Haar measure on G can be given by

$$\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(k_1 a k_2)\xi(\log a)d\nu(k_1)da d\nu(k_2), \quad \psi \in C_c(G), \tag{2}$$

where da denotes the Lebesgue measure on A,

$$\xi(s) = \prod_{\alpha\in\Sigma^+} \sinh(\alpha(s))^{m_\alpha}, \quad s \in \mathfrak{a},$$

and m_α denotes the dimension of the root space for the root $\alpha \in \Sigma^+$.

Let $\tilde{g} \in G$ be such that $x\tilde{g} = \tilde{x}$. Then $G = \tilde{g}^{-1}KA^+K$, $G_T(\tilde{x}) = \tilde{g}^{-1}KA_T^+K$, and

$$\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(\tilde{g}^{-1}k_1 a k_2)\xi(\log a)d\nu(k_1)da d\nu(k_2), \quad \psi \in C_c(G). \tag{3}$$

In particular, it follows that

$$\mathrm{Vol}(G_T(\tilde{x})) = \mathrm{Vol}(G_T) = \int_{A_T^+} \xi(\log a)da. \tag{4}$$

2.2. **Reduction to maximal parabolics.** Fix a system of simple roots

$$\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Sigma^+.$$

Here $r = \dim A$ is the $\mathbb{R}$-rank of G. It is well-known that the closed subgroups of G that contain P are in one-to-one correspondence with the subsets of Π (see [W, Sec. 1.2]). In particular, $P_i = P_{\{\alpha_i\}}$, $i = 1, \dots, r$, are the maximal parabolic subgroups of G and

$$P = \bigcap_{i=1}^r P_i.$$

We consider the projection maps

$$\pi_i : G/P \times G/P \to G/P_i \times G/P_i, \quad i = 1, \dots, r.$$

Let Δ and Δ_i denote the diagonals in $G/P \times G/P$ and $G/P_i \times G/P_i$ respectively. Then

$$\Delta = \bigcap_{i=1}^r \pi_i^{-1}(\Delta_i).$$

Since

$$\prod_{i=1}^{r}\pi_i:\ G/P\times G/P\to\prod_{i=1}^{r}G/P_i\times G/P_i$$

is a continuous injective map from a compact space to a Hausdorff space, it is a homeomorphism onto its image. It follows that for any neighborhood $\mathcal{O}$ of Δ in $G/P\times G/P$, there exist neighborhoods $\mathcal{O}_i$ of Δ_i in $G/P_i\times G/P_i$ such that

$$\mathcal{O}\supset\bigcap_{i=1}^{r}\pi_i^{-1}(\mathcal{O}_i).$$

Then for every $(u,v)\in G/P\times G/P$,

$$\{g\in G: g\cdot(u,v)\notin\mathcal{O}\}\subset\bigcup_{i=1}^{r}\{g\in G: g\cdot\pi_i(u,v)\notin\mathcal{O}_i\}.$$

This inclusion shows that it suffices to prove Theorem 2 under the assumption that P is a maximal parabolic subgroup of G. We keep this assumption until the end of this section.

2.3. **Dynamics on projective space.** By a result from [T], there is an irreducible representation $G\to\mathrm{GL}(V)$ such that the highest weight space is one-dimensional, and the stabilizer of this space is P. We consider the induced action of G on the projective space $\mathbb{P}(V)$, and let $w^+\in\mathbb{P}(V)$ be the direction of the highest weight space. The map $g\mapsto gw^+$ defines an embedding of G/P in $\mathbb{P}(V)$. Note that if λ is the highest weight, the other weights of the representation are of the form $\lambda-\sum_{\alpha\in\Sigma^+}n_\alpha\alpha$ for integers $n_\alpha\ge 0$. We denote by $V^<$ the sum of all root spaces with weights other than λ. We fix a K-invariant scalar product on V, which gives rise to a metric d on $\mathbb{P}(V)$, which is K-invariant. Put $\tilde{d}(w_1,w_2)=d(\tilde{g}w_1,\tilde{g}w_2)$. Let $V^<_\varepsilon$ be the open ε-neighborhood of $V^<$ in $\mathbb{P}(V)$ with respect to the metric $\tilde{d}$.

For $w\in\mathbb{P}(V)$ and $\tau>0$, define

$$K_\tau(w)=\{k\in K: kw\notin V^<_\tau\}.$$

Lemma 3. *For every $w\in G\cdot w^+$,*

$$\lim_{\tau\to 0^+}\nu(K-K_\tau(w))=0.$$

Proof. It follows from the Iwasawa decomposition that $G\cdot w^+=K\cdot w^+$. Thus, without loss of generality, we may assume that $w=w^+$. By the continuity of the measure, it suffices to prove that

$$\nu(\{k\in K: kw^+\in V^<\})=0.$$

Suppose that this is false. For a subspace W of V, define

$$K_W=\{k\in K: kw^+\in W\}.$$

Let W be a minimal subspace of $V^<$ such that $\nu(K_W) > 0$. We claim that $\mathrm{Stab}_K(W) = K$. If $\mathrm{Stab}_K(W)$ has infinite index in K, then there exist $k_i \in K$, $i \geq 1$, such that $k_i W \neq k_j W$ for $i \neq j$. Since all sets $k_i K_W \subset K$, $i \geq 1$, have the same positive measure, it follows that for some $i \neq j$, $k_i K_W \cap k_j K_W$ has positive measure. Then $k_j^{-1} k_i K_W \cap K_W$ has positive measure too, and for $k \in k_j^{-1} k_i K_W \cap K_W$,

$$kw^+ \in k_j^{-1} k_i W \cap W.$$

Since $k_j^{-1} k_i W \cap W$ is a proper subspace of W, this contradicts the choice of W. Thus, $\mathrm{Stab}_K(W)$ is a closed subgroup of finite index in K. Since K is connected, it follows that $K = \mathrm{Stab}_K(W)$. Then $w^+ \in K_W^{-1} W \subset V^<$. This contradiction proves the lemma. □

We consider the sets

$$\begin{aligned} A_T^\eta &= \{a \in A_T : \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+\}, \\ G_{T,\varepsilon}(u,v) &= \{g \in G_T(\tilde{x}) : \tilde{d}(gu, gv) > \varepsilon\}, \\ \Omega_{T,\tau}^\eta(u,v) &= \tilde{g}^{-1} K A_T^\eta (K_\tau(u) \cap K_\tau(v)) \end{aligned} \tag{5}$$

defined for $T, \eta, \tau, \varepsilon > 0$ and $u, v \in \mathbb{P}(V)$.

Lemma 4. *For every $\varepsilon > 0$ and $\tau > 0$, there exists $\eta > 0$ such that for every $T > 0$ and $u, v \in G \cdot w^+$,*

$$\Omega_{T,\tau}^\eta(u,v) \cap G_{T,\varepsilon}(u,v) = \emptyset. \tag{6}$$

Proof. Note that an element $a \in A_T^\eta$ acts by diagonal transformations on V with respect to some fixed basis, and the eigenvalue associated to the vector w^+ is at least e^η times greater than the other eigenvalues. Therefore, for all $w \notin V_\tau^<$ and sufficiently large η (depending only on τ and ε), we have $d(aw, w^+) < \varepsilon/2$ when $a \in A_T^\eta$. Thus, for

$$\tilde{g}^{-1} k_1 a k_2 \in \Omega_{T,\tau}^\eta(u,v) = \tilde{g}^{-1} K A_T^\eta (K_\tau(u) \cap K_\tau(v)),$$

we have

$$\tilde{d}(\tilde{g}^{-1} k_1 a k_2 u, \tilde{g}^{-1} k_1 a k_2 v) = d(a k_2 u, a k_2 v) \leq d(a k_2 u, w^+) + d(a k_2 v, w^+) < \varepsilon,$$

This proves the lemma. □

2.4. **Completion of the proof.**

By (3),

$$\mathrm{Vol}(\Omega_{T,\tau}^\eta(u,v)) = \left(\int_{A_T^\eta} \xi(\log a) da \right) \cdot \nu(K_\tau(u) \cap K_\tau(v)). \tag{7}$$

Let $\varepsilon, \delta \in (0,1)$. Using Lemma 3, we choose $\tau > 0$ such that

$$\nu(K_\tau(u) \cap K_\tau(v)) > 1 - \delta.$$

Let $\eta > 0$ be as Lemma 4. By Lemma 9(a), for sufficiently large T,

$$\int_{A_T^\eta} \xi(a)da \geq (1-\delta)\int_{A_T^+} \xi(\log a)da.$$

Thus, it follows from (4) and (7) that

$$\mathrm{Vol}(\Omega_{T,\tau}^\eta(u,v)) \geq (1-\delta)^2 \mathrm{Vol}(G_T(\tilde{x})).$$

for sufficiently large $T > 0$. Therefore, by (6),

$$\mathrm{Vol}(G_{T,\varepsilon}(u,v)) \leq (1-(1-\delta)^2)\mathrm{Vol}(G_T(\tilde{x}))$$

for all $\delta \in (0,1)$ and sufficiently large $T > 0$. Since the sets

$$\{(g_1P, g_2P) : \tilde{d}(g_1w^+, g_2w^+) < \varepsilon\}, \quad \varepsilon > 0,$$

form a base of the neighborhoods of the diagonal in $G/P \times G/P$, this proves the first part of Theorem 2.

To prove the second part of Theorem 2, we choose a neighborhood $\mathcal{P}$ of e in G and a neighborhood $\mathcal{Q}$ of the diagonal in $G/P \times G/P$ such that

$$\mathcal{P}^{-1}\mathcal{P} \cap \Gamma = \{e\}, \tag{8}$$
$$\mathcal{P}^{-1} \cdot \mathcal{Q} \subset \mathcal{O}, \tag{9}$$
$$\mathcal{P} \cdot G_T(\tilde{x}) \subset G_{T+c}(\tilde{x}). \tag{10}$$

for fixed $c > 0$ and all $T > 0$. Here we use that Γ is discrete, the space G/P is compact, and the metric on the symmetric space is uniformly continuous. By (9), for every $\gamma \in \Gamma$ such that $\gamma \cdot (u,v) \notin \mathcal{O}$, we have $\mathcal{P}\gamma \cdot (u,v) \cap \mathcal{Q} = \emptyset$. Thus, using (10), we deduce that

$$\mathcal{P} \cdot \{\gamma \in \Gamma_T(\tilde{x}) : \gamma \cdot (u,v) \notin \mathcal{O}\} \subset \{g \in G_{T+c}(\tilde{x}) : g \cdot (u,v) \notin \mathcal{Q}\}.$$

Then by (8), $\mathcal{P}\gamma_1 \cap \mathcal{P}\gamma_2 = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, and

$$\begin{aligned} |\{\gamma \in \Gamma_T(\tilde{x}) : \gamma \cdot (u,v) \notin \mathcal{O}\}| &\leq \frac{1}{\mathrm{Vol}(\mathcal{P})}\mathrm{Vol}(\{g \in G_{T+c}(\tilde{x}) : g \cdot (u,v) \notin \mathcal{Q}\}) \\ &= o(\mathrm{Vol}(G_{T+c}(\tilde{x}))) \end{aligned}$$

as $T \to \infty$. Now the second statement of Theorem 2 follows from Lemma 9(d) and (1).

3. Equidistribution on $\Gamma\backslash G$

Recall that K is a maximal compact subgroups of G, and ν is the probability Haar measure on K. Denote by ϱ a right Haar measure on the minimal parabolic subgroup P. For a suitable normalization of ϱ, the Haar measure on G is given by

$$\int_G \psi(g)dg = \int_K\int_P \psi(kp)d\varrho(p)d\nu(k), \quad \psi \in C_c(G). \tag{11}$$

We also define a measure μ on G by

$$\int_G \psi(g)d\mu(g) = \int_K \int_P \psi(kp^{-1})d\varrho(p)d\nu(k), \quad \psi \in C_c(G). \tag{12}$$

Note that μ is left K-invariant.

The first step in the proof of Theorem 1 is the following result.

Proposition 5. *For every $\Psi \in C_c(\Gamma\backslash G)$ and $z \in \Gamma\backslash G$,*

$$\lim_{T\to\infty} \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg)d\mu(g) = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi\, dg$$

where $G_T = \{g \in G : d(x, xg) < T\}$.

Proposition 5 is a consequence of the equidistribution of translates of K in $\Gamma\backslash G$ proved by Eskin and McMullen in [EM] (see also [S] for a more general result). They showed that for every strongly divergent sequence $\{g_n\} \subset G$,

$$\lim_{n\to\infty} \int_K \Psi(zkg_n)d\nu(k) = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi\, dg. \tag{13}$$

Recall that a sequence $\{g_n\} \subset G$ is *strongly divergent* if the projection of $\{g_n\}$ on every noncompact simple factor of G is divergent. Note that (13) was proved in [EM] under the condition that the lattice Γ is irreducible. Since the proof of (13) is based on mixing properties of the action of G on $\Gamma\backslash G$, it is applicable to the case of a reducible lattice Γ provided that the sequence $\{g_n\}$ is strongly divergent.

Denote by $\pi_i : G \to G_i$, $i = 1, \dots, s$, the projections of G onto its simple factors. Let $C_{i,j} \subset G_i$, $j \ge 1$, be an increasing sequence of compact subsets such that $G_i = \cup_{j\ge1} C_{i,j}$. Define

$$G_{T,n} = G_T - \bigcup_{1\le i\le s} \pi_i^{-1}(C_{i,n}). \tag{14}$$

Lemma 6. *For every $n \ge 1$, $\mu(G_{T,n}) \sim \mu(G_T)$ as $T \to \infty$.*

Proof. It suffices to show that for every $i = 1, \dots, s$ and $n \ge 1$,

$$\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = o(\mu(G_T)) \quad \text{as} \quad T \to \infty.$$

Fix $i = 1, \dots, s$ and $n \ge 1$. Note that $G = DH$, where D and $H = \ker(\pi_i)$ are normal connected semisimple Lie subgroups with finite centers, and D and H commute. We have $\pi_i^{-1}(C_{i,n}) = D_{i,n}H$ for some compact set $D_{i,n} \subset D$. There is a constant $\delta > 0$ such that

$$D_{i,n}H_{T-\delta} \subset (D_{i,n}H)_T \subset D_{i,n}H_{T+\delta} \quad \text{for all } T > 0. \tag{15}$$

We define measures μ_D and μ_H for the groups D and H respectively as in (12). With appropriate normalization, $\mu = \mu_D \otimes \mu_H$. Thus, it follows from (15) that

$$\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = \mu((D_{i,n}H)_T) \ll \mu_H(H_{T+\delta}). \tag{16}$$

Since $G_T = KP_T$ and $P_T^{-1} = P_T$, using (11) and (12), we conclude that

$$\mu(G_T) = \varrho(P_T^{-1}) = \varrho(P_T) = \mathrm{Vol}(G_T). \tag{17}$$

Similarly, $H = LQ_T$ where L is a maximal compact subgroup of H contained in K, and Q is a minimal parabolic subgroup of H. As in (17), we deduce that $\mu_H(H_T) = \mathrm{Vol}_H(H_T)$. By (16), it is sufficient to show that

$$\mathrm{Vol}_H(H_{T+\delta}) = o(\mathrm{Vol}(G_T)) \quad \text{as} \quad T \to \infty. \tag{18}$$

Note that with appropriate normalization the Haar measure on G is the product of Haar measures on D and H. Without loss of generality, $\mathrm{Vol}_D(D_{i,n}) > 0$. Then by (15),

$$\mathrm{Vol}_H(H_{T+\delta}) \ll \mathrm{Vol}(D_{i,n}H_{T+\delta}) \leq \mathrm{Vol}((D_{i,n}H)_{T+2\delta}).$$

Let G_T^η be defined as in (24). Since the set $D_{i,n}$ is compact, there exists $\eta > 0$ such that

$$(D_{i,n}H)_{T+2\delta} \subset G_{T+2\delta} - G_{T+2\delta}^\eta.$$

Thus, (18) follows from Lemma 9(b). □

Proof of Proposition 5. The map $K \times A^+ \times K \to G$ is a diffeomorphism on an open set of full measure. Since the measure μ is left K-invariant and smooth, for some $\sigma \in C(A^+ \times K)$,

$$\int_G \psi(g)d\mu(g) = \int_K \int_{A^+} \int_K \psi(k_1ak_2)\sigma(a,k_2)d\nu(k_1)dad\nu(k_2), \quad \psi \in C_c(G).$$

Let $G_{T,n}$ be defined as in (14), and it is K-bi-invariant (equivalently, all $C_{i,j}$ are $\pi_i(K)$-bi-invariant). Then

$$G_{T,n} = KA_{T,n}^+K \text{ and } \mu(G_{T,n}) = \int_K \int_{A_{T,n}^+} \sigma(a,k_2)dad\nu(k_2),$$

where $A_{T,n}^+ = G_{T,n} \cap A^+$.

Let $\varepsilon > 0$. By (13),

$$\left| \int_K \Psi(zk_1ak_2)d\nu(k_1) - \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi \, dg \right| < \varepsilon$$

for $a \in A^+_{T,n}$ and $k_2 \in K$ when $n > n_0(\varepsilon)$. Thus, for $n > n_0(\varepsilon)$,

$$\begin{aligned}
(19) \qquad & \left| \int_{G_{T,n}} \Psi(zg)d\mu(g) - \frac{\mu(G_{T,n})}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi \, dg \right| \\
= \; & \left| \int_K \int_{A^+_{T,n}} \int_K \Psi(zk_1ak_2)d\nu(k_1)\sigma(a,k_2)dad\nu(k_2) \right. \\
- \; & \left. \frac{\mu(G_{T,n})}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi \, dg \right| \le \int_K \int_{A^+_{T,n}} \left| \int_K \Psi(zk_1ak_2)d\nu(k_1) \right. \\
- \; & \left. \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi \, dg \right| \sigma(a,k_2)dad\nu(k_2) < \varepsilon\mu(G_{T,n}).
\end{aligned}$$

By Lemma 6, for every $n \ge 1$,

$$\int_{G_T} \Psi(zg)d\mu(g) = \int_{G_{T,n}} \Psi(zg)d\mu(g) + o(\mu(G_{T,n}))$$

as $T \to \infty$. Thus, it follows from (19) that

$$\limsup_{T\to\infty} \left| \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg)d\mu(g) - \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi \, dg \right| < \varepsilon$$

for every $\varepsilon > 0$. This proves the proposition. □

4. Equidistribution on average

In this section we prove that Theorem 1 holds "on average". In the case of hyperbolic spaces, the following proposition is a consequence of the work of Roblin [R].

Proposition 7. *For every $f \in C(G/P)$ and $y \in G/P$,*

$$\lim_{T\to\infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma\in\Gamma_T(\tilde{x})} \int_K f(\gamma k y)d\nu(k) = \int_{G/P} f dm_{\tilde{x}}$$

where $\Gamma_T(\tilde{x}) = \{\gamma \in \Gamma : d(x, \tilde{x}\gamma) < T\}$.

Proof. There exists $\tilde{p} \in P$ such that $\tilde{x} = x\tilde{p}$. Then $\tilde{K} = \tilde{p}^{-1}K\tilde{p}$, and it follows from (11) that

$$(20) \qquad \int_G \psi(g)dg = \int_{\tilde{K}} \int_P \psi(k\tilde{p}^{-1}p)d\varrho(p)d\tilde{\nu}(k), \quad \psi \in C_c(G).$$

Without loss of generality, $f \ge 0$, and since $G = KP$, we may assume that $y = eP$. Let $\varepsilon > 0$, $\mathcal{O}_\varepsilon = \{z \in X : d(x,z) < \varepsilon\}$, and $\phi_\varepsilon \in C_c(X)$ such that

$$\phi_\varepsilon \ge 0, \quad \mathrm{supp}(\phi_\varepsilon) \subset \mathcal{O}_\varepsilon, \quad \int_P \phi_\varepsilon(xp^{-1})d\varrho(p) = 1.$$

Since $X = \tilde{x}P$ and ϱ is right invariant, it follows that

$$\int_P \phi_\varepsilon(zp^{-1})d\varrho(p) = 1 \quad \text{for every } z \in X. \tag{21}$$

Let

$$\psi_\varepsilon(g) = f(gP)\phi_\varepsilon(\tilde{x}g), \quad g \in G.$$

Clearly, $\psi_\varepsilon \in C_c(G)$ and

$$\Psi_\varepsilon(\Gamma g) \overset{def}{=} \sum_{\gamma\in\Gamma} \psi_\varepsilon(\gamma g) \in C_c(\Gamma\backslash G).$$

By Proposition 5,

$$\lim_{T\to\infty} \frac{1}{\mu(G_T)} \sum_{\gamma\in\Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)d\mu(g) = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \Psi_\varepsilon(\Gamma g)dg \tag{22}$$

and by (20),

$$\begin{aligned} \mathrm{Vol}(\Gamma\backslash G) \int_{\Gamma\backslash G} \Psi_\varepsilon(\Gamma g)dg &= \int_G \psi_\varepsilon(g)dg = \int_{\tilde{K}} f(kP)d\tilde{\nu}(k) \cdot \int_P \phi_\varepsilon(\tilde{x}\tilde{p}^{-1}p)d\varrho(p) \\ &= \int_{G/P} f dm_{\tilde{x}} \cdot \int_P \phi_\varepsilon(xp)d\varrho(p). \end{aligned}$$

Denote by δ the modular function of P. By (21),

$$\begin{aligned} \left| \int_P \phi_\varepsilon(xp)d\varrho(p) - 1 \right| &= \left| \int_P \phi_\varepsilon(xp^{-1})(\delta(p)-1)d\varrho(p) \right| \\ &\leq \max\{|\delta(p)-1| : xp^{-1} \in \mathcal{O}_\varepsilon\}. \end{aligned}$$

The sets $\{p \in P : xp^{-1} \in \mathcal{O}_\varepsilon\}$, $\varepsilon > 0$, form a base of neighborhoods of $P \cap K$ in P. Since $\delta|_{P\cap K} = 1$ and $P \cap K$ is compact,

$$\max\{|\delta(p)-1| : xp^{-1} \in \mathcal{O}_\varepsilon\} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Thus, it follows from (22) that

$$\lim_{\varepsilon\to 0^+} \lim_{T\to\infty} \frac{1}{\mu(G_T)} \sum_{\gamma\in\Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)d\mu(g) = \int_{G/P} f dm_{\tilde{x}}. \tag{23}$$

Since $G_T = KP_T$,

$$\begin{aligned} &\sum_{\gamma\in\Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)d\mu(g) \\ \overset{(12)}{=}\; &\sum_{\gamma\in\Gamma} \int_{K\times P_T^{-1}} \psi_\varepsilon(\gamma kp^{-1})d\nu(k)d\varrho(p) \\ =\; &\sum_{\gamma\in\Gamma} \int_K f(\gamma kP) \left(\int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) \right) d\nu(k). \end{aligned}$$

For $\gamma \in \Gamma - \Gamma_{T+\varepsilon}(\tilde{x})$, $k \in K$, and $p \in P_T^{-1}$,

$$d(x, \tilde{x}\gamma kp^{-1}) = d(xpk^{-1}, \tilde{x}\gamma) \geq d(x, \tilde{x}\gamma) - d(x, xpk^{-1}) \geq \varepsilon.$$

This implies that $\int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) = 0$ for $\gamma \in \Gamma - \Gamma_{T+\varepsilon}(\tilde{x})$. Thus,

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \int_{G_T} \psi_\varepsilon(\gamma g) d\mu(g) \\
= &\sum_{\gamma\in\Gamma_{T+\varepsilon}(\tilde{x})} \int_K f(\gamma kP) \left(\int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) \right) d\nu(k) \\
\leq &\sum_{\gamma\in\Gamma_{T+\varepsilon}(\tilde{x})} \int_K f(\gamma kP) \left(\int_P \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) \right) d\nu(k) \\
\overset{(21)}{=} &\sum_{\gamma\in\Gamma_{T+\varepsilon}(\tilde{x})} \int_K f(\gamma kP) d\nu(k).
\end{aligned}$$

Combining (23), (17), (1) and Lemma 9(c), we deduce that

$$\liminf_{T\to\infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma\in\Gamma_T(\tilde{x})} \int_K f(\gamma kP) d\nu(k) \geq \int_{G/P} f dm_{\tilde{x}}.$$

On the other hand, for $\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})$, $k \in K$, and $p \in P$ such that $d(x, \tilde{x}\gamma kp^{-1}) < \varepsilon$,

$$d(x, xp^{-1}) \leq d(x, \tilde{x}\gamma kp^{-1}) + d(xp^{-1}, \tilde{x}\gamma kp^{-1}) < T.$$

This shows that for $\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})$,

$$\int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) = \int_P \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) \overset{(21)}{=} 1.$$

Hence,

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \int_{G_T} \psi_\varepsilon(\gamma g) d\mu(g) \\
\geq &\sum_{\gamma\in\Gamma_{T-\varepsilon}(\tilde{x})} \int_K f(\gamma kP) \left(\int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1}) d\varrho(p) \right) d\nu(k) \\
= &\sum_{\gamma\in\Gamma_{T-\varepsilon}(\tilde{x})} \int_K f(\gamma kP) d\nu(k).
\end{aligned}$$

By (23), (17), (1), and Lemma 9(c),

$$\limsup_{T\to\infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma\in\Gamma_T(\tilde{x})} \int_K f(\gamma kP) d\nu(k) \leq \int_{G/P} f dm_{\tilde{x}}.$$

This proves the proposition. □

5. Proof of Theorem 1

Now the proof can be completed using the argument from [Ma]. Let $\varepsilon > 0$. Since the space $G/P \times G/P$ is compact, there exists a neighborhood $\mathcal{O}$ of the diagonal in $G/P \times G/P$ such that for every $(z_1, z_2) \in \mathcal{O}$, we have $|f(z_1) - f(z_2)| < \varepsilon$. Then for every $k \in K$,

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma_T(\tilde{x})} f(\gamma y) - \sum_{\gamma\in\Gamma_T(\tilde{x})} f(\gamma k y)\right| \\
\leq\ & \sum_{\gamma\in\Gamma_T(\tilde{x}):(\gamma y,\gamma k y)\in\mathcal{O}} |f(\gamma y) - f(\gamma k y)| + \sum_{\gamma\in\Gamma_T(\tilde{x}):(\gamma y,\gamma k y)\notin\mathcal{O}} |f(\gamma y) - f(\gamma k y)| \\
\leq\ & \varepsilon|\Gamma_T(\tilde{x})| + 2\sup|f| \cdot |\{\gamma\in\Gamma_T(\tilde{x}) : (\gamma y, \gamma k y)\notin\mathcal{O}\}|.
\end{aligned}$$

Thus, it follows from Theorem 2 that

$$\lim_{T\to\infty} \frac{1}{|\Gamma_T(\tilde{x})|}\left|\sum_{\gamma\in\Gamma_T(\tilde{x})} f(\gamma y) - \sum_{\gamma\in\Gamma_T(\tilde{x})} f(\gamma k y)\right| = 0$$

for all $k \in K$. Hence, by the dominated convergence theorem,

$$\lim_{T\to\infty}\left|\frac{1}{|\Gamma_T(\tilde{x})|}\sum_{\gamma\in\Gamma_T(\tilde{x})} f(\gamma y) - \frac{1}{|\Gamma_T(\tilde{x})|}\sum_{\gamma\in\Gamma_T(\tilde{x})}\int_K f(\gamma k y)d\nu(k)\right| = 0.$$

Finally, Theorem 1 follows from Proposition 7.

6. Appendix: volume estimates

In this section, we give proofs of volume estimates, which are used in Theorems 1 and 2. There are other ways to establish these volume estimates. For example, one can use the exact asymptotics of the volume of Riemannian balls from [Kn] (see also [GO]). We present a straightforward proof that does not use asymptotics.

Let $\mathfrak{a}$ be the Lie algebra of the Cartan subgroup A and $\mathfrak{a}^+$ the positive Weyl chamber with respect to the root system Σ^+. The Riemannian metric defines a scalar product on $\mathfrak{a}$ and, by duality, on the dual space of $\mathfrak{a}$. For $\alpha \in \Sigma^+$, we denote by m_α the dimension of the corresponding root space and put $\rho = \frac{1}{2}\sum_{\beta\in\Sigma^+} m_\beta\beta$.

Lemma 8. *The maximum of ρ on $\{a \in \mathfrak{a} : \|a\| \leq 1\}$ is achieved at a unique point in the interior of $\mathfrak{a}^+$.*

Proof. Since the set $\{a \in \mathfrak{a} : \|a\| = 1\}$ is strictly convex, it is clear that the point of maximum is unique. It is sufficient to show that $(\rho, \alpha) > 0$ for every $\alpha \in \Sigma^+$. Denote by σ_α the reflection with respect to the hyperplane $\{\alpha = 0\}$. The map σ_α permutes the elements of the set $\Sigma^+ - \{\alpha, 2\alpha\}$ and $\sigma_\alpha(\alpha) = -\alpha$. Since $m_{\sigma_\alpha(\beta)} = m_\beta$, we have

$$\sigma_\alpha(\rho) = \rho - 2m_\alpha\alpha - 4m_{2\alpha}\alpha.$$

Thus,

$$(\rho,\alpha) = (\sigma_\alpha(\rho),\sigma_\alpha(\alpha)) = 2m_\alpha(\alpha,\alpha) + 4m_{2\alpha}(\alpha,\alpha) - (\rho,\alpha)$$

and $(\rho,\alpha) = (m_\alpha + 2m_{2\alpha})(\alpha,\alpha)$ is positive. □

For $T,\eta > 0$, define

$$\begin{aligned} A_T^\eta &= \{a \in A_T \,:\, \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+\} \\ &= \{a \in A \,:\, \|\log a\| < T,\, \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+\}, \\ G_T^\eta &= K A_T^\eta K. \end{aligned} \tag{24}$$

Lemma 9. *For every $\eta > 0$,*

(a)

$$\int_{A_T^\eta} \xi(\log a)da \sim_{T\to\infty} \int_{A_T^+} \xi(\log a)da,$$

(b)

$$\mathrm{Vol}(G_T^\eta) \sim_{T\to\infty} \mathrm{Vol}(G_T),$$

(c)

$$\liminf_{\varepsilon\to 0^+}\left(\limsup_{T\to\infty} \frac{\mathrm{Vol}(G_{T+\varepsilon})}{\mathrm{Vol}(G_T)}\right) = 1,$$

(d)

$$\mathrm{Vol}(G_{T+\eta}) \ll \mathrm{Vol}(G_T).$$

Proof. We have

$$\int_{\mathfrak{a}_T^+} \xi(a)da = 2^{-|\Sigma^+|}\sum_{i\in I}\int_{\mathfrak{a}_T^+} e^{\lambda_i(a)}da \tag{25}$$

where λ_i's the characters of the form $2\rho - \sum_{\alpha\in\Sigma^+} n_\alpha \alpha$ for some $n_\alpha \geq 0$. Let

$$\begin{aligned} \delta &= \max\{2\rho(a) : a \in \mathfrak{a}_1^+\}, \\ \delta_i &= \max\{\lambda_i(a) : a \in \mathfrak{a}_1^+\}, \quad i \in I, \\ \delta_\alpha &= \max\{2\rho(a) : a \in \mathfrak{a}_1^+, \alpha(a) = 0\}, \quad \alpha \in \Sigma^+. \end{aligned}$$

It follows from Lemma 8 that for $\lambda_i \neq 2\rho$ and $\alpha \in \Sigma^+$, $\delta > \max\{\delta_i,\delta_\alpha\}$. Thus,

$$\int_{\mathfrak{a}_T^+} e^{\lambda_i(a)}da \leq \mathrm{Vol}(\mathfrak{a}_T^+)e^{\delta_i T} \ll T^r e^{\delta_i T} \tag{26}$$

where $r = \dim \mathfrak{a}$. Let $\varepsilon > 0$ be such that

$$\delta - \varepsilon > \max\{\delta_i, \delta_\alpha : i \in I, \alpha \in \Sigma^+\}.$$

Then

$$\begin{aligned} \int_{\mathfrak{a}_T^+} e^{2\rho(a)}da &= T^r \int_{\mathfrak{a}_1^+} e^{2T\rho(a)}da \\ &\geq T^r e^{(\delta-\varepsilon)T}\mathrm{Vol}(\{a \in \mathfrak{a}_1^+ : 2\rho(a) \geq \delta - \varepsilon\}) \gg T^r e^{(\delta-\varepsilon)T}. \end{aligned} \tag{27}$$

Combining (25), (26), and (27), we deduce that

$$\int_{\mathfrak{a}_T^+} \xi(a)da \gg T^r e^{(\delta-\varepsilon)T}. \tag{28}$$

On the other hand, for $\alpha \in \Sigma^+$,

$$\begin{aligned}\int_{\mathfrak{a}_T^+\cap\{\alpha<\eta\}} \xi(a)da &\le \int_{\mathfrak{a}_T^+\cap\{\alpha<\eta\}} e^{2\rho(a)}da \ll \int_{\mathfrak{a}_T^+\cap\{\alpha=0\}} e^{2\rho(a)}da \\ &= T^{r-1}\int_{\mathfrak{a}_1^+\cap\{\alpha=0\}} e^{2T\rho(a)}da \ll T^{r-1}e^{\delta_\alpha T} = o(e^{(\delta-\varepsilon)T}).\end{aligned}$$

Since

$$\mathfrak{a}_T^+ - \mathfrak{a}_T^\eta \subset \bigcup_{\alpha\in\Sigma^+} \mathfrak{a}_T^+ \cap \{\alpha < \eta\}.$$

This proves part (a) of the lemma. Part (b) follows from (2).

To prove part (c), we note that

$$\mathrm{Vol}(G_{T+\varepsilon}) = \int_{\mathfrak{a}_{T+\varepsilon}^+} \xi(a)da = (T+\varepsilon)^r \int_{\mathfrak{a}_1^+} \xi((T+\varepsilon)a)da$$

It is easy to check that there exist $b > 0$ such that $\sinh(t+\varepsilon) \le e^\varepsilon \sinh(t) + b$ for every $\varepsilon \in (0,1)$ and $t \ge 0$. Thus, for $a \in \mathfrak{a}_1^+$ and sufficiently small $\varepsilon > 0$,

$$\xi((T+\varepsilon)a) \le \prod_{\alpha\in\Sigma^+} (a_\varepsilon \sinh(\alpha(Ta)) + b)^{m_\alpha} \le d_\varepsilon \xi(Ta) + C\sum_{i\in I} e^{\lambda_i(a)}$$

where $d_\varepsilon \to 1$ as $\varepsilon \to 0^+$, $C > 0$, and λ_i's are characters such that $2\rho - \lambda_i < 0$ in the interior of $\mathfrak{a}^+$. Thus, it follows from (26) that

$$\int_{\mathfrak{a}_T^+} \xi((T+\varepsilon)a)da \le d_\varepsilon \int_{\mathfrak{a}_T^+} \xi(Ta)da + o(e^{(\delta-\varepsilon)T}).$$

Using (4) and (28), we deduce that

$$\limsup_{T\to\infty} \frac{\mathrm{Vol}(G_{T+\varepsilon})}{\mathrm{Vol}(G_T)} \le d_\varepsilon,$$

and part (c) of the lemma follows. The last part of lemma can be proved similarly. □

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